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AN EXOTIC INVOLUTION OF S^4

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CAPPELL and Shaneson[1] construct a family of smooth 4-manifolds which are simple homotopy equivalent to real projective 4-space RP^4, but not even smoothly h-cobordant to RP^4. (It is possible they are homeomorphic to RP^4.) It is natural to ask whether their double covers are S^4 or not.

THEOREM. *Let Q^4 be the fake RP^4 built with the matrix* $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$, *as described below. Then the double cover of Q^4, called Σ^4, is diffeomorphic to S^4. Hence there is an exotic free involution on S^4.*

The proof that Σ^4 is S^4 begins with a precise picture of the zero, one and two-handles of Σ^4. After sliding some handles over other handles, we cancel appropriate pairs of one and two-handles. The result is seen to be $S^2 \times B^2 \# S^2 \times B^2 \# S^2 \times B^2$. Their complement in Σ^4 consists of the 4-handle and three-handles, which is diffeomorphic to $S^1 \times B^3 \# S^1 \times B^3 \# S^1 \times B^3$. These pieces are glued together by a diffeomorphism h of $S^1 \times S^2 \# S^1 \times S^2 \# S^1 \times S^2$ to get Σ^4. Laudenbach and Poenaru have shown[2] that Σ^4 must be diffeomorphic to S^4.

Proof. Recall the construction of Q^4: divide RP^4 into two pieces; one is the normal B^2-bundle over RP^2, called $RP^2 \tilde{\times} B^2$; the other is its complement which is the non-trivial bundle over S^1; namely $S^1 \tilde{\times} B^3$. Cappell and Shaneson replace $S^1 \tilde{\times} B^3$ with a manifold C with $\partial C = S^1 \tilde{\times} S^2$. Let A: $R^3 \to R^3$ be the orientation reversing linear map given by the matrix $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$. (This is one of many matrices that work in Cappell and Shaneson's general construction, and the only one we consider here.) Taking quotients, we get an orientation reversing diffeomorphism A: $T^3 \to T^3$, where $T^3 = R^3/(2Z)^3$. Via a small isotopy in a neighborhood of $0 \in T^3$, we can assume A is a reflection on $U = \{x \in R^3 \| x| < \varepsilon\}$. Let C be the mapping torus of $A|T^3 - V$, i.e. $C = ((T^3 - V) \times [-1, 1])/(x, -1) \sim (A(x), 1)$, where V is a small open ball inside U.

It follows that the double cover Σ^4 of Q^4, can be constructed from the double cover of $RP^2 \tilde{\times} B^2$ which is $S^2 \times B^2$ and the double cover of C which is $\hat{C} = \{$mapping torus of $A^2|T^3 - V\}$.

To build Σ^4 as a handlebody we construct $\hat{C}$ and then turn $S^2 \times B^2$ "upside down" and add its 2-handle and 4-handle. To get $\hat{C}$, start with $T^3 \times [-1, 1]$ which has a 0-handle, three 1-handles, a_1, a_2 and a_3, three 2-handles α_1, α_2 and α_3 and a 3-handle. These handles are attached to the boundary of the 0-handle $B^3 \times [-1, 1]$ which is pictured as $R^3 \cup \infty$ where $\partial B^3 \times 0$ is $\partial[-1, 1]^3$, 0×1 is the origin of R^3 and $0 \times (-1)$ is ∞ (see Fig. 1). The 1-handle $a_1 (= B^1 \times B^3)$ is attached by its ends $S^0 \times B^3$ to small 3-balls at the ends of the unit vectors on the x-axis. Similarly with a_2 and the y-axis, and a_3 and the z-axis. The 1-handles are not drawn, just the attaching maps. The 2-handles α_i are attached along the indicated circles, where the gaps are filled in by going over 1-handles. In particular, the attaching circle of α_1 "lies in" the yz-plane, α_2 in zx-plane, and α_3 in the xy-plane. We don't need the 3-handle, and its attaching map is not drawn.

