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Algebraic Equations for a Class of P. L. Spaces

Selman Akbulut
Department of Mathematics, University of Wisconsin, Madison, Wisc. 53706, USA

Our main purpose is to discuss the question: "Can closed P. L. manifolds be represented as real algebraic varieties?" It is known by Nash that all closed smooth manifolds are diffeomorphic to components of real algebraic variables [7]. He also conjectured that they are diffeomorphic to real algebraic varieties. Since all closed P. L. manifolds of dimension less than 8 have smooth structures, they are P. L. homeomorphic to components of real algebraic varieties. For closed P. L. manifolds of dimension 8, Kuiper proves that they are represented by components of real algebraic varieties [3]. Then recently Tognoli proved the Nash conjecture [10]. Here we define a class of polyhedra and prove that the elements of this class are represented as real algebraic varieties (i.e., we prove a P. L. version of the Nash conjecture for these polyhedra). Elements of this class roughly are: Singular spaces which are smooth in the complements of disjoint union of smooth manifolds with certain nice properties. In particular, it contains all P. L. 8-manifolds. All our methods produce algebraic varieties, which are locally complete intersections. We produce a nonsmoothable P. L. 9-mainfold, which is not such a variety.

I would like to thank R. Kirby for his infinite help and encouragement. I also want to thank H. King for reading earlier versions of this work, making valuable comments, and suggesting that I prove a P. L. version of the Nash conjecture.

1. Introduction

Throughout the paper we will consider partially smoothed polyhedra; and a homeomorphism between two such polyhedron $M \approx M'$ is defined to be a P. L. homeomorphism $M \to M'$ which is a diffeomorphism when restricted to the smooth part of M. B^n_r, S^{n-1}_r denotes $\{x \in \mathbb{R}^n : |x| < r\}$ and $\{x \in \mathbb{R}^n : |x| = r\}$ respectively, and for $V \subset \mathbb{R}^n$ we denote $B^n_\varepsilon(V) = \{x \in \mathbb{R}^n :$ distance $(x, V) < \varepsilon\}$. $c\Sigma$, $\mathring{c}\Sigma$ denote the closed and the open cones on Σ respectively. A polynomial $f:(\mathbb{R}^n, 0) \to (\mathbb{R}^k, 0)$ is called an isolated singularity if f has rank k on $f^{-1}(0) - \{0\}$; and Σ is called the link of f if $f^{-1}(0) \cap B^n_\varepsilon \approx \mathring{c}\Sigma$ for some $\varepsilon > 0$. Let $U \subset W \subset \mathbb{R}^n$ be open sets, and $f: W \to \mathbb{R}^k$ be a smooth map, we denote $U_f(\varepsilon, s) = \{g: W \to \mathbb{R}^k : g$ is smooth $|D_\alpha(f-g)|_{\bar{U}}| < \varepsilon$, $|\alpha| \leqq s\}$. If

$g \in U_f(\varepsilon, s)$ for ε small and s large, we say g is close to f on U, and denote it by $g \sim f$, when there is no confusion.

Definition. Let $\Lambda_k = \{\Sigma^k : \Sigma^k$ is the link of a polynomial isolated singularity $f:(\mathbb{R}^{k+r+1}, 0) \to (\mathbb{R}^r, 0)\}$.

Remark 1. For $a = (a_0, a_1, \ldots, a_m)$, let $\Sigma(a)$ denote the link of the singularity $z_0^{a_0} + z_1^{a_1} + \ldots + z_m^{a_m}$ in $\mathbb{C}^{m+1}$, then $\Sigma(a) \in \Lambda_{2m-1}$. In particular $bP_{k+1} \subset \Lambda_k$, where bP_{k+1} is the group of exotic k-spheres which bound parallelizable manifolds.

Definition. Let $\mathfrak{C} = \{M : M$ is compact and $M = M_0 \cup \coprod_{i=1}^{r} c\Sigma_i \times N_i$, where M_0 is a smooth manifold with $\partial M_0 = \coprod \Sigma_i \times N_i$ and where the N_i's are closed smooth manifolds and $\Sigma_i \in \Lambda_{k_i}\}$ where $\coprod$ denotes disjoint union.

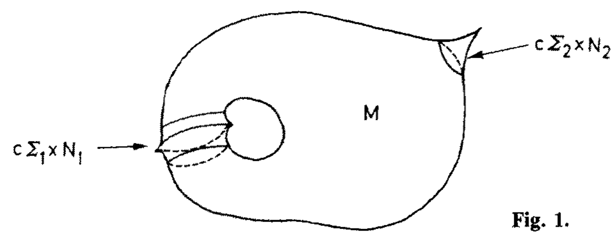


Fig. 1.

Theorem. *If $M \in \mathfrak{C}$ then M can be represented as a real algebraic variety.*

Example 1. The Kervaire and Milnor manifolds, K^{4n+2} and M^{4n}, are elements of $\mathfrak{C}$.

Example 2. Let M^8 be the 3-connected nonsmoothable Milnor manifold. Then the manifold obtained from $Q^8 \times S^5$ by killing π_4 via surgery is an element of $\mathfrak{C}$. Furthermore it has no smooth structure.

The proof of the theorem proceeds by the following steps: Let $M = M_0 \cup c\Sigma \times N \in \mathfrak{C}$.

1. We first construct a cobordism W such that

(i) $\partial W = M \coprod V$, where V is a variety,
(ii) $N(\mathrm{Sing}\, V) \approx \mathring{c}\Sigma \times N$,
(iii) $N(\mathrm{Sing}\, W) \approx \mathring{c}\Sigma \times N \times I$,

where $N(\mathrm{Sing}\, V)$, $N(\mathrm{Sing}\, W)$ denote the neighborhood of the nonsmooth points of V and W respectively.

2. Let Q be the double of W. Since the neighborhood of the nonsmooth points of Q is already a variety ($\approx \mathring{c}\Sigma \times N \times S^1$) we can do a relative Nash approximation to the smooth part of Q. Since the neighborhood of the singular points of Q is a zero set of sufficiently stable polynomials, during the Nash approximation the topological picture of the singular part of Q is not altered. (Here we appeal to the method of [3].)

3. Nash's approximation will make Q a component of the zero set of an algebraic function (not a polynomial). Then we do the Nash normalization to make Q a component of an algebraic variety [7].

4. Finally, we apply Tognoli's trick to get the equations for M from the equations of Q [10, 1].

2. Preliminary Lemmas and the Grassman Variety

Let $G_{n,n-m}=(n-m)$-planes in $\mathbb{R}^n$.

