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A POTENTIAL SMOOTH COUNTEREXAMPLE IN DIMENSION 4 TO THE POINCARE CONJECTURE, THE SCHOENFLIES CONJECTURE, AND THE ANDREWS–CURTIS CONJECTURE

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WE DESCRIBE a homotopy 4-sphere Σ^4, built with the usual zero and 4-handle and two 1-handles and two 2-handles (Figure 28). Of course Σ^4 is homeomorphic to S^4 [Freedman] but considerable effort has not led to a proof that Σ^4 is diffeomorphic to S^4. Σ^4 has the following virtues:

(A) Although it is easy to construct smooth homotopy 4-spheres (e.g. the Gluck construction on knotted 2-spheres or via non-trivial presentations of the trivial group), this is the only (except S^4) example we know without 3-handles and with so few handles altogether.

(B) The presentation of the trivial group arising from Σ^4 (see §2) is $\{x, y \mid xyx = yxy, x^5 = y^4\}$; it is easy to show that this group is trivial, but it seems difficult to do so using Andrews–Curtis moves ([1] or [10] 5.1).

(C) Let Σ_0 be Σ without the 4-handle; then we can add two 2-handles and two 3-handles and a 4-handle to get (smoothly) S^4 (see §2). Applying the topological Schoenflies theorem to $\partial\Sigma_0$ in S^4, we see directly that Σ_0 is homeomorphic to B^4.

(D) $\partial\Sigma_0$ is an interesting smooth S^3 in S^4. The smooth Schoenflies conjecture is unsettled in dimension 4 and $\partial\Sigma_0$ is a good test case. So in §4, we give a critical level imbedding of $\partial\Sigma_0$ in S^4 (Fig. S1–S11). Scharlemann [12] has used critical level imbeddings to prove the conjecture for genus 2 imbeddings; this one is genus 51.

(E) Σ_0 is the result of the Gluck construction on a knot K in S^4 (Fig. 16). K is constructed from two distinct ribbons for the 8_9 knot (see [11]).

Σ was first defined as the double cover of a certain exotic RP^4 of Cappell and Shaneson [7]. It was built by decomposing RP^4 into a 2-disk bundle over RP^2 and the non-trivial 3-disk bundle over S^1, and then replacing the latter by a punctured 3-torus bundle over S^1 with monodromy $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$. We thought we proved (in [4]) that the double cover Σ of this exotic RP^4 was diffeomorphic to S^4. However Iain Aitchison and J. H. Rubenstein ([2], [3]) discovered an error (the last sentence on page 77 of [4] states that a framing is zero when it should be odd). We actually proved that Σ^4 is the Gluck construction on a knotted 2-sphere in S^4 (this was discussed in Remarks 2 and 3 of [4]). As mentioned above, we are still unable to prove that Σ^4 is diffeomorphic to S^4. In the meantime however, Fintushel and Stern [8] have constructed, by different methods, an exotic RP^4 whose double cover is S^4. Note that both these exotic RP^4's are homotopy RP^4's which are s-cobordant to RP^4, and then homeomorphic to RP^4 by Freedman's recent proof of the topological s-cobordism theorem for many fundamental groups including $Z/2$.

After some definitions in §1, we begin in §2 with a handlebody description of Σ^4 from [4, Fig. 5]. We simplify this handlebody presentation by sliding handles over other handles and by handle cancellations and births to get the properties of Σ^4 mentioned above in (A), (B), (C) and (E). It is worth remarking that it is usually hard to see how to add a cancelling pair of handles (a birth) in any useful way; this is done with a (2–3)-pair in Fig. 19 and later in Fig. 35.

Some problems are suggested by this work:

(1) Does every homotopy 4-ball with boundary S^3 smoothly imbed in S^4?

(2) Do the results of this paper hold for the other fake RP^4's of [3]?

(3) Is S^4 the double of Σ_0?

(4) Classify the different ribbons for a ribbon knot.

We would like to thank Andrew Casson for helping us in simplifying the fundamental group calculations.

§1

First, here are some definitions and notation. 1-handles were described in [4] by drawing their attaching maps, i.e. by drawing two 3-balls in $S^3 = \partial B^4$. They may also be described by an unknotted circle with a dot on it to distinguish it from the attaching map of a 2-handle (see [5], p. 260). This dotted circle means: delete from B^4 the thickened, unknotted 2-ball which the circle bounds, obtaining $B^4 \cup$ 1-handle. Any arc going through the dotted circle goes over the 1-handle. This notation has the virtue that replacing the dot by a zero is the same as replacing the 1-handle by a 2-handle, i.e. surgering $S^1 \times B^3$ to $S^2 \times B^2$. This notation can be extended to the case where the dotted circle is a slice knot and we are meant to delete the thickened slice from B^4. Since a slice knot has more than one slice (just connect sum any knotted S^2) it is necessary to somehow specify the slice to be used; when the dotted knot is ribbon, this may be done by carrying along dotted arcs indicating which ribbon moves on the knot give the ribbon disk.

§2

Figure 1 is our starting point; it is a picture of the zero, one and two-handles of Σ^4, the double cover of Q (the fake RP^4). It comes from Fig. 5 of [4] by changing the notation for the one-handle a_* (which is attached to the balls of the origin and at ∞) to the "dotted circle", and by adding the last 2-handle γ with framing-1 (not 0 as erroneously claimed). The framing of the α_i and β_i, $i = 1, 2, 3$, are given by the normal vector field lying in the plane of the paper. There are three 3-handles and a 4-handle which are not drawn.