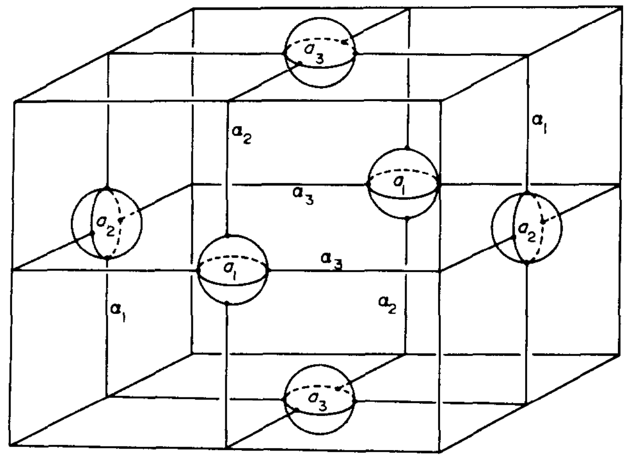

Fig. 1.

To see how the mapping torus is formed, reflect on the analogous picture for $M^3 = (T^2\text{-(2-handle)}) \times [-1, 1]/(x, -1) \sim (B(x), 1)$ where $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We need to isotope B so that it takes 1-handles into 1-handles. The isotopy is analogous to $\begin{pmatrix} 1 & 1-t \\ 0 & 1 \end{pmatrix}$, $t \in [0, 1]$. It is drawn in Fig. 2.

In Fig. 3, we form M^3 by adding a 1-handle a_* to a small 2-ball centered at $0 \times (-1)$ and another at 0×1. Then we add two 2-handles, β_1 and β_2 along the arcs indicated. The result is M^3.

Working analogously in one higher dimension, we isotop A^2 so that it takes 1-handles into 1-handles (Fig. 4). Since $A^2 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$, we begin with the isotopy $\begin{pmatrix} t & 0 & 1-t \\ t-1 & 1 & 0 \\ 0 & t-1 & 1 \end{pmatrix}$, $t \in [0, 1]$, near the origin, to get the second picture in Fig. 4. In the remaining pictures, we push the 1-handles into the 1-handles.

Now we attach a 1-handle a_* to a small 3-ball centered at ∞ and one centéred at the origin (this connects $0 \times (-1)$ and 0×1 in $T^3 \times [-1, 1]$). Then we attach 2-handles β_1, β_2 and β_3 along the indicated arcs in Fig. 5 (this has the effect of identifying the 1-handles in $T^3 \times (-1)$ with their image under A^2 in $T^3 \times 1$). This finishes the construction of the handles of $\hat{C}$ up to index 2. (But note that we have slightly changed the definition of $\hat{C} = (T^3 - V) \times [-1, 1]/A^2$. V is no longer a small open ball centered at $0 \in T^3$, but is now a "blister" on the 0-handle B^3, missing the attaching maps of the 1- and 2-handles. It is described more precisely below where the attaching map of γ is defined.) It would require three 3-handles to identify the three 2-handles in $T^3 \times (-1)$ with their images in $T^3 \times 1$, but these are not drawn; instead they (together with the 4-handle) produce the $S^1 \times B^3 \# S^1 \times B^3 \# S^1 \times B^3$ mentioned at the beginning.

Finally, we must add the dual 2-handle γ of the $S^2 \times B^2$, i.e. a thickened point $\times B^2$. The $S^2 \times B^2$ is added to $\partial V \times [-1, 1]/A^2 = S^2 \times S^1$, so α is attached along a thickened point $\times S^1$, which we can assume lies on ∂ {0-handle $\cup a_*$}. In the lower dimensional

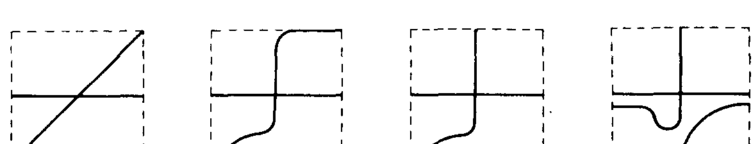

Fig. 2.

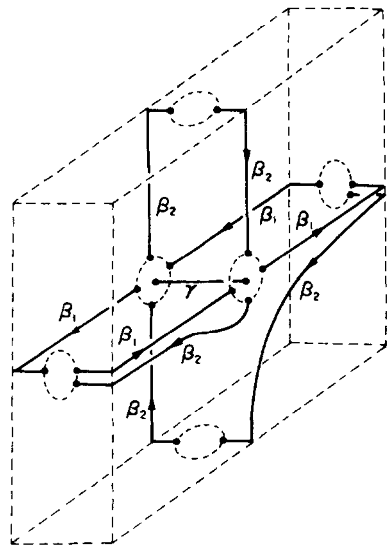

Fig. 3.

