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A note on homology surgery and the Casson–Gordon invariant

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We give a 'picture' proof to a theorem of M. Freedman (2) which shows the failure of the 4-dimensional homology surgery theory and the homology splitting theorem. Our proof employs the language of the framed links (3) and it involves calculating Casson–Gordon invariant of a certain algebraically slice knot. We use framed links to represent 1-connected 4-manifolds with boundary by attaching 2-handles along them onto B^4 via the framings. We adapt the notation $\approx$ for diffeomorphisms, and $\overset{\partial}{\approx}$ for diffeomorphisms between the boundaries of manifolds. Here manifolds refer to smooth manifolds.

THEOREM. *There exists a compact spin* 4-*manifold* M^4 (*with boundary*) *which is homotopy equivalent to* $M_0^4 \# S^2 \times S^2$ *where* M_0^4 *is a homology* $S^1 \times D^3$ *such that there are hyperbolic classes* $\xi_1, \xi_2 \in H_2(M: \mathbb{Z}[\mathbb{Z}])$ *that cannot be represented by* P.L. *imbedded* $S^2 \vee S^2$.

Remark. There is not even an imbedded codimedsion zero submanifold, which is homology equivalent to $S^2 \times S^2 - \operatorname{int} B^4$, representing $\{\xi_1, \xi_2\}$.

(1) *Casson–Gordon invariant.* Fix a surjection $\chi: \mathbb{Z}_{m^2} \to \mathbb{Z}_m$. Let M^3 be a closed 3-manifold such that:

(i) $M^3 = \partial W^4$, W^4 is compact,

(ii) $H_1(M^3) = \mathbb{Z}_{m^2} \xrightarrow{i_* = \chi} H_1(W) = \mathbb{Z}_m$ is onto where i is the inclusion, m is a prime.

Let $\tilde{W}$ be the m-fold covering of W induced by χ. Then the chain complex $C_*(\tilde{W})$ is a $\mathbb{Z}[\mathbb{Z}_m]$-module. Let k be the cyclotomic field $\mathbb{Q}(\mathbb{Z}_m) \subset \mathbb{C}$ ($\mathbb{Z}_m$ is identified by the group of m-th roots of 1), then define:

$$H_*(W; k) = H_*(C_*(\tilde{W}) \underset{\mathbb{Z}[\mathbb{Z}_m]}{\otimes} k).$$

There is a hermitian pairing:

$$H_2(W; k) \otimes H_2(W; k) \to k$$

defined by

$$(x, y) = \sum_{t \in \mathbb{Z}_m} \langle x, ty \rangle t^{-1},$$

where $\langle\ ,\ \rangle$ is the ordinary intersection pairing of $H_2(\tilde{W}: \mathbb{Z})$. Now define

$$\sigma(M^3, \chi) = \text{signature } H_2(W; k) - \text{signature } H_2(W; \mathbb{Z}).$$

It is shown in (1) that $\sigma(M, \chi)$ is well defined, and if $H_*(W; \mathbb{Q}) = 0$, then

$$\sigma(M^3: \chi) = \pm 1.$$

Now let K be a knot in a homology sphere N^3 which is a boundary of a homology ball V^4. If the 2-fold branched covering space M^3 of N^3 branched along K satisfies (i) and (ii), we can define $\sigma(K:\chi) = \sigma(M^3:\chi)$. In (1) it is shown that if $\partial K^0 = \partial L^4$, where L^4 compact manifold which is a homology $S^1 \times D^3$ and K^0 is the 4-manifold obtained by attaching a 2-handle onto V^4 along K with 0-framing, then $|\sigma(K:\chi)| \leqslant 1$. (*)

Remark. In fact, one can deduce from (1) that in the above statements one can replace the smooth manifolds W^4 and L^4 by integral homology manifolds with the same properties.

