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On Representing Homology Classes of 4-Manifolds

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We discuss a P.L. non-imbedding theorem for 2-dimensional homology classes in certain smooth 4-manifolds, which arises from an invariant of Atiyah-Singer.

Theorem. *There exists a compact smooth 4-manifold W^4 (with boundary) which is homotopy equivalent to $W_0^4 \# \mathbb{C}P^2$ where W_0^4 is a smooth homology ball* (i.e. $\tilde{H}_*(W_0;\mathbb{Z})=0$) *such that the generator $\xi \in H_2(W;\mathbb{Z}) \cong \mathbb{Z}$ cannot be represented by a P.L. imbedded 2-sphere (possibly non-locally flat).*

We construct W^4 by attaching a 2-handle onto a homology ball W_0 along a knot $K \subset \partial W_0$, such that K is null homotopic in ∂W_0. Therefore we get the following Corollary which shows the failure of Dehn's lemma for homology 4-balls.

Corollary. *The knot K cannot bound a P.L. disc in W_0^4.*

In particular this means that homotopic knots in the homology sphere ∂W_0 are not necessarily P.L. concordant (in $\partial W_0 \times I$). We are informed that these results are also obtained by C.McA. Gordon.

The main point of this paper is to describe geometrically the handlebody pictures of non-abelian covering spaces of 3 and 4 manifolds. The above theorem follows by explicitly calculating a certain invariant from these pictures.

1. Introduction

Let M^3 be a closed oriented 3-manifold and let G be a finite group. Let $M \xrightarrow{f} K(G,1)$ be a map such that $f_*: \pi_1(M) \to G$ is onto. Since the bordism group $\Omega_3 K(G,1)$ is finite, rM^3 bounds a smooth compact oriented 4-manifold W^4 over $K(G,1)$, for some $r>0$. In other words $\partial W = rM$, and f extends to $W \to K(G,1)$. Let $\tilde{W}$ be the covering space of W corresponding to the subgroup kernel f_* of

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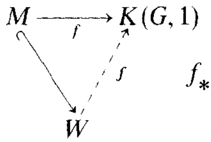
$\pi_1(W)$. As in [CG] define $\tau(M^3,\alpha)=\frac{1}{r}\left(\frac{1}{d}\text{Signature}(\tilde{W})-\text{Signature}(W)\right)$, where d = order of G, and $\alpha=f_*$. Since $\text{Sign}(\tilde{W})=d\,\text{Sign}(W)$ for closed 4-manifolds W [AS], and the Novikov additivity theorem implies that $\tau(M,\alpha)$ is well defined. One can easily see that to calculate this invariant one doesn't have to choose W to be manifold, it only has to be a homology manifold (i.e. a smooth manifold in the complements of a finite number of interior points, where the links of these points are homology 3-spheres). Now recall ([CG]).

Lemma. *Let M^3 be as above, suppose M^3 bounds a connected oriented homology manifold W over $K(G,1)$ with $H_*(W;\mathbb{Q})=0$, and $\pi_1(W)$ finite. Then $|\tau(M,\alpha)|<1$.*

Proof. Let W^4 be such a homology manifold. We have, $\partial W=M$



$f_*=\alpha.$

We can use W to compute $\tau(M,\alpha)$. Let $\tilde{W}$ be the corresponding covering. $\pi_1(\tilde{W})$, being a subgroup of $\pi_1(W)$, is finite. Hence rank $H_1(\tilde{W})=0=$ rank $H_3(\tilde{W})$ (duality). $\chi(\tilde{W})=d\chi(W)$ (χ denotes the Euler characteristic) implies that rank $H_2(\tilde{W})$ $=d-1$. Hence $|\text{Sign } H_2(\tilde{W})|\leqq d-1$, therefore $|\tau(M,\alpha)|\leqq\frac{d-1}{d}<1$. $\square$

2. Proof of the Theorem

Let W_0^4 be the homology 4-ball of Fig. 1. The handlebody of W_0 contains one 0-handle ($=B^4$), two 1-handles and two 2-handles, as shown in the figure. In