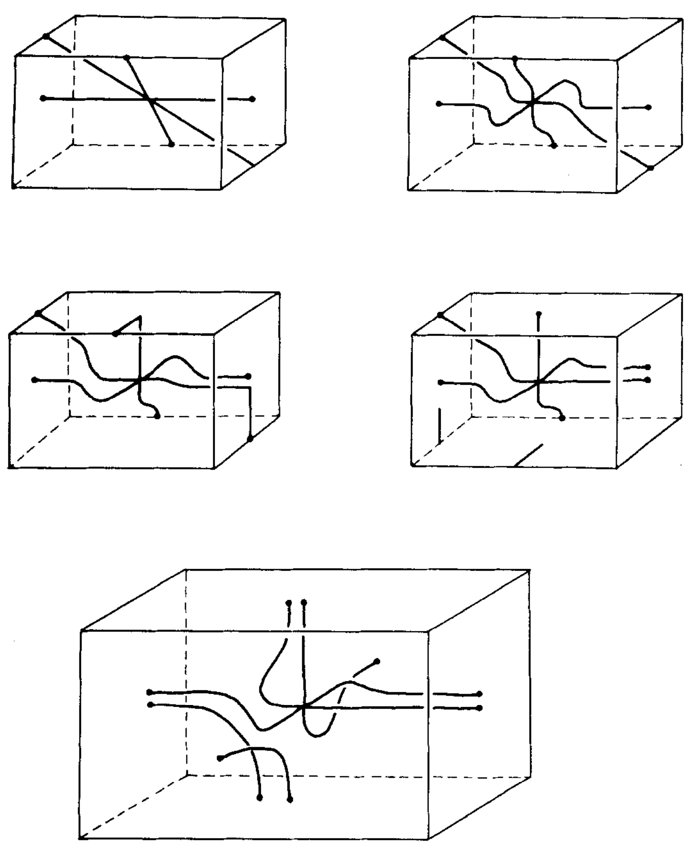

Fig. 4.

case, Fig. 3, this 2-handle γ would be attached along the indicated arc, which continues around the 1-handle a_*. In our case the arc is analogous to a ray from the origin to ∞, which avoids the attaching maps of other handles. For convenience, we pick a line given by the vector $(\frac{1}{2}, \frac{1}{2}, 1)$, and connect its ends via the 1-handle a_*. The $S^2 \times B^2$ can be added to $\hat{C}$ with or without a twist on $S^1 \times S^2 = \partial\hat{C} = \partial(S^2 \times B^2)$. In our case there is no twist, so the framing for the attaching map for γ is the untwisted one.

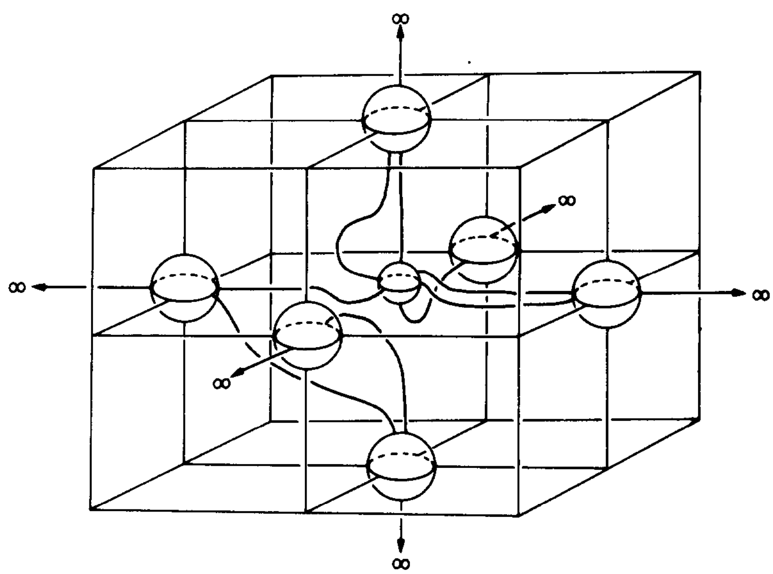

Fig. 5.