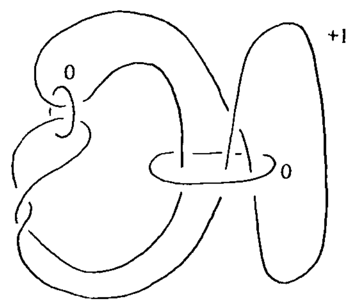
$M_1^4 =$ 

Fig. 1

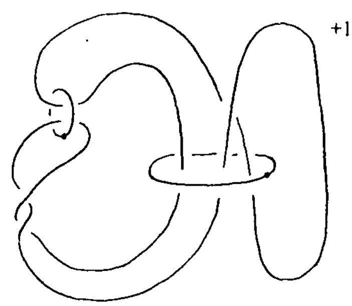
$M_0^4 =$ 

Fig. 2

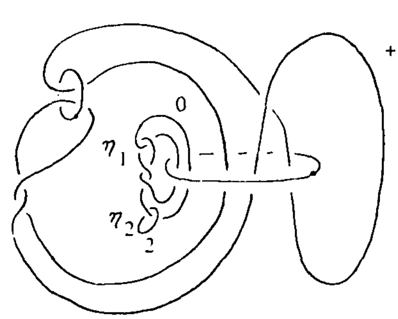
$M^4 =$ 

Fig. 3

(2) *Proof of the theorem.* Define Fig. 1. Then by surgering the two 2-spheres corresponding to the 0-framed circles in M_1^4 (i.e. changing the 2-handles $S^2 \times D^2$ with the 1-handles $B^3 \times S^1$ in the interior of M_1^4) we get Fig. 2. We put a dot on the 0-framed circles in the picture to denote the surgered 2-spheres. We adapt this notation for the

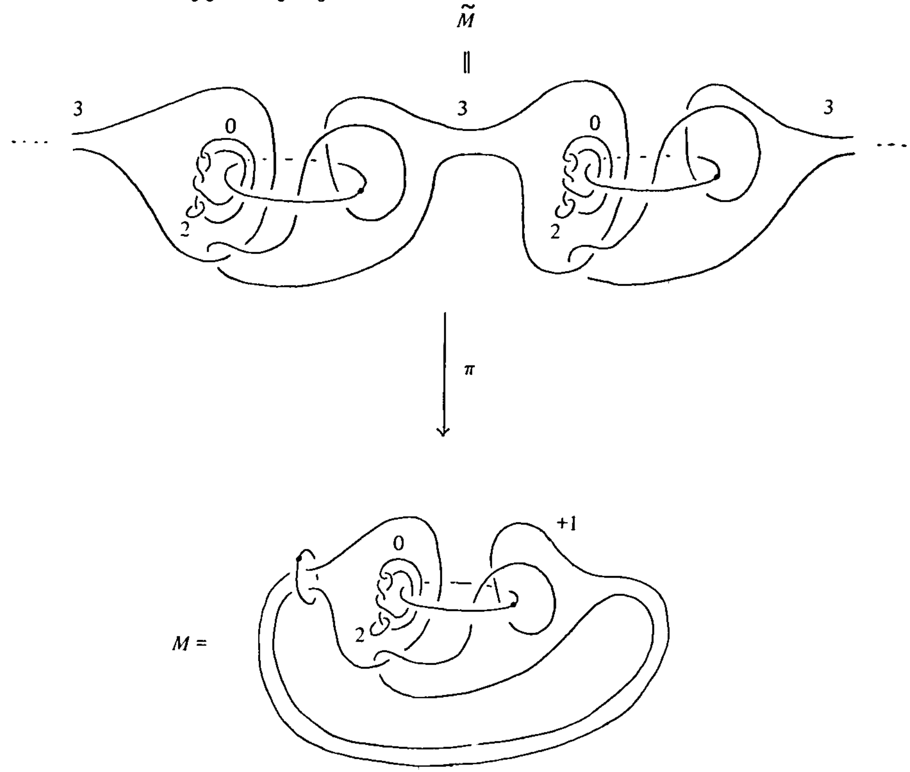


Fig. 4

rest of the paper. M_0^4 has two 1-handles and one 2-handle, and

$$H_*(M_0^4\colon \mathbb{Z}) = H_*(S^1 \times D^3\colon \mathbb{Z}).$$

Define M^4 by attaching a pair of 2-handles onto M_0^4 as shown in Fig. 3. Since the new 2-handles are attached along the link $\{\eta_1, \eta_2\}$ which is homotopic to the link { } in ∂M_0, $M^4 \simeq M_0 \# S^2 \times S^2$.