$G_{n,n-m}$ is a nonsingular algebraic variety ([8], p. 39) and is given by the equations:

$$G_{n,n-m}=\{T\in\mathfrak{R}_n : T^2=T,\ \mathrm{trace}(T)=n-m\}$$

where $\mathfrak{R}_n=$symmetric $(n\times n)$-matrices$=\mathbb{R}^b$, $b=\frac{n(n+1)}{2}$. Mow let Y be an arbitrarily small open neighborhood of $G_{n,n-m}$ in $\mathbb{R}^b$. Since Y is the set of symmetric matrices close to $G_{n,n-m}$; $z\in Y$ implies that all the characteristic values of z are close to $\{0,1\}$ (since the characteristic values of matrices in $G_{n,n-m}$ are $\{0,1\}$). Therefore

$$C_z(\lambda)=\det(\lambda I-z)=A_z(\lambda)\,B_z(\lambda)$$

where $A_z(\lambda)$, $B_z(\lambda)$ are monic polynomials, and

$$\begin{aligned} A_z(\lambda)&=(\lambda-\lambda_1(z))\dots(\lambda-\lambda_m(z))\\ &=\lambda^m+a_1(z)\,\lambda^{m-1}+\dots+a_m(z) \quad (*)\\ &\sim\lambda^m\ (\sim \text{ means close to})\\ B_z(\lambda)&=(\lambda-\lambda_{m+1}(z))\dots(\lambda-\lambda_n(z))\\ &\sim(\lambda-1)^{n-m}, \end{aligned}$$

and the λ_i's are real numbers such that $\lambda_i\sim 0$ for $1\leqq i\leqq m$, and $\lambda_i\sim 1$ for $m+1\leqq i\leqq n$. Let $z\in Y$ and consider the matrix $A_z(z)$; any eigenvector of z is an eigenvector of $A_z(z)$. For any eigenvalue λ of z, the corresponding eigenvalue of $A_z(z)$ is $A_z(\lambda)$.

Therefore, the m-eigenvalues of $A_z(z)$ corresponding to the m-small eigenvalues of z [$=$roots of $A_z(\lambda)$] are zero. Also, since $A_z(z)\sim z^m\sim z_1^m=z_1\sim z$ (where z_1 is a point in $G_{n,n-m}$ close to z), then $A_z(z)\in Y$; the remaining $n-m$ eigenvalues of $A_z(z)$ are close to the large eigenvalues of z_1, which are equal to 1. Hence $\mathrm{rank}(A_z(z))=n-m$. Therefore, image $(A_z(z))$ defines a plane. Now define a retraction $\varrho: Y\to G_{n,n-m}$ by $\varrho(z)=\mathrm{image}(A_z(z))$. In particular $\varrho(z)\sim z$, and $A_z(z)(x)$ is the almost (or close to) orthogonal projection of x onto the plane $\varrho(z)$.

We need the following two lemmas, which play an important role in Tognoli's proof of the Nash conjecture. I would like to thank H. King for communicating them to me.

Lemma 1. *Let $V^m\subset\mathbb{R}^n$ be a compact nonsingular real algebraic variety. Let $\alpha: V\to G_{n,n-m}$ be the map which assigns each point of V to the normal plane of V at that point ($=$Gauss map). Then there is a rational function $\frac{P(x)}{Q(x)}$ such that $\alpha(x)=\frac{P(x)}{Q(x)}$ for $x\in V$, where P, Q are polynomials and $Q(x)\neq 0$ for all x.*

Proof. Let V^m be given nonsingularly by the polynomials $g(x)=(g_1(x),\dots,g_k(x))$. Then for any $x\in V^m$ there is a small neighborhood U_x of x in $\mathbb{R}^n$, and $(n-m)$-polynomials $\{g_1,\dots,g_{n-m}\}$ among $\{g_1,\dots,g_k\}$ such that $(g_1,\dots,g_{n-m})^{-1}(0)\cap U_x$

$=V\cap U_x$ and $\operatorname{rank}(g_1,\ldots,g_{n-m})=n-m$ on U_x. Applying the Gram-Schmidt process to $\{dg_1(u),\ldots,dg_{n-m}(u)\}$ we get the orthonormal vectors $\{v_1(u),\ldots,v_{n-m}(u)\}$ which are rational functions of $u\in U_x$. Then

$$\alpha(u)(y)=\sum_{i=1}^{n-m}\langle y,v_i(u)\rangle v_i(u)$$

for $u\in U_x\cap V$. Therefore, $\alpha(u)=\frac{1}{h_x(u)}R_x(u)$ where $R_x(u)$ is a $(n\times n)$-symmetric matrix whose coordinates are polynomials in u, and $h_x(u)$ is a polynomial in u.

Since V is compact, we can find $x_1,\ldots,x_r$ so that $V\subset\bigcup_{i=1}^{r}U_{x_i}$. Then

$$\alpha(u)=\frac{\sum_{i=1}^{r}R_{x_i}(u)h_{x_i}(u)}{\sum_{i=1}^{r}h_{x_i}^2(u)+|g(u)|^2}. \quad\square$$

Lemma 2. *Let $F:(\bar U,V)\to(\mathbb{R},0)$ be a smooth function, $U\subset\mathbb{R}^n$ be a bounded open set, and $V^m\subset U$ be a real algebraic variety given by the equations $\beta_1(x)=0,\ldots,\beta_r(x)=0$. Let F be in the form $\sum_{i=1}^{r}\alpha_i\beta_i$ in a neighborhood* $\operatorname{Sing}(V)$ $=\{x\in V:\operatorname{rank}(\beta_1,\ldots,\beta_r)<n-m\}$ *where $\alpha_1,\ldots,\alpha_r$ are smooth functions. Then for any ε, $s>0$, there is a polynomial $\Phi:(\mathbb{R}^n,V)\to(\mathbb{R},0)$ such that $\Phi\in U_F(\varepsilon,s)$.*

Proof. We first claim that $F(x)=\Sigma\bar\alpha_i(x)\beta_i(x)$ for $x\in U$ where $\bar\alpha_1,\ldots,\bar\alpha_r$ are some smooth functions. It suffices to express F locally as above; then by partitions of unity we put the local expressions together to get the required global expression. If $x\notin V$, there exists i such that $\beta_i(x)\neq 0$; then we simply write $F(x)=(F(x)/\beta_i(x))\beta_i(x)$ locally. If $x\in\operatorname{Sing}(V)$ then by hypothesis $F(x)$ is already in that form. If $x\in V-\operatorname{Sing}(V)$ we can pick $\beta_1,\ldots,\beta_{n-m}$ among $\beta_1,\ldots,\beta_r$ such that $\operatorname{rank}(\beta_1,\ldots,\beta_{n-m})=n-m$ near $x\in\mathbb{R}^n$. We can complete $\beta_1,\ldots,\beta_{n-m}$ to a local coordinate system $\beta=(\beta_1,\ldots,\beta_{n-m},\beta'_{n-m+1},\ldots,\beta'_n)$. Then

$$F(x)=\sum_{i=1}^{n-m}\bar\alpha_i(x)\beta_i(x)$$

where

$$\bar\alpha_i(x)=\int_0^1\frac{\partial\hat F}{\partial u_i}(0,\ldots,0,t\beta_i,\ldots,\beta'_n)dt$$

and

$$\hat F=F\circ\beta^{-1}.$$

To conclude the proof, we simply C^s approximate each $\bar\alpha_i$ by a polynomial $\bar{\bar\alpha}_i$ on $\bar U$ and let $\Phi=\Sigma\bar{\bar\alpha}_i\cdot\beta_i$, $\square$