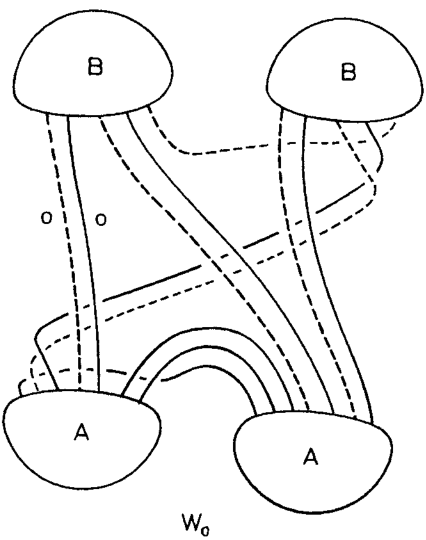


Fig. 1

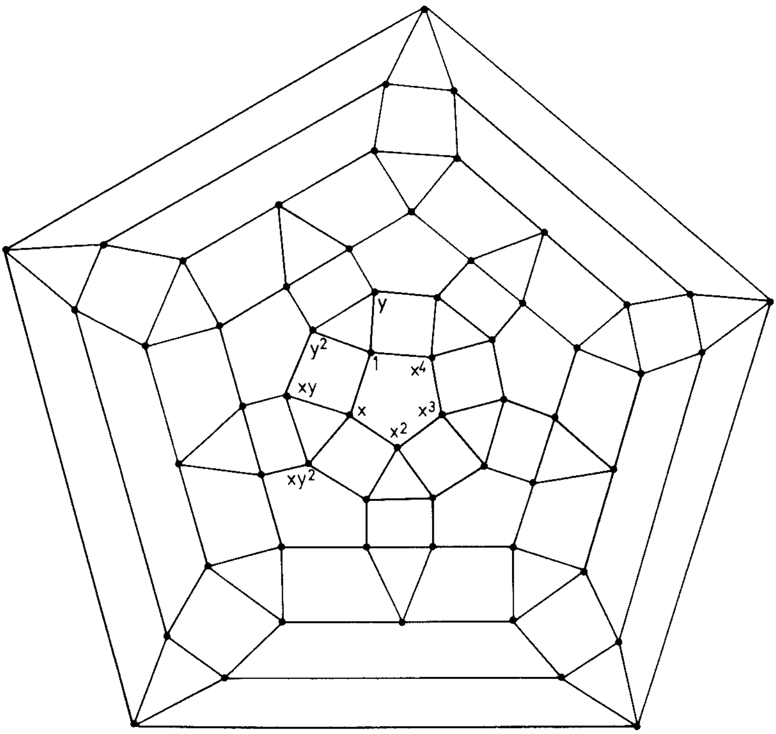


Fig. 2

the Fig. 1 1-handles are not drawn, just the attaching maps. The 2-handles are attached along the indicated loops, where they continue over 1-handles. The framings on these loops, by which the 2-handles are attached, are indicated in the picture. One can easily compute, from the picture, that $\pi_1(W_0)=\{x,y|x^5=y^3=(xy)^2\}$. Let $\alpha\colon \pi_1(W_0)\to A_5$ be the obvious quotient map, where A_5 is the symmetric group $\{x,y|x^5=y^3=(xy)^2=1\}$ of order 60.

To visualize the 60-fold cover of W_0, we first draw the picture of the group A_5. This means that we draw a 1-complex built by assigning vertices to the elements of A_5 and by joining any two vertices a,b by a 1-simplex if $a=b\,z$ where z is a generator of A_5 (x or y). This is shown in Fig. 2. We then thicken this 1-complex; and think of it as a union of sixty 4-balls as in Fig. 3 (it consists of sixty 0-handles and hundred twenty 1-handles). Clearly this is a 60-fold cover of one 0-handle $\cup$ two 1-handles ($=\overset{2}{\#}\, S^1\times B^3$) as indicated in the figure. To extend this to the 60-fold cover $\tilde{W}_0$ of W_0, we simply lift the attaching circles of the 2-handles of W_0 to the 60-fold cover of $\overset{2}{\#}\, S^1\times B^3$; and attach 2-handles along them with the obvious framings (the framings of these circles upstairs are uniquely determined by the lifting). For brevity we haven't drawn the pictures of lifted circles in the Fig. 3.

If $\tau(\partial W_0,\alpha)\geqq 0$, we define W by attaching a 2-handle onto W_0 as in Fig. 4 (only a part of the attaching circle of this 2-handle is visible since it goes over the 1-handle). Since we attached this 2-handle along a null homotopic loop ξ in