The following demonstrates the homology equivalence between the infinite cyclic coverings of M and $M_0 \# S^2 \times S^2$ as $\mathbb{Z}[\mathbb{Z}]$-modulos. Let $\tilde{M} \xrightarrow{\pi} M$ denote the infinite cyclic covering of M. The handlebody picture is Fig. 4. The self-intersections in $H_2(\tilde{M}; \mathbb{Z})$ are easily computed by the formula $\langle \tilde{C}_i, \ldots, \tilde{C}_{-1} + \tilde{C}_0 + \tilde{C}_1 + \tilde{C}_2 + \ldots \rangle = \langle C, C \rangle$, where $\tilde{C}_i \in H_2(\tilde{M}; \mathbb{Z})$, $-\infty < i < \infty$, such that $\pi_*(\tilde{C}_i) = C$. The above picture demonstrates how the classes η_1, η_2 come from $H_2(\tilde{W}; \mathbb{Z})$. Now by changing 1-handles with 2-handles in the interior of M^4 (= surgery), and by a series of operations in Figs. 5–8 we get a knot K in S^3 with $\partial M \approx \partial K^0$. Below the invariant $\sigma(K, \chi)$ is computed to be 4 for this knot K, for some χ.

We can now conclude the proof of the Theorem. We have the classes

$$\eta_1, \eta_2 \in H_2(M; \mathbb{Z}[\mathbb{Z}])$$

which have the intersection matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

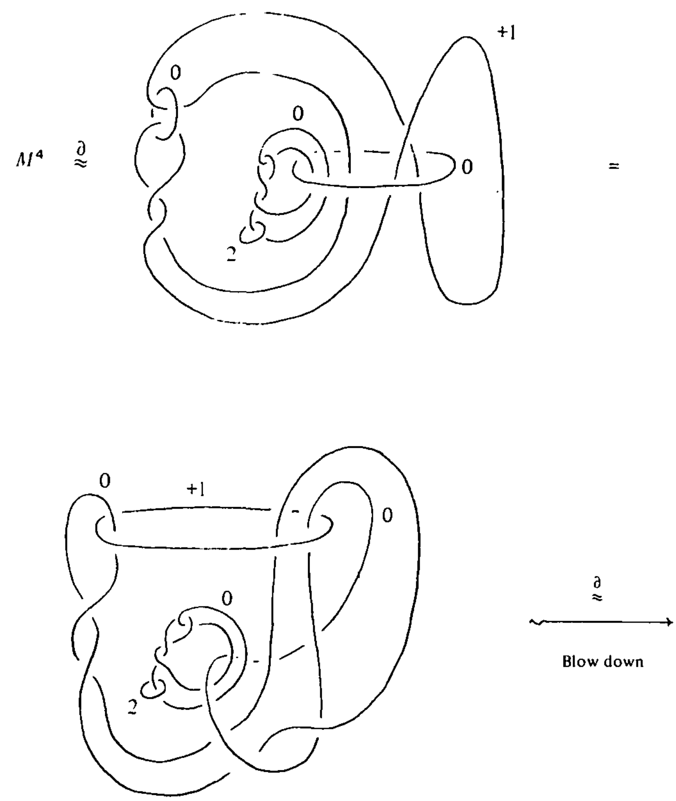


Fig. 5

by doing the basis change $\eta_1 \to \eta_1$ $\eta_2 \to \eta_2 - \eta_1$, we get classes $\xi_1, \xi_2 \in H_2(M; \mathbb{Z}[\mathbb{Z}])$ which are hyperbolic. Suppose ξ_1, ξ_2 were represented by P.L. imbedded $S^2 \vee S^2$ in M^4. Let $N(\xi_1, \xi_2)$ be a regular neighbourhood of $S^2 \vee S^2$ in M, by duality $\partial N(\xi_1, \xi_2)$ is an integral homology sphere. Since $\partial M^4 \approx \partial K^0$, by constructing the homology madifold

$$M' = (M - N(\xi_1, \xi_2)) \cup \text{cone}\,(\partial N(\xi_1, \xi_2)),$$

we get a contradiction to (*) and the Remark. This proves the Theorem.