Lemma 3. *Suppose $F:(\bar U,V)\to(\mathbb{R},0)$ satisfies the hypothesis of Lemma 2. Furthermore assume*

(i) $0\in U\subset\mathbb{R}^n=\mathbb{R}^m\times\mathbb{R}^k$,

(ii) *$F(x,y)$ restricted to $(\mathfrak{D}\times\mathbb{R}^k)\cap U$ is equal to a polynomial $f(x)$ which vanishes on V, where $\mathfrak{D}$ is a neighborhood of 0 in $\mathbb{R}^m$, $(x,y)\in\mathbb{R}^m\times\mathbb{R}^k$.*

Then the polynomial Φ in the conclusion of Lemma 2 can have the additional property: $\Phi(x,y)=f(x)+x^{s+1}\sigma(x,y)$, *where* x^{s+1} *means a linear combination* $x_1^{i_1},\ldots,x_m^{i_m}$, $i_1+\ldots+i_m\geqq s+1$.

Proof. First replace U by $U\cup(\mathfrak{D}\times U_1)$ where U_1 is a bounded open set in $\mathbb{R}^k$ containing $\pi(U)$ where π is the projection $\mathbb{R}^m\times\mathbb{R}^k\to\mathbb{R}^k$. F extends over the new U by $f(x)$. Now, as in the proof of Lemma 2, we can write $F-f=\Sigma\bar{\alpha}_i\beta_i$, and our hypothesis implies that we can assume $\bar{\alpha}_i=0$ on $\mathfrak{D}\times U_1$. Approximate each $\bar{\alpha}_i$ by a polynomial ψ_i on $\bar{U}$ such that:

(a) $\psi_i\in U_{\bar{\alpha}_i}(\delta,2s)$

(b) $j^{(s)}(\psi_i)=0$ (see [3], p. 147 for example).

Let

$$\psi_i(x,y)=\sum_{|I|\leqq s}\theta_I(y)x^I+x^{s+1}\sigma(x,y)\qquad\text{(fixing } i\text{)},$$

where $I=(i_1,\ldots,i_m)$, $|I|=i_1+\ldots+i_m$, $x^I=x_1^{i_1},\ldots,x_m^{i_m}$. $|D_\alpha(\bar{\alpha}_i-\psi_i)(0,y)|<\delta$ for $|\alpha|\leqq 2s$ and $(0,y)\in U$; since $\bar{\alpha}_i$ vanishes on $\mathfrak{D}\times U_1$ then $|D_\alpha(\psi_i)(0,y)|<\delta$ for $|\alpha|\leqq 2s$. Therefore $|D_\alpha(\theta_I(y))|<\delta$ for $|\alpha|\leqq s$, $(0,y)\in U$; the hypothesis on the domain U implies that we can replace the condition $(0,y)\in U$ by $(x,y)\in U$. Then it is easily seen that there is a constant $C\geqq 1$ depending on s and U such that $|D_\alpha(\Sigma\theta_I(y)x^I)|<\delta C$ for $|\alpha|\leqq s$, $(x,y)\in U$. Choose δ so that $\delta C<\varepsilon/2$; and let $\bar{\bar{\alpha}}_i=\psi_i-\Sigma\theta_I x^I$. Clearly

$$\begin{aligned}|D_\alpha(\bar{\alpha}_i-\bar{\bar{\alpha}}_i)|&\leqq|D_\alpha(\bar{\alpha}_i-\psi_i)|+|D_\alpha(\Sigma\theta_I x^I|\\&<\varepsilon/2+\varepsilon/2=\varepsilon\quad\text{for}\quad|\alpha|\leqq s,\ (x,y)\in U.\end{aligned}$$

We do this for each i and let $\Phi=f+\Sigma\bar{\bar{\alpha}}_i\beta_i$. □

Definition. For convenience, throughout the paper we will use the notation

$$j_x^{(s)}(\Sigma\alpha_{I,J}x^Iy^J)=\sum_{|I|\leqq s}a_{I,J}x^Iy^J$$

and

$$j_x^{(s)}(f(x,y))=j_x^{(s)}\qquad[\text{Taylor series expansion of } f(x,y)]$$

for a smooth function $f(x,y)$. This should be viewed as a "partial s-jet" of a function; and it is a symbolic definition because the sum might be infinite.

Lemma 4. *If $\Sigma^k\in A_k$, then Σ^k is a link of a polynomial map $F:(\mathbb{R}^{k+r+1},0)\to(\mathbb{R}^r,0)$ such that:*

(i) *$F^{-1}(0)$ is compact,*

(ii) *F has* rand *r on $F^{-1}(0)-\{0\}$.*

Proof. Let $f:(\mathbb{R}^{k+r+1},0)\to(\mathbb{R}^r,0)$ be the polynomial map having Σ^k as its link, and let y_0 be a small regular value for the function $f(x)/|x|^{2l}:\mathbb{R}^{k+r+1}-\{0\}\to\mathbb{R}^r$ where l is an arbitrarily large integer. Let $F(x)=f(x)-y_0|x|^{2l}$. From Lemma 5, $F^{-1}(0)\cap B_\varepsilon^{k+r+1}\approx f^{-1}(0)\cap B_\varepsilon^{k+r+1}=\mathring{c}\Sigma$ for a small $\varepsilon>0$. By construction (ii) is satisfied. For $x\in F^{-1}(0)$

$$|x|^2\leqq|f(x)|/|y_0|\,|x|^{2(l-1)}\to 0\ \text{as}\quad x\to\infty$$

hence $F^{-1}(0)$ is compact. □

Lemma 5. (T. Kuo [4].) *Let $f:(\mathbb{R}^n,0)\to(\mathbb{R}^k,0)$ be a polynomial isolated singularity. Then there is an integer s, such that any smooth map g, with $j^{(s)}g(x)=f(x)$, has the property that $g^{-1}(0)\cap B^n_\varepsilon \approx f^{-1}(0)\cap B^n_\varepsilon$ for some small $\varepsilon>0$ ($\approx$ is a canonical homeomorphism which can be made to be arbitrarily close to the identity map in $\mathbb{R}^n$ by taking s to be arbitrarily large).*