The remainder of the proof involves cancelling a_* with γ and then a_1, a_2 and a_3 with β_1, $\beta_2-\beta_1$, and β_3 respectively. In general, if a 2-handle δ cancels a 1-handle d, and if other 2-handles $\delta_1, \delta_2 \ldots$ go over d, then we slide δ_i off d using δ. A typical case of this is shown in Fig. 6, where we have drawn the actual one-handle (usually this one-handle is only imagined).

To cancel a_* and γ, we push off 6 copies of γ and slide β_1, β_2 and β_3 off a_*. Erasing a_* and γ, we get Fig. 7.

We redraw Fig. 7, simplifying the attaching maps (and deleting the boundary of the cube, leaving only the attaching curves). This gives Fig. 8, except for some framings which need explanation. Until now, the choice of a framing for a normal tube has been the obvious one. But in passing from Fig. 7 to Fig. 8, some twisting occurred as illustrated in Fig. 9. Plus (minus) one means one full right (left) handed twist in the framing.

In Fig. 8, the short arcs at the top and right can be pushed onto the boundary of the attaching 3-balls of a_3 and a_2, around those 1-handles, and then off the attaching 3-balls at the bottom and left, as in Fig. 10.

To get β_2 to cancel a_2, we must slide it over β_1. Push off a copy of β_1 using the -1

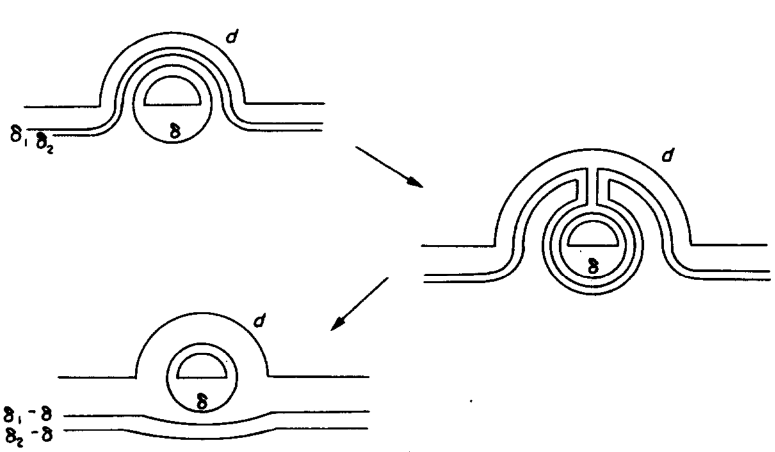

Fig. 6. d is a 1-handle, and δ, δ_1 and δ_2 are 2-handles. Slide δ_i over δ by pushing off a copy of δ and taking a band connected sum with δ_i. Then cancel d and δ by erasing them.

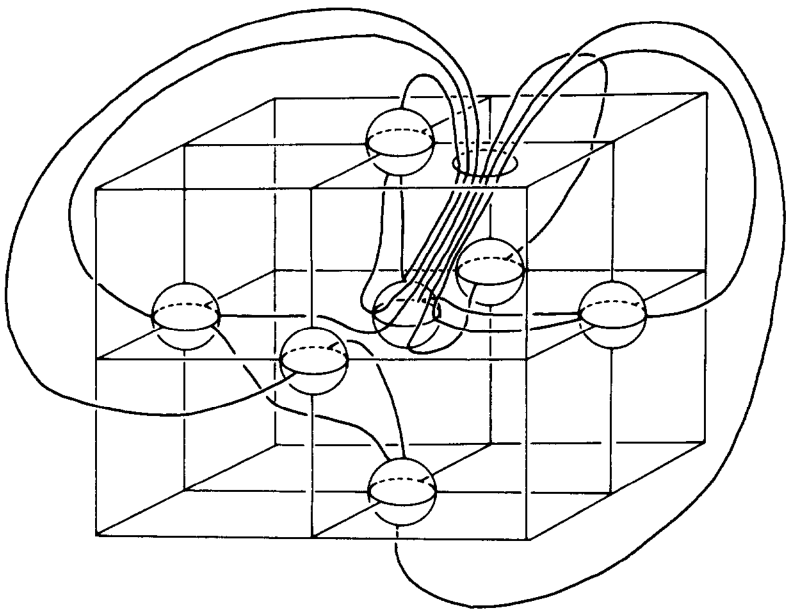

Fig. 7.