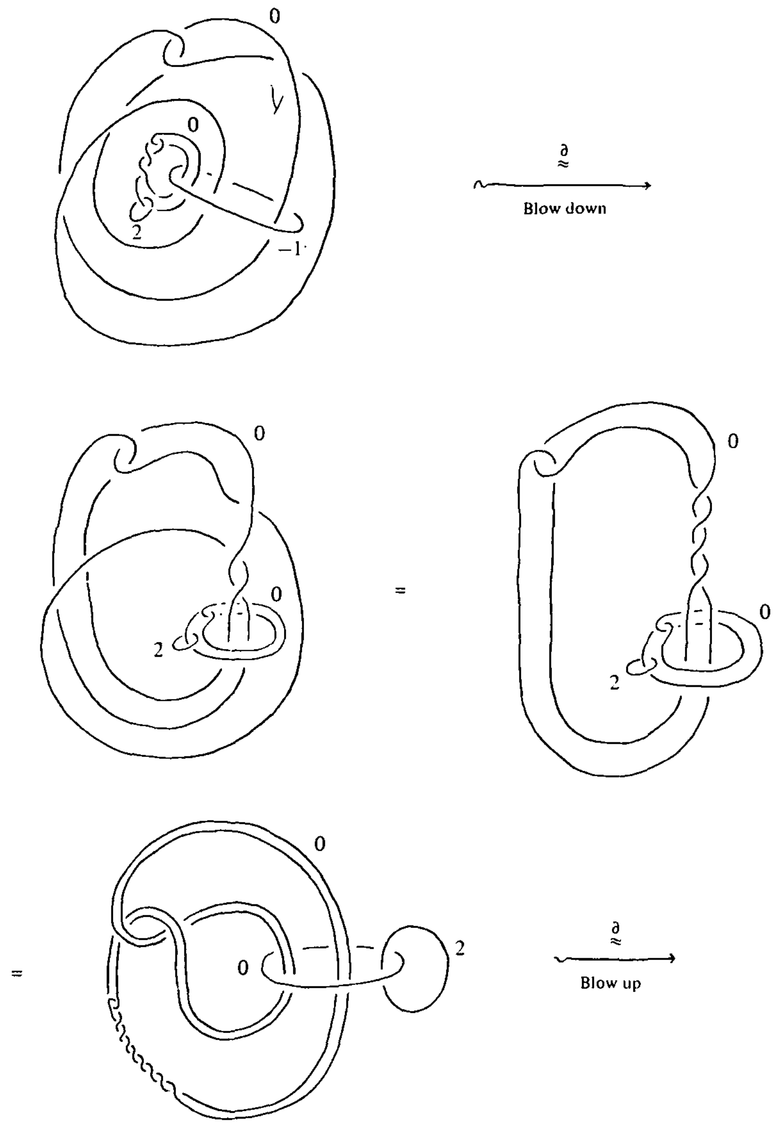


Fig. 6

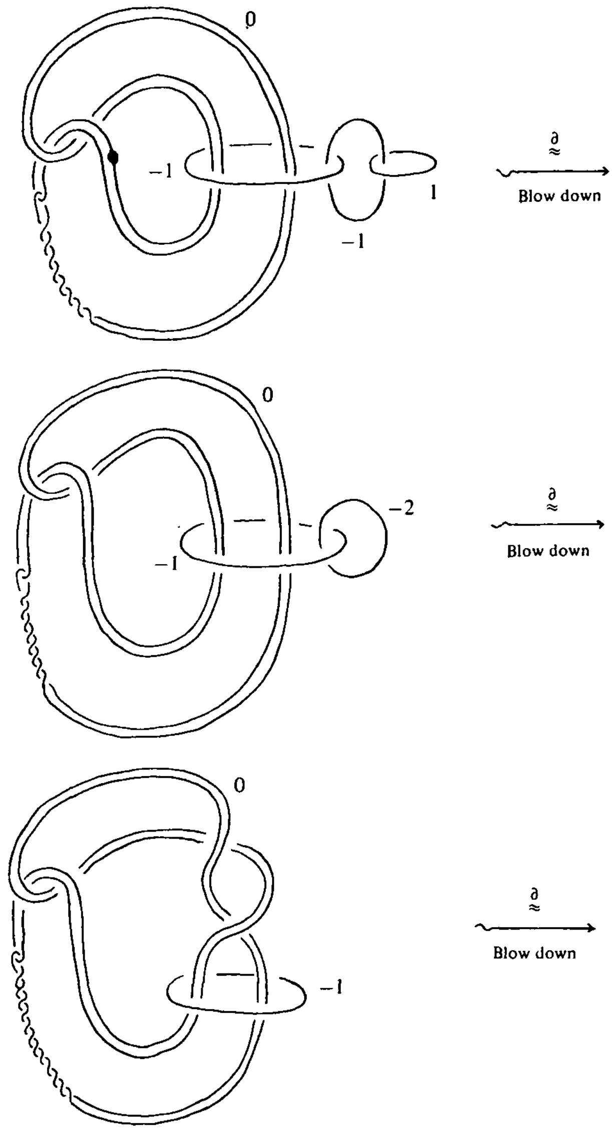


Fig. 7

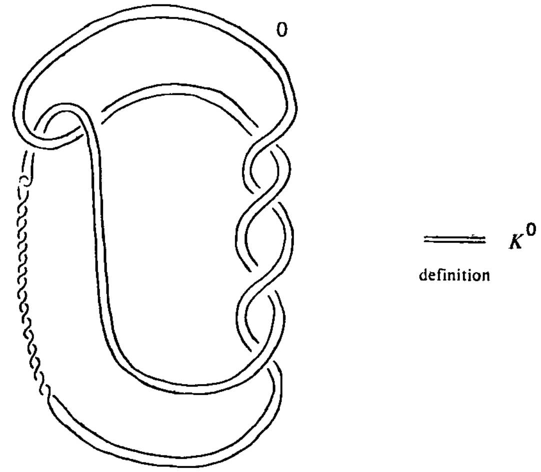