Remark 2. Let $f:(\mathbb{R}^n,0)\to(\mathbb{R}^k,0)$ be a polynomial isolated singularity, and $\gamma:\mathbb{R}^2\to\mathbb{R}$ be the stable polynomial $y_1^2+y_2^2-y_1$. Let $F:\mathbb{R}^n\times\mathbb{R}^2\to\mathbb{R}^k\times\mathbb{R}$ be a smooth map $F(x,y)=(F_1(x,y),F_2(x,y))$ such that $F\sim f\times\gamma$ and $j_x^{(s)}F_1(x,y)=f(x)$ for an arbitrarily large s. Then Lemma 5, without much difficulty, implies that $F^{-1}(0)\cap(B^n_\varepsilon\times\mathbb{R}^2)\approx(f^{-1}(0)\cap B^n_\varepsilon)\times S^1$ for some $\varepsilon>0$. This is what we will use in the proof of the theorem. □

3. Proof of the Theorem

Let $M^m\in\mathfrak{C}$; without loss of generality we can assume $M^m=M_0\cup(c\Sigma^k\times N^n)$ where $\Sigma^k\in\Lambda_k$. From Lemma 4 there is a compact algebraic variety $f:(\mathbb{R}^p,0)\to(\mathbb{R}^r,0)$ with the isolated singularity at $0\in\mathbb{R}^p$, $p=r+k+1$, such that $f^{-1}(0)\cap B^p_\delta\approx\mathring{c}\Sigma^k$ for a small $\delta>0$. Note that p can be taken to be arbitrarily large by simply replacing f by $f\times\mathrm{id}:\mathbb{R}^p\times\mathbb{R}^l\to\mathbb{R}^p\times\mathbb{R}^l$. N, being a closed smooth manifold, is a nonsingular algebraic variety by Tognoli's theorem (if N is stably parallelizable or even cobordant to zero we don't need to evoke Tognoli's theorem). Let $h:\mathbb{R}^q\to\mathbb{R}^u$ be the polynomials giving N as a nonsingular variety.

Then $f\times h:\mathbb{R}^p\times\mathbb{R}^q\to\mathbb{R}^r\times\mathbb{R}^u$ gives a compact real algebraic variety V^m, with its singular set $0\times N$. We have:

(a) $V=(f\times h)^{-1}(0)=f^{-1}(0)\times N$

(b) $V\cap(B^p_\delta\times\mathbb{R}^q)\approx\mathring{c}\Sigma^k\times N$.

Define $Q_1^m=(M^m-\mathring{c}\Sigma\times N)\cup(\Sigma\times N\times I)\cup(V^m-\mathring{c}\Sigma\times N)$. Then since Q_1 is a closed smooth manifold, it is cobordant to a nonsingular algebraic variety (because the unoriented cobordism ring is generated by nonsingular algebraic varieties [11]); call it H^m, and denote the cobodism by Q_2^{m+1}, i.e., $\partial Q_2^{m+1}=Q_1\cup H$. Let $Q_3^{m+1}=Q_2\cup c\Sigma\times N\times I$, where the union is taken along $\Sigma\times N\times I\subset Q_1$. Then let Q^{m+1} be the double of Q_3^{m+1}.

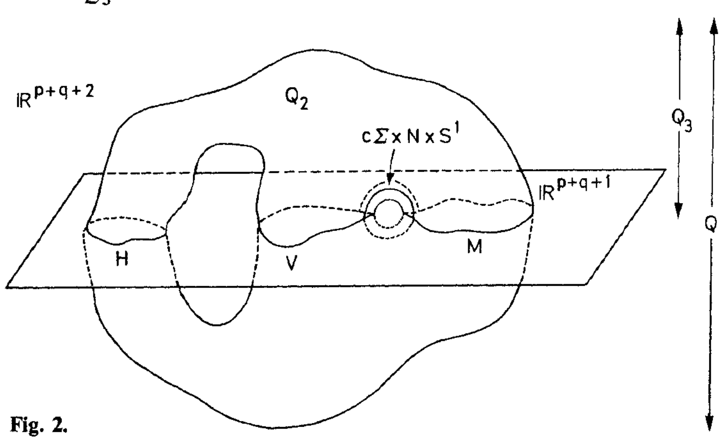


Fig. 2.

We can assume $H^m \subset \mathbb{R}^q$ and $H \cap N = \emptyset$, by taking q large to begin with. Imbed Q^{m+1} into $\mathbb{R}^{p+q+2}$ as follows: identify $H \cup V \subset \mathbb{R}^{p+q} \times (0,0) \subset \mathbb{R}^{p+q} \times \mathbb{R}^2$ and imbed $c\Sigma \times N \times S^1 \subset \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^2$ via $f^{-1}(0) \times h^{-1}(0) \times \gamma^{-1}(0)$ where $\gamma : \mathbb{R}^2 \to \mathbb{R}$, $\gamma(x,y) = x^2 + y^2 - x$. Extend this to an imbedding of $M \subset \mathbb{R}^{p+q} \times (1,0)$; then extend it to an imbedding of Q^{m+1} in $\mathbb{R}^{p+q+2}$ such that:

(i) $Q \cap (\mathbb{R}^{p+q+1} \times 0) = H \cup V \cup M$,

(ii) $Q \cap (B^p_\delta \times \mathbb{R}^{q+2}) = \mathring{c}\Sigma \times S^1 \times N$.

since p can be taken to be large. Let W be a small open neighborhood of Q, and let $W_\delta = W - B_\delta \times \mathbb{R}^{q+2}$, $Q_\delta = Q - B_\delta \times \mathbb{R}^{q+2}$. Let $U_\delta = B^p_\delta \times B^2_\delta(\gamma^{-1}(0))$; then $W = W_\delta \cup U_\delta \times \mathfrak{D}$ where $\mathfrak{D}$ is an open tubular neighborhood of N in $\mathbb{R}^q$.

We now want to define the Gauss map over the singular space Q as in [3]: let $(\bar{\beta}, \beta)$ be the normal bundle map

$$\begin{array}{ccc} W|_{Q-Z_\delta} & \xrightarrow{\bar{\beta}} & E = \{(g,x) \in G \times \mathbb{R}^{p+q+2} : x \in g\} \\ \downarrow & & \downarrow \\ Q - Z_\delta & \xrightarrow{\beta} & G \end{array}$$

where $G = G_{p+q+2, r+q-n+1}$,

$$Z_\delta = V \times I_\delta \cup (Q - Q_\delta) \approx V \times I_\delta \cup (c\Sigma \times N \times S^1)$$

and $V \times I_\delta$ is a δ-neighborhood of V in Q. Notice $\partial Z_\delta \approx (f^{-1}(0) \# f^{-1}(0)) \times N$.