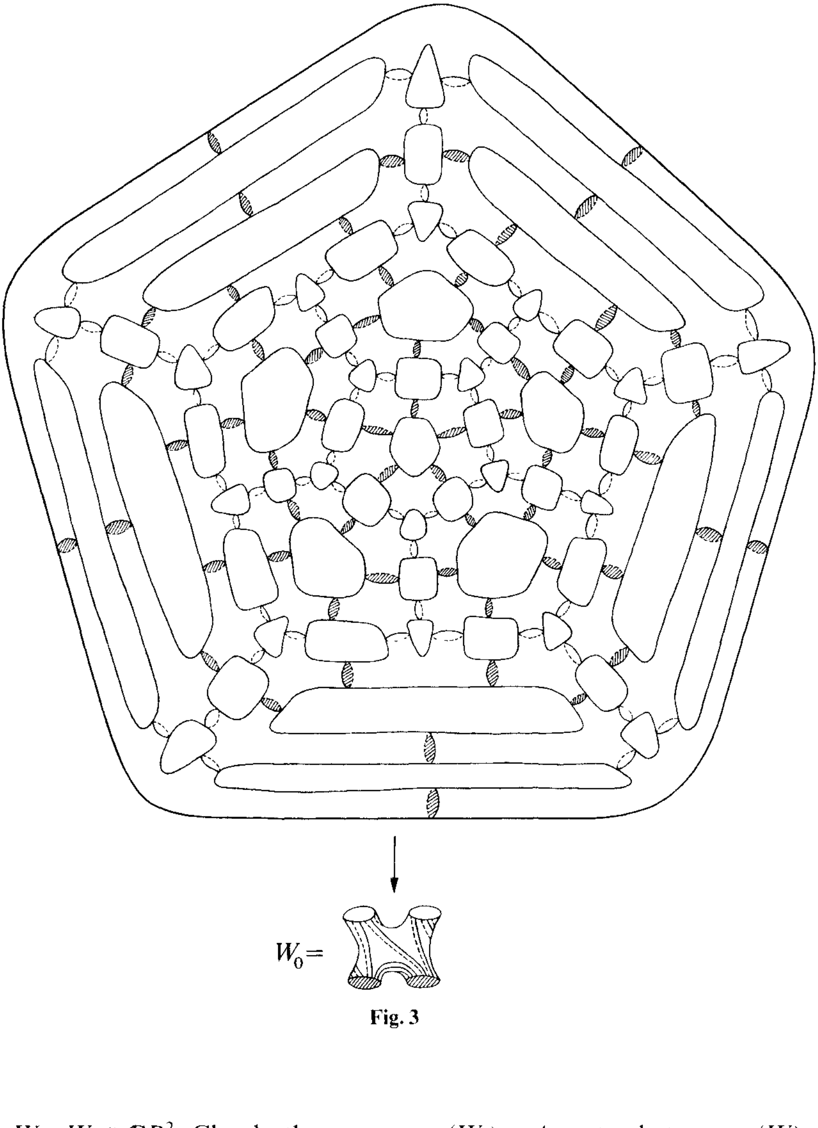
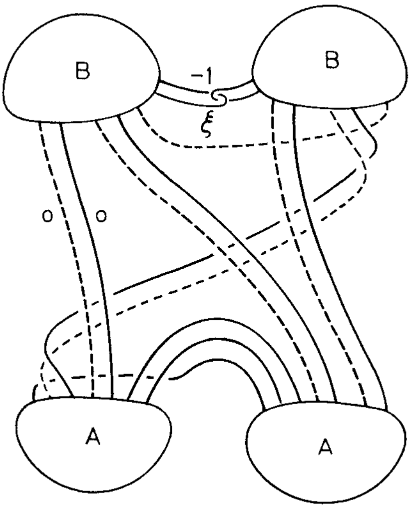


Fig. 3

∂W_0, $W \simeq W_0 \# \mathbb{C}P^2$. Clearly the map $\alpha\colon \pi_1(W_0) \to A_5$ extends to $\alpha\colon \pi_1(W) \to A_5$ since $W \simeq W_0 \vee S^2$. Hence we can compute $\tau(\partial W, \alpha)$: Extend the 60-fold cover $\tilde{W}_0$ of W_0 to the 60-fold cover $\tilde{W}$ of W by simply lifting ξ to the loops ξ_i $i = 1, 2, \ldots, 60$ in $\partial \tilde{W}_0$ and attaching 2-handles onto $\tilde{W}_0$ along the loops ξ_i's with the well defined framings. We draw this in Figure 5, and for simplicity we don't draw the attaching circles of the 2-handles of W_0 and $\tilde{W}_0$ even though they are there. Part of the intersection matrix of $\tilde{W}$ corresponding to the homology classes coming from ξ_i's can easily be read off from Fig. 5, for example one can use the equation $\xi_i \cdot (\xi_1 + \cdots + \xi_{60}) = \xi \cdot \xi$ (considering ξ, ξ_i's as homology classes) to solve $\xi_i \cdot \xi_i$. One can see from the figure that ξ_i's don't meet the homology