Fig. 8

Computation of $\sigma(K, \chi)$

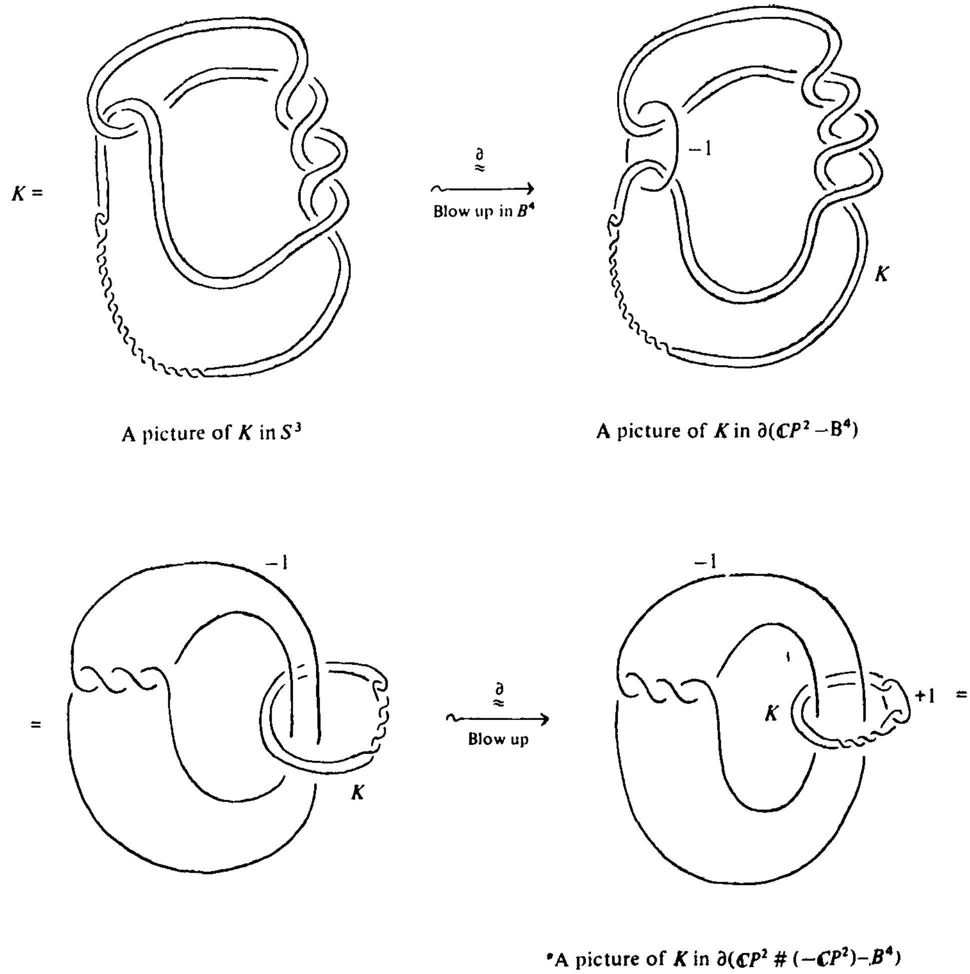


Fig. 9