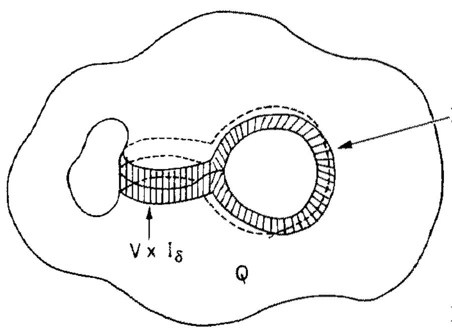


Fig. 3.

$Z_\delta \subset f^{-1}(0) \times S^1 \times N \subset \mathbb{R}^p \times \mathbb{R}^2 \times \mathbb{R}^q$, and $\beta|_{\partial Z_\delta}$ is given by

$$(x,y,t) \to df(x) \oplus d\gamma(y) \oplus (\text{the normal plane to } N \text{ at } t \text{ in } \mathbb{R}^q)$$
$$= \beta_1(x,y) \oplus \beta_2(t) \qquad \text{(definition)}$$

$\beta|_{\partial Z_\delta}$ is in the image of

$$\begin{array}{c} [f^{-1}(0) \# f^{-1}(0);\ V_{p+2,r+1}] \times [N; G_{q,q-n}] \\ \downarrow \\ [f^{-1}(0) \# f^{-1}(0);\ G_{p+2,r+1}] \times [N; G_{q,q-n}] \\ \downarrow \\ [\partial Z_\delta;\ G_{p+q+2,r+q-n+1}] \end{array}$$

where $V_{p+2,r+1}=(r+1)$-frames in $\mathbb{R}^{p+2}$. Since $V_{p+2,r+1}$ is $(p-r)$-connected, $[f^{-1}(0)\#f^{-1}(0);\, V_{p+2,r+1}]=0$. β_1 is null homotopic, therefore we can extend β over Z_δ such that:

$$\beta|_{Z_{\delta/2}}=L\oplus\beta_2=\left(\begin{array}{c|c} I_{r+1} & 0 \\ \hline 0 & \beta_2 \end{array}\right)$$

where L is the fixed $(r+1)$-plane $\mathbb{R}^{r+1}\times 0\subset\mathbb{R}^{p+2}$. Since the trivializations $f\times\gamma$ and $d(f\times\gamma)$ of the normal bundle of $f^{-1}(0)\#f^{-1}(0)$ in $\mathbb{R}^{p+2}$ are equivalent, we can assume that

$$(\bar\beta,\beta)|_{B_{\delta/2}\times\mathbb{R}^{q+2}}=(f\times\gamma,\mathrm{L})\oplus(\bar\beta_2,\beta_2)=((f\times\gamma)\oplus\bar\beta_2,\, L\oplus\beta_2)$$

where $\bar\beta,\bar\beta_2$ are the maps covering β and β_2.

Call $i\circ\bar\beta=\psi=(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5): W\to Y\times\mathbb{R}^r\times\mathbb{R}^1\times\mathbb{R}^{p-r+1}\times\mathbb{R}^q$ where i is the inclusion $E\hookrightarrow G\times\mathbb{R}^{p+q+2}\hookrightarrow Y\times\mathbb{R}^{p+q+2}$ (see §2 for the definition of Y).

By construction:

(1) $\psi|_{B_{\delta/2}\times\mathbb{R}^{q+2}}=(\psi_1,f,\gamma,0,\psi_5)$

(2) $\psi_1|_{B_{\delta/2}\times\mathbb{R}^{q+2}}=L\oplus\beta_2$

(3) $\psi_i=0$ on Q, for $2\leqq i\leqq 5$.

Using Lemmas 2 and 3, when $2\leqq i\leqq 5$, we can approximate ψ_i's by polynomials Φ_i which vanish on $H^m\cup V^m$, and $j_x^{(s)}(\Phi_2)=f(x)$ for $x\in\mathbb{R}^p$. Restriction of the map ψ_1 to $H^m\cup V^m$ comes from:

$$H^m\cup N^n\to H^m\cup(f^{-1}(0)\times N)\to G$$

if $x\in H^m$, $x\to$(the normal plane to H^m at x in $\mathbb{R}^q$)$\oplus\mathbb{R}^{p+1}$ if $x\in N^n$, $x\to$(the normal plane to N^n at x in $\mathbb{R}^q$)$\oplus\mathbb{R}^{r+1}$. Since $H\cup N$ is a nonsingular variety, Lemma 1 implies that $\psi_1|_{H\cup V}$ is equal to a rational function, call it θ. Clearly θ is in the form $L\oplus(*)$. By applying Lemmas 2 and 3, we can approximate $\psi_1-\theta$ by a polynomial Φ_1' which vanishes on $H\cup V$, and $j_x^{(s)}(\Phi_1')=0\oplus(*)$, for $x\in\mathbb{R}^p$, where 0 denotes the $(r+1)\times(r+1)$ matrix with zero entries; let $\Phi_1=\Phi_1'+\theta$. Let $\Phi=(\Phi_1,\Phi_2,\Phi_3,\Phi_4,\Phi_5)$. We have:

(1) Φ is a rational function which is C^s close to ψ.

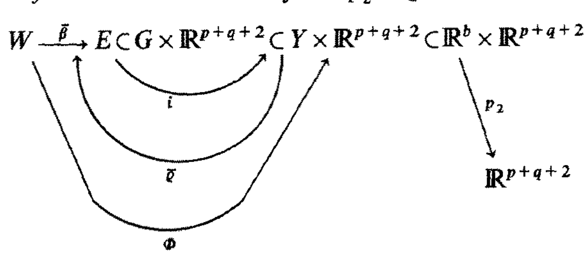
(2) $\Phi_1|_{H\cup V}=\psi_1|_{H\cup V}$.

(3) $j_x^{(s)}(\Phi_1)=\left(\begin{array}{c|c} I_{r+1} & 0 \\ \hline 0 & * \end{array}\right)$.

(4) $\Phi_i|_{H\cup V}=0$ for $2\leqq i\leqq 5$.

(5) $j_x^{(s)}(\Phi_2)=f(x)$ where $x\in\mathbb{R}^p$.

Define $F: W\to\mathbb{R}^{p+q+2}$ by $F=p_2\circ i\circ\bar\varrho\circ\Phi$



where p_2 is the obvious projection, and $\bar{\varrho}(z,x)=(\varrho(z), A_z(z)(x))$, i.e., $F(x) = A_{\Phi_1(x)}(\Phi_1(x))p_2(\Phi(x))$ [see §2 (*)].

We claim:

(a) $F^{-1}(0)\cap W \approx Q^{m+1}$

(b) $F^{-1}(0)\cap W\cap(\mathbb{R}^{p+q+1}\times 0)=H\cap V\cup M'$, where $M'\approx M$.