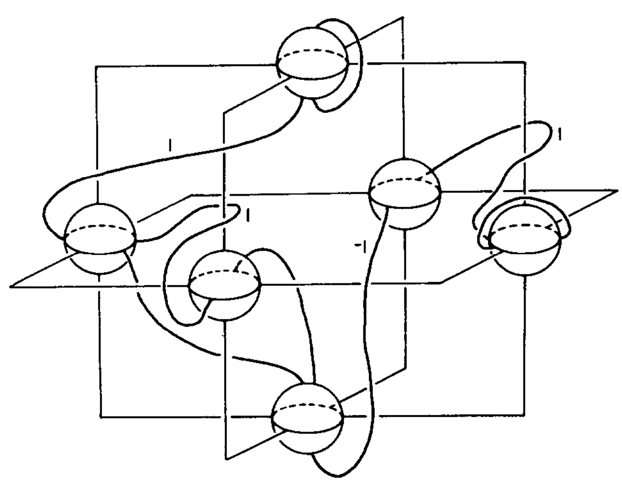

Fig. 8.

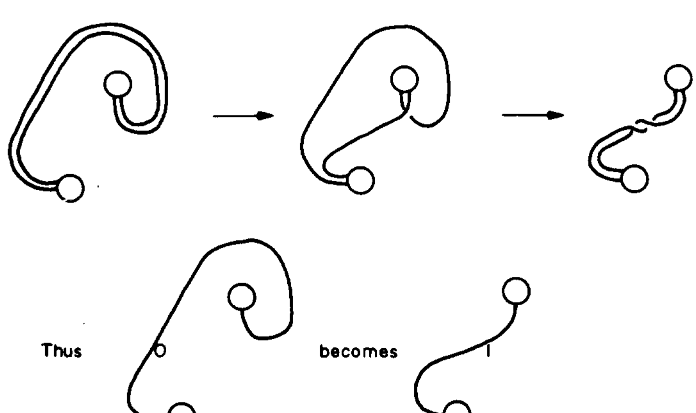

Fig. 9.

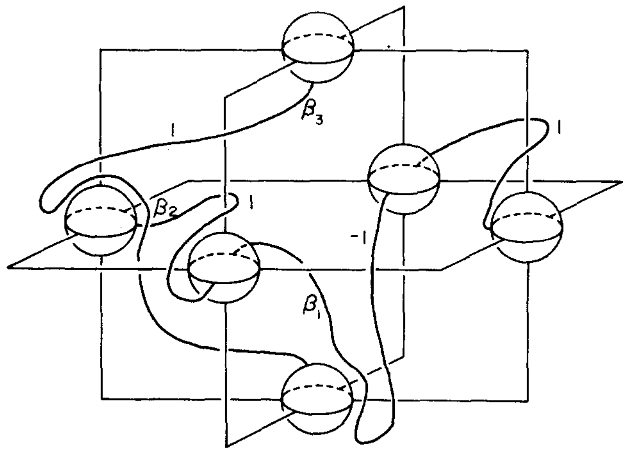

Fig. 10.

framing, and band connect sum, as in Fig. 11. We redraw to get Fig. 12 (the +1 framing on β_2 is changed to 0 in the redrawing).

Now we cancel a_1, a_2, a_3 and $\beta_1, \beta_2-\beta_1, \beta_3$ simultaneously, in analogy with Fig. 6. The result is Fig. 13, where the 1 and −1 denote full right and left-handed twists. Figure 13 is the unlink! Get 3 colors of chalk and a large blackboard; have fun.

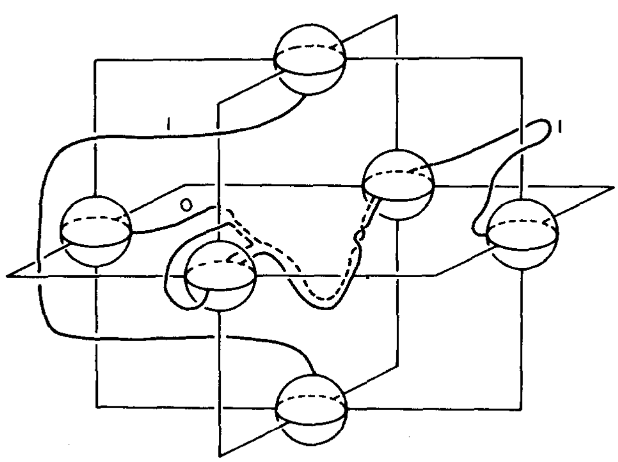

Fig. 11.