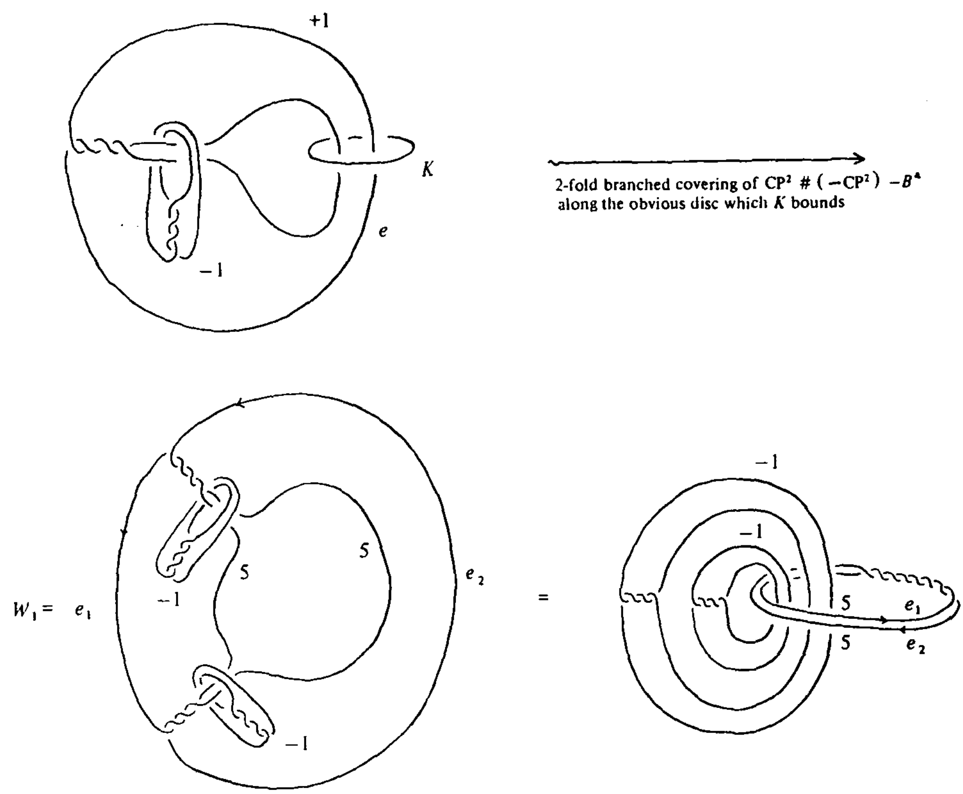


Fig. 10

The self-intersections in $H_2(W_1, \mathbb{Z})$ are easily computed by the formula