Proof of the claim. Call $F(u,t)=(\xi_1(u,t), \xi_2(u,t))\in\mathbb{R}^{r+1}\times\mathbb{R}^{p-r+q+1}$, for $(u,t)\in U_{\delta/2}\times\mathfrak{D}$ and let $u=(x,y)\in\mathbb{R}^p\times\mathbb{R}^2$. Since $A_{\Phi_1}(\Phi_1)\sim A_{\psi_1}(\psi_1)=\psi_1^{m+1}=\psi_1$ (recall $\psi_1\in G$), a point on the plane $A_{\Phi_1}(\Phi_1)$ is zero if and only if its projection onto the plane ψ_1 is zero. Hence $F=0$ if and only if $\psi_1(F)=0$; also

$$\psi_1|_{U_{\delta/2}\times\mathfrak{D}}=L\oplus\beta_2=\left(\begin{array}{c|c|c} \overbrace{I_{r+1}}^{r+1} & \overbrace{0}^{p-r+1} & \overbrace{0}^{q} \\ \hline 0 & 0 & 0 \\ \hline 0 & 0 & \beta_2 \end{array}\right)$$

where β_2 is the normal bundle map to N in $\mathbb{R}^q$. Therefore $F^{-1}(0)\cap(U_{\delta/2}\times\mathfrak{D})$ is described by the equations $\xi_1(u,t)=0$, and $\beta_2(t)(\xi_2(u,t))=0$.

Consider

$$u\times\mathfrak{D}\xrightarrow{\tau_u}E_{q,q-n}\subset G_{q,q-n}\times\mathbb{R}^q$$

$$\tau_u(u,t)=(\beta_2(t),\beta_2(t)(\xi_2(u,t))).$$

τ_u is close to the map which classifies the normal bundle of $u\times N\subset u\times\mathfrak{D}$, for each $u\in U_{\delta/2}\cdot\hat{\xi}_2^{-1}(0)\cap(u\times\mathfrak{D})=\tau_u^{-1}(G_{q,q-n})$ where $\hat{\xi}_2(u,t)=\beta_2(t)(\xi_2(u,t))$. By transversality $\tau_u^{-1}(G_{q,q-n})$ is an n-manifold which is diffeomorphic to N via the diffeomorphism α_u, which assigns each point of $u\times N$ to the closest point on $\tau_u^{-1}(G_{q,q-n})$ in $u\times\mathbb{R}^q$. α_u extends to a diffeomorphism by $U_{\delta/2}\times N\to\hat{\xi}_2^{-1}(0)\cap(U_{\delta/2}\times\mathfrak{D})$ by $\alpha(u,t)=(u,\alpha_u(t))$. Then

$$\begin{aligned} F^{-1}(0)\cap(U_{\delta/2}\times\mathfrak{D})&\approx\alpha^{-1}(F^{-1}(0)\cap(U_{\delta/2}\times\mathfrak{D}))\\ &=\alpha^{-1}(\xi_1^{-1}(0)\cap(U_{\delta/2}\times\mathfrak{D})\cap\alpha^{-1}(\hat{\xi}_2^{-1}(0)\cap(U_{\delta/2}\times\mathfrak{D}))\\ &=\{(u,t):\xi_1(u,\alpha_u(t))=0\}\cap(U_{\delta/2}\times N).\end{aligned}$$

Since

$$j_x^{(s)}F=j_x^{(s)}(A_{j_x^{(s)}\Phi_1}(j_x^{(s)}\Phi_1)\,p_2(\Phi)),\quad x\in\mathbb{R}^p$$

$$=j_x^{(s)}\left(A_{\left(\begin{array}{c|c} I_{r+1} & 0\\ \hline 0 & \Phi_{11}\end{array}\right)}\left(\begin{array}{c|c} I_{r+1} & \\ \hline 0 & \Phi_{11}\end{array}\right)\right)p_2(\Phi)$$

where Φ_{11} is a symmetric matrix.

$$=j_x^{(s)}\left(A_{\Phi_{11}}\left(\begin{array}{c|c} I_{r+1} & 0\\ \hline 0 & \Phi_{11}\end{array}\right)\right)p_2(\Phi).$$

(From the definition of A see §2.)

$$=j_x^{(s)}\left(\left(\begin{array}{c|c} A_{\Phi_{11}}(1)\, I_{r+1} & 0 \\ \hline 0 & * \end{array}\right)\right)(\Phi_2, \Phi_3, \Phi_4, \Phi_5).$$

Since $A_{\Phi_{11}}(\lambda)\sim\lambda^{m+1}$, $A_{\Phi_{11}}(1)$ is a some nonzero number c.

$$=j_x^{(s)}(c\Phi_2, c\Phi_3, \Phi_4, \Phi_5)=(cf(x), \ldots),$$

we have $\xi_1(x, y, t)=(cf(x)+x^{s+1}\sigma(x, y, t), \lambda(x, y, t))$, $u=(x, y)$, where $\lambda\sim\gamma(y)$.

Remark 2 implies that there is a homeomorphism α_t' when δ small

$$\alpha_t':\{(u,t):\xi_1(u,\alpha_u(t))=0\}\cap(U_{\delta/2}\times t)\to(\mathring{c}\Sigma\times S^1)\times t$$

which depends continuously on t, hence α_t' extends to a homeomorphism

$$\{(u,t):\xi_1(u,\alpha_u(t))=0\}\cap(U_{\delta/2}\times N)\to(\mathring{c}\Sigma\times S^1)\times N$$

by $(u,t)\to(\alpha_t'(u),t)$. Hence there is a homeomorphism

$$\zeta:\mathring{c}\Sigma\times S^1\times N\to F^{-1}(0)\cap(U_{\delta/2}\times\mathfrak{D})$$

which in particular can be made to be arbitrarily C^s close to the identity map.

Since $\bar{\beta}|_{W_0}$ is transversal to $G\subset E$ and $\bar{\varrho}\circ\Phi\sim\bar{\beta}$ on W, $\bar{\varrho}\circ\Phi|_{W_\delta}$ is transversal to G for any $\delta>0$. $\bar{\beta}^{-1}(G)=Q$ and $(\bar{\varrho}\circ\Phi)^{-1}(G)=F^{-1}(0)$; hence the map which assigns a point p of $Q\cap W_{\delta/4}$ to the unique point of $F^{-1}(0)\cap W_{\delta/4}$ given by intersecting the normal plane of Q at p with $F^{-1}(0)$ is a diffeomorphism which is C^s close to the identity map in $\mathbb{R}^{p+q+2}$. Then this diffeomorphism can be extended to a homeomorphism $Q\approx F^{-1}(0)\cap W$ which equals to ζ near $0\times S^1\times N$. Similarly (and using the properties of Φ) (b) can be easily verified.