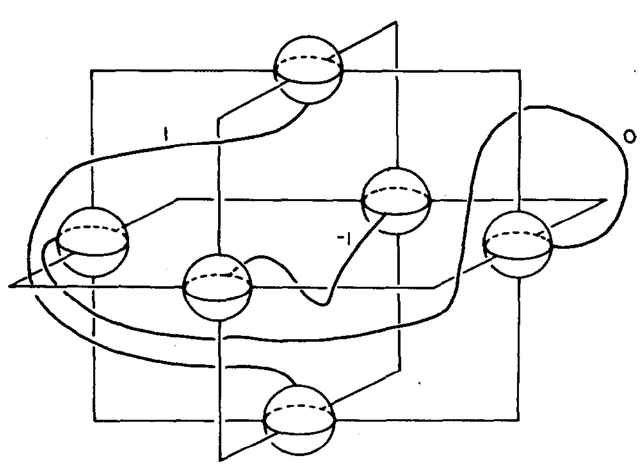

Fig. 12.

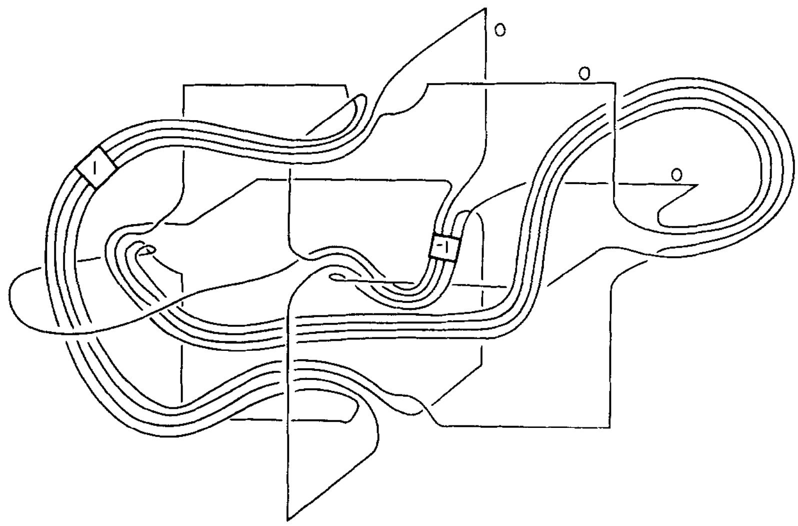

Fig. 13.

Remarks. (1) We have not seriously tried to make this argument work for other examples of Cappell and Shaneson. Other matrices have more non-zero entries, making the attaching maps more complicated.

(2) It is clear from the theorem that there is a knotted 2-sphere K in S^4 which is fibered by the punctured 3-torus with monodromy A^2. The exotic involution on S^4 restricts to the antipodal map on $K^2 \times B^2$ and to the map $(x, t) \to (Ax, t + \pi)$ on $T_0^3 \times_{A^2} S^1 = \hat{C}$. It would be a nontrivial exercise to draw K.

(3) Conjecture (Gluck): If a tubular neighborhood of a knot is removed from S^4 and sewn back in by the nontrivial diffeomorphism of $S^1 \times S^2$ coming from $\pi_1(SO(3))$, then the result is S^4.

If this operation is performed on K, then the resulting picture differs in that the 2-handle γ is attached with a framing having one full twist. After cancelling a_* with γ, we get Fig. 7 except with a full twist in the six lines parallel to the vector (1/2, 1/2, 1). We have not been able to show that this manifold is S^4; indeed, it may be a counterexample to the conjectures of Gluck and Poincaré.

REFERENCES

1. S. Cappell and J. Shaneson: Some new four-manifolds, *Ann. Math.* **104** (1976), 61–72.
2. F. Laudenbach and V. Poenaru: A note on 4-dimensional handlebodies, *Bull. Soc. math. Fr.* **100** (1972), 337–344.

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