Before γ is added, but after the 3-handles are added, the boundary is $S^1 \times S^2$ and what is missing is $B^2 \times S^2$, i.e. γ and the 4-handle. The core, $0 \times S^2$, is the "knotted" 2-sphere K in Σ^4. If we perform the Gluck construction on K we get S^4, since removing $K \times B^2$ means removing γ and the 4-handle, and replacing $K \times B^2$ with a twist means adding γ with 0-framing and the 4-handle; but this was what was proven in [4] to be S^4. Conversely then, Σ^4 is the Gluck construction on K in S^4.

We can see K by seeing $0 \times S^2$ in $S^1 \times S^2$, the boundary before γ is added. Since the $S^1 \times S^2$ is just $S^1 \times \partial T_0^3$ where T_0^3 is the punctured 3-torus, we can take $O \times S^2$ to be $\partial[-1, 1]^3$

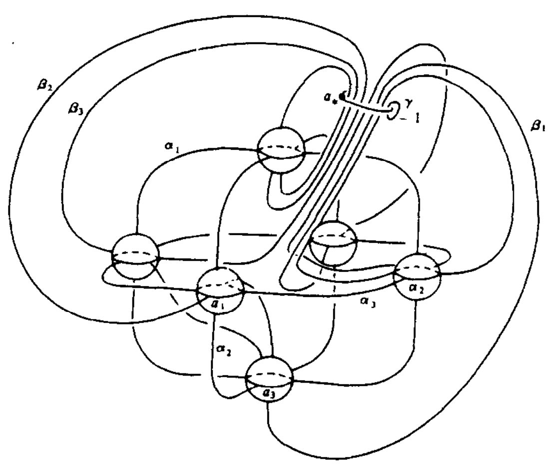


Fig. 1.

followed by surgery on a_1, a_2, a_3 and $\alpha_1, \alpha_2, \alpha_3$, the three 1-handles and 2-handles of $T_0^3 \times [-1, 1]$. Thus an equator for $0 \times S^2 = K$ is a circle close to and parallel to the dotted circle representing a_*; call this equator A_*. One of the hemispheres it bounds in K is just the obvious flat 2-ball bounded by a_*, call it H_-; the other, H_+, is harder to see because it goes over the handles. H_- looks unknotted, but it isn't. If we cancel a_1, a_2, and a_3 with $\beta_1, \beta_2 - \beta_1$, and β_3, then H_- turns into a ribbon disk for the 8_9 knot in S^3, Fig. 16, as we shall see later. Similarly, if we turned our handlebody upside down and cancelled a_* with γ (with framing 0 so as to see K in S^4) and α_1, α_2, and α_3 with the 3-handles, then H_+ would become a different ribbon disk for 8_9, (Figs. 15, 16).

We want to begin simplifying the description in Fig. 1 of Σ^4. It turns out that α_2 and α_3 are cancelled by 3-handles, and if we show this first then in later versions of Fig. 1 we will not have to draw α_2 and α_3. To see that α_2 and α_3 are cancelled, go through the steps in [4], but carrying along a_* and γ. In going from Fig. 7 to Fig. 8 of [4], a_* (as a dotted circle) will be pulled over 1-handles, and then as a_1, a_2 and a_3 are cancelled, a_* becomes a knot (8_9 in fact). Remember from the introduction that a knotted dotted circle must have a preferred ribbon disk which is to be removed (the same as adding a 1-handle if the dotted circle is unknotted). The three circles, α_1, α_2 and α_3, in Fig. 13 of [4] become the unlink tangled with a_*; we get Fig. 2.

If we slide α_2 over α_1 it also becomes parallel to α_1. Then slide both α_2 and α_3 over α_1 so that they become unlinked (with zero framings). Then α_2 and α_3 contribute two copies of $S^1 \times S^2$ to the boundary which is $S^1 \times S^2 \# S^1 \times S^2 \# S^1 \times S^2$. Thus α_2 and α_3 must be cancelled by 3-handles. To put it another way, in Fig. 1, α_2 and α_3 may be slid over the other handles (excepting a_* and γ) until they become an unlink separated from the rest.

Now we change the notation for the 1-handles in Fig. 1, switching to dotted circles. This is shown in Fig. 3, where α_1 is not drawn even though it is there. The attaching circles in Fig. 3 are drawn with care for their framings are determined by the push off which lies parallel in the figure. Thus the push off for β_1 has one left crossing which, when straightened out, gives one full left twist with its pushoff, hence framing -1. Similarly β_2 has framing $+2$ and β_3 gets -1.

To go from Fig. 3 to Fig. 4, we do some simple isotopies and slide β_1 over a_1 and β_3 over a_3, thereby changing their framings. Next, slide β_1 over β_2 as indicated in Fig. 5, cancel a_1 and β_2 by erasing them, and isotop to Fig. 6.

We continue in Fig. 7 by sliding α_1 and β_1 twice (algebraically zero) over β_3 and then cancelling β_3 and a_3 by erasing them. Further isotopies give Figs. 8 and 9.

Now we want to describe K as the union of two ribbons for the 8_9 knot. Remember that Fig. 9 plus a 3 and 4-handle is Σ^4, and that if we remove the 4-handle *and* γ then we have the knot complement $S^4 - (K \times \text{int } B^2) = \Sigma^4 - (K \times \text{int } B^2)$. Also recall that a copy of K consists of a hemisphere H_- equal to the obvious disk bounded by the equator $A_* = a_*$ of K, and a hemisphere H_+ which is harder to see. Shrink β_1, turning a_* into a ribbon knot, as in Fig. 10;