Unfortunately, F is not polynomial, it is only an algebraic function (recall the definitions); to find a polynomial with the same properties as F we appeal to Nash's original technique [7].

Nash's Normalization

For given $y=(y_1,\ldots,y_{m+1})\in\mathbb{R}^{m+1}$, define $y(\lambda)$ to be the polynomial $\lambda^{m+1}+y_1\lambda^m+\ldots+y_{m+1}$. Now define $\hat{\varrho}:Y\times\mathbb{R}^{p+q+2}\times\mathbb{R}^{m+1}\to\mathbb{R}^b\times\mathbb{R}^{p+q+2}\times\mathbb{R}^{m+1}$, by $\hat{\varrho}(z, x, y)=(y(z), y(z)x, y)$ (recall that z is a matrix); also define $\mathcal{N}_i:\mathbb{R}^{p+q+2}\times\mathbb{R}^{m+1}\to\mathbb{R}$, $i=1, 2, \ldots, m+1$ by:

$$\mathcal{N}_1(x,y)\,\lambda^m+\mathcal{N}_2(x,y)\,\lambda^{m-1}+\ldots+\mathcal{N}_{m+1}(x,y)=\text{Remainder}\,(C_{\Phi_1(x)}(\lambda)/_{y(\lambda)})$$

(see §2 to recall the definitions). Let $\mathcal{N}=(\mathcal{N}_1,\ldots,\mathcal{N}_{m+1})$. Define

$$\hat{F}=\pi\circ\hat{\varrho}\circ(\Phi\times\text{id}):\mathbb{R}^{p+q+2}\times\mathbb{R}^{m+1}\to\mathbb{R}^{m+1}$$

where

$$W\times\mathbb{R}^{m+1}\xrightarrow{\Phi\times\text{id}}Y\times\mathbb{R}^{p+q+2}\times\mathbb{R}^{m+1}\xrightarrow{\hat{\varrho}}\mathbb{R}^b\times\mathbb{R}^{p+q+2}\times\mathbb{R}^{m+1}\xrightarrow{\pi}\mathbb{R}^{p+q+2}$$

Let $F^*=(\hat{F},\mathcal{N})$. We claim that the rational function F^* has the following properties:

(a) $F^{*-1}(0)\cap(W\times W')\approx Q$

(b) $F^{*-1}(0)\cap(W\times W')\cap(\mathbb{R}^{p+q+1}\times 0\times\mathbb{R}^{m+1})=H\cup V\cup M''$

where W' is a neighborhood of $0\in\mathbb{R}^{m+1}$, and $M''\approx M$.

Proof of the claim. $F^*(x,y)=0$, $(x,y)\in\mathbb{R}^{p+q+2}\times\mathbb{R}^{m+1}$ means $y(\Phi_1(x))(p_2(\Phi(x)))=0$ and $\mathcal{N}(x,y)=0$; In particular, $y(\lambda)$ divides $C_{\Phi_1(x)}(\lambda)$. If we restrict $(x,y)\in W\times W'$ where W' is a sufficiently small neighborhood of $0\in\mathbb{R}^{m+1}$, all roots of $y(\lambda)$ are close to zero. Hence $y(\lambda)=A_{\Phi_1(x)}(\lambda)$ [see §2 (*)]. i.e., $y=(a_1(\Phi_1(x)),\ldots,a_{m+1}(\Phi_1(x))$. Therefore $y(\Phi_1(x))(p_2(\Phi(x)))=F(x)$, hence the map

$$F^{-1}(0)\cap W\to F^{*-1}(0)\cap(W\times W')$$
$$x\to(x,a_1(\Phi_1(x)),\ldots,a_{m+1}(\Phi_1(x))$$

is a homeomorphism, i.e., (a) is proved.

Since

$$\Phi_1|_{H\cup V}=\psi_1|_{H\cup V}$$
$$a_i(\Phi_1(x))=a_i(\psi_1(x)) \quad \text{for} \quad x\in H\cup V$$
$$=0 \text{ (from definitions of } a_i\text{'s in §2, and } \psi_1\in G)$$

Hence

$$F^{*-1}(0)\cap(W\times W')\cap(H\cup V\times\mathbb{R}^{m+1})=H\cup V$$
$$F^{*-1}(0)\cap(W\times W')\cap(\mathbb{R}^{p+q+1}\times 0\times\mathbb{R}^{m+1})=H\cup V\cup M''$$

where $M''\approx M$.

Call $Q'=F^{*-1}(0)\cap(W\times W')$. In general, $F^{*-1}(0)$ might have components other than Q' denote them D, i.e., $D=F^{*-1}(0)-Q'$. We can get rid of these extra components by appealing to Tognoli.

Tognoli's Trick

We first claim that there is a rational function $g:\mathbb{R}^{p+q+2}\times\mathbb{R}^{m+1}\to\mathbb{R}$ such that g is C^s close to zero on Q', is equal to zero on $H\cup V$, and is >1 on D.

Proof of the claim. (Following [1].) For brevity call $v=p+q+2+m+1$. Consider the imbedding

$$\mathbb{R}^v\xrightarrow{j_2}\mathbb{R}_v(\mathbb{R})\xrightarrow{j_1}\mathbb{R}^{(v+1)^2}$$

$j_2(x_1,\ldots,x_v)=[x_1,\ldots,x_v,1]$, $j_2[x_1,\ldots,x_{v+1}]=(\ldots,u_{ij},\ldots)$, $u_{ij}=x_ix_j/x_1^2+\ldots+x_{v+1}^2$ (Veronese imbedding)

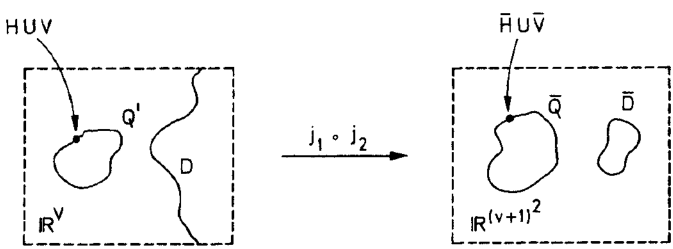


Fig. 4.