W

Fig. 4

classes of $\tilde{W}_0$, hence

$$\tau(\partial W, \alpha) = \frac{\operatorname{Sign}(\tilde{W})}{60} - \operatorname{Sign}(W)$$

$$= \frac{1}{60}\Bigl[\operatorname{Sign}\overbrace{\begin{pmatrix} 1 & -1 & -1 & & & & & & \\ -1 & 1 & -1 & & & & & & \\ -1 & -1 & 1 & & & & & & \\ & & & 1 & -1 & -1 & & & \\ & & & -1 & 1 & -1 & & & \\ & & & -1 & 1 & -1 & & & \\ & & & & & & 1 & -1 & -1 \\ & & & & & & -1 & 1 & -1 \\ & & & & & & -1 & -1 & 1 \end{pmatrix}}^{\text{twenty } 3\times 3\text{-blocks}} + \operatorname{Sign}(\tilde{W}_0)\Bigr] - (-1)$$

$$\geqq \tfrac{20}{60} + 1 = \tfrac{4}{3} > 1. \qquad (\text{Since } \tau(\partial W_0, \alpha) \geqq 0)$$

If $\tau(\partial W_0, \alpha) \leqq 0$, we use the mirror image of ξ (left handed self linking) to attach a 2-handle along ξ with $+1$ framing to obtain W. In that case we get $\tau(\partial W, \alpha) < -1$. In any case we have $|\tau(\partial W, \alpha)| > 1$.

We now ready to conclude the proof:

Suppose the generator ξ of $H_2(W{:}\,\mathbb{Z})$ is represented by possibly non-locally flat P.L. imbedded S^2. Let $N(\xi)$ be a regular neighborhood of ξ, then by duality $\partial N(\xi)$ is an integral homology sphere. Then by constructing the homology

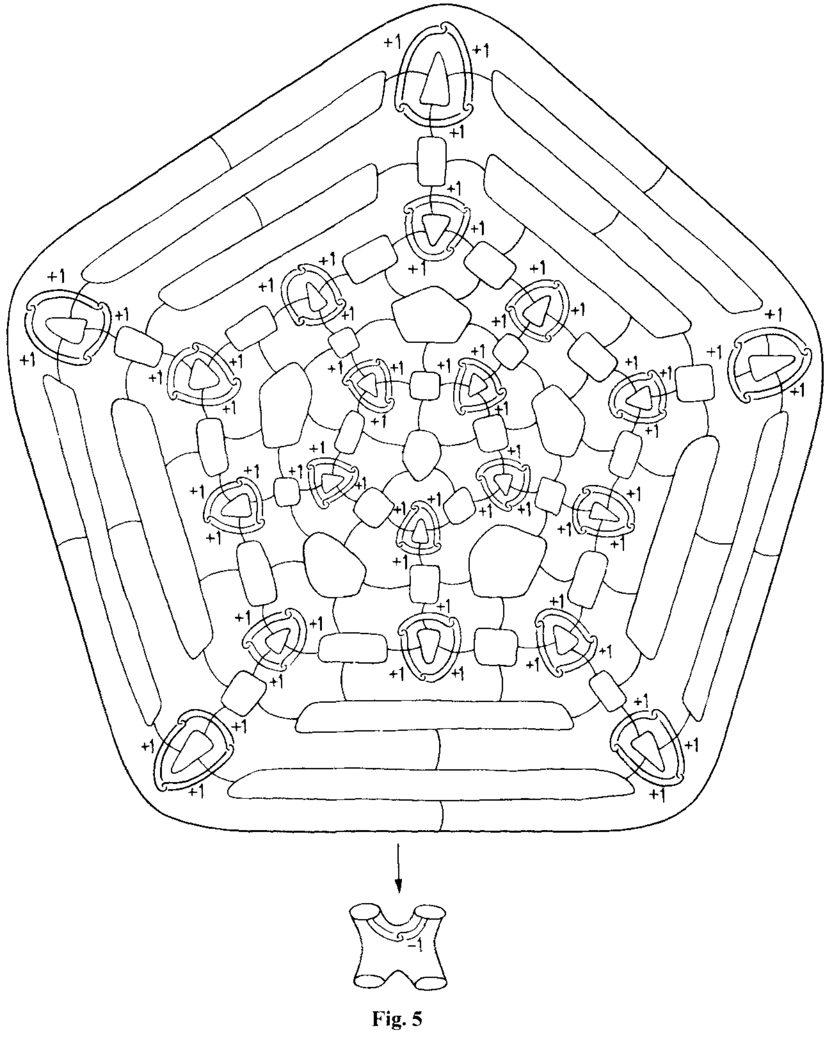


Fig. 5

manifold $W' = (W - N(\xi) \cup \mathrm{cone}(\partial N(\xi))$ we get a contradiction to the Lemma of Sect. 1; since W' is bounded by $\partial W \xrightarrow{f} K(A_5, 1)$ and $H_*(W': \mathbb{Q}) = 0$. □

Remark. By making ξ to link (geometrically) itself twice, one can arrange the Rohlin invariant of the homology sphere ∂W to be zero, if one wishes.

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