$$\langle e_i, (e_1 + e_2)\rangle = \langle e, e\rangle = 1, \quad i = 1, 2.$$

∂W_1 is the 2-fold branched covering space of K (Fig. 10).

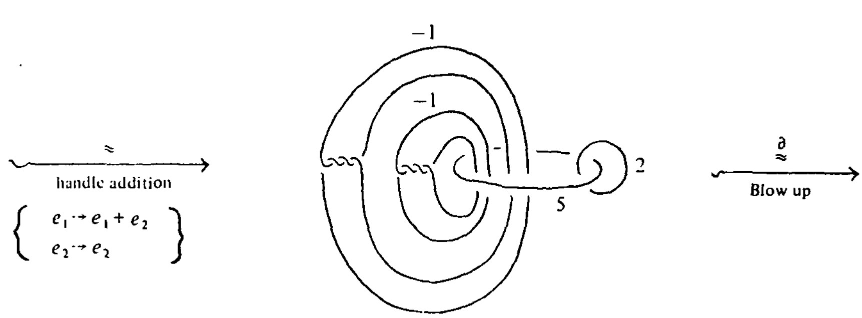


Fig. 11

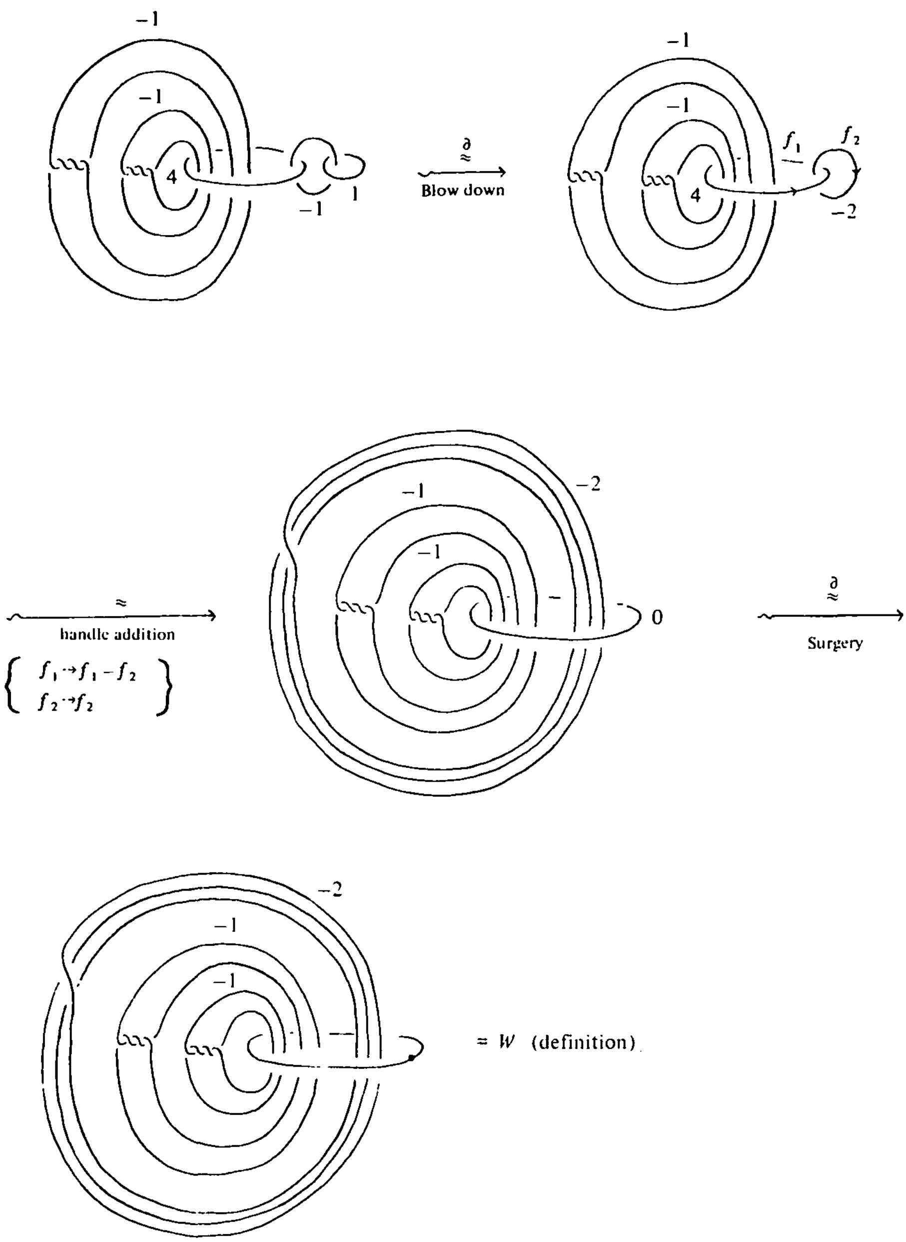

−1
−1
4
−1
1
∂
≈
Blow down
−1
−1
f_1
f_2
4
−2
≈
handle addition
$f_1 \to f_1 - f_2$
$f_2 \to f_2$
−2
−1
−1
0
∂
≈
Surgery
−2
−1
−1
= W (definition)

Fig. 12

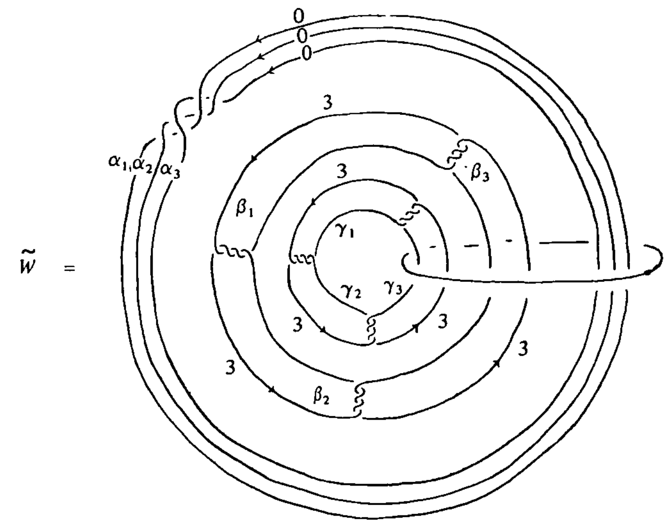


Fig. 13

Since W satisfies (i), (ii) of Section 1 (with $m = 3$) we can use it to compute $\sigma(K:\chi)$. The 3-fold covering space $\tilde{W}$ of W is Fig. 13. The intersection form $H_2(\tilde{W}: \mathbb{Z})$ is computed as before. Clearly

$$\alpha_j = t^{j-1}\alpha_1, \quad \beta_j = t^{j-1}\beta_1, \quad \gamma_j = t^{j-1}\gamma_1,$$

where $t = e^{\frac{1}{3}(2\pi i)}$

$$(\beta_1, \beta_1) = \sum_{j=0}^{2} \langle \beta_1, t^j\beta_1 \rangle t^{-j} = 3 - 2t^2 - 2t = 5,$$

similarly $(\gamma_1, \gamma_1) = 5$. The other pairwise intersections are all zero, and α_1 is cancelled by the 1-handle. Hence:

$$\sigma(K:\chi) = \sigma(\partial W:\chi) = \text{signature}\begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix} - \text{signature}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$$

$$= 2 - (-2) = 4.$$

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