Let $\bar{Q}=j_1\circ j_2(Q')$, $\bar{D}=j_1\circ j_2(D)$, $\bar{H}\cup\bar{V}=j_1\circ j_2(H\cup V)$. Then clearly $\bar{D}$ is compact and $\bar{H}\cup\bar{V}$ is a nonsingular algebraic variety (since $H\cup V$ is). Let $\bar{g}$ be a C^∞ function which is zero near $\bar{Q}$, and is equal to 2 on $\bar{D}$. Then using Lemma 2, C^s approximate $\bar{g}$

with a polynomial $\bar{\bar{g}}$ near $\bar{Q}\cup\bar{D}$ which vanishes on $\bar{H}\cup\bar{V}$. Let $g=\bar{\bar{g}}\circ j_1\circ j_2$. We are now ready to conclude the proof:

Let $\tilde{g}(x_1,\ldots,x_v)=g(x_1,\ldots,x_v)+x_{p+q+2}/1+x_{p+q+2}^2$; $\tilde{g}^{-1}(0)$ is just a C^s perturbation of the plane $\mathbb{R}^{p+q+1}\times 0\times\mathbb{R}^{m+1}$ near Q'. Also $\tilde{g}=0$ on $H\cup V$, and $\tilde{g}^{-1}(0)$ misses D.

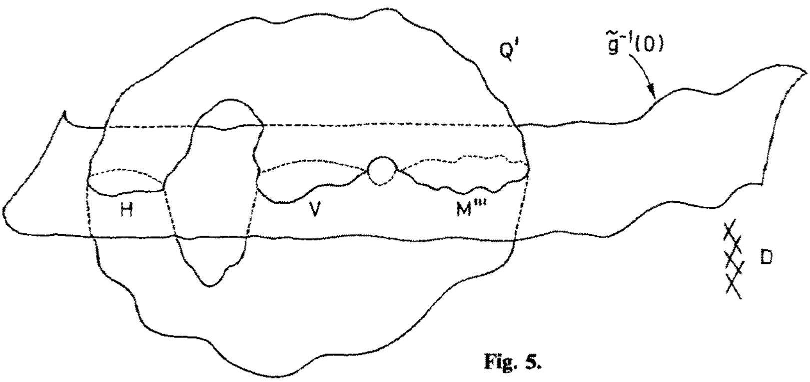


Fig. 5.

It is easily seen that $\tilde{g}^{-1}(0)\cap F^{*-1}(0)=H\cup V\cup M'''$, where $M'''\approx M$. Define the rational function

$$H(x,t)=|F^*(x)|^2+|tk(x)-1|^2+|\tilde{g}(x)|^2$$

where k is a polynomial such that $k^{-1}(0)=H\cup V$, and $t\in\mathbb{R}$.

$$H^{-1}(0)=\{(x,t):x\in H\cup V\cup M''', k(x)\neq 0, t=1/k(x)\}$$

$$=\{(x,1/_{k(x)}):x\in M'''\}\approx M'''\approx M.$$

Therefore M is a zero set of rational functions whose denominators are never zero (from the constructions); hence it is a zero set of polynomials.

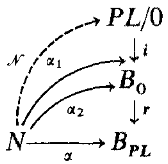
A Discussion of a 9-Manifold

Consider the half exact sequence $0\to\Omega_8^{\text{spin}}\to\Omega_8$ where Ω_8, Ω_8^{spin} are 8-dimensional oriented and spin cobordism groups respectively ([5], p. 55). Let $\Sigma^8\in\Gamma_8=\mathbb{Z}_2$ (=group of homotopy 8-spheres) $\Sigma^8\neq 0$, then since homotopy spheres orientably bound, we have $[\Sigma]=0$ ($[\Sigma]$ = oriented cobordism class of Σ). Therefore, from the half exact sequence we get $[\Sigma]_{\text{spin}}=0$ ($[\Sigma]_{\text{spin}}$ = spin cobordism class of Σ), i.e., Σ^8 bounds a spin 9-manifold N'. Hence N' is parallelizable over 3-skeleton. Then by surgery kill $\pi_i(N')$ for $i\leqq 3$. Call the new manifold by N. Define $M=N\underset{\partial N=\Sigma}{\cup} c\Sigma$.

Claim. M has no smooth structure.

Proof of the Claim. (The following proof was explained to me by D. Sullivan and R. Kirby.) Call the smooth structure of N by α_1. If M had a smooth structure, N would have a smooth structure α_2 such that $(\partial N)_{\alpha_2}=S^8$ (=standard 8-sphere). N, being 3-

connected, has a unique P. L. structure. We have the fibration:



$\alpha = r(\alpha_1) = r(\alpha_2)$.

Since $r(\alpha_1 \oplus \alpha_2^{-1}) = 0$, there is $\mathcal{N}: N \to PL/0$ such that $i(\mathcal{N}) = \alpha_1 \oplus \alpha_2^{-1}$. It is easily seen that $\mathcal{N}|\partial N: S^8 \to PL/0$ is represented by the nonzero element of $\pi_8 PL/0 = \mathbb{Z}_2$. But this is a contradiction. Since N is 3-connected it has homotopy type of a 5-complex and $PL/0$ is 6-connected, we cannot have a map from N to $PL/0$ which restricts to an essential map on the boundary.

We will show that M cannot be a real algebraic variety, which is a locally complete intersection. In fact, a slightly stronger statement is true.

Claim. M cannot be represented locally by a polynomial $f:(\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ such that f has rank $n-9$ on $B_\varepsilon^n - \{0\}$ where B_ε is the open ball of radius ε at 0.

Proof. Suppose M is represented near the cone point by such a polynomial f (i.e., link of f is Σ^8). By Tarski-Seidenberg theorem $f(B_\varepsilon)$ is a semialgebraic set of dimension $n-9$ [12, 9]. Therefore, we can find a $m-n+9$ plane L, passing through 0 in $\mathbb{R}^m$, such that $L \cap f(B_\varepsilon) = \{0\}$ for ε small. Now let $\varphi: \mathbb{R}^m \to \mathbb{R}^m$ be a rotation such that $\varphi(L) = \mathbb{R}^{m-n+9} \times 0 \subset \mathbb{R}^m$. Then let $F: \mathbb{R}^n \to \mathbb{R}^{n-9}$, $F = \pi \circ \varphi \circ f$ where π is the projection $\mathbb{R}^m = \mathbb{R}^{m-n+9} \times \mathbb{R}^{n-9} \to \mathbb{R}^{n-9}$. Then clearly $F^{-1}(0) \cap B_\varepsilon = f^{-1}(0) \cap B_\varepsilon$ and F has rank $n-9$ on $B_\varepsilon - \{0\}$. Then pick a regular value $y_0 \in \mathbb{R}^{n-9}$ of F, near 0. Therefore, one easily checks that $F^{-1}(y_0) \cap \bar{B}_\varepsilon$ is a parallelizable manifold whose boundary is diffeomorphic to Σ^8. Contradiction.

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Note added in proof: The recent solution of the double suspension problem by R. Edwards and J. Cannon, along with the Kirby Siebenman theory implies that $\mathfrak{C}$ contains topological manifolds which are not P. L., e.g. $M^5 = (M_0^4 \cup c\Sigma(2,3,5)) \times S^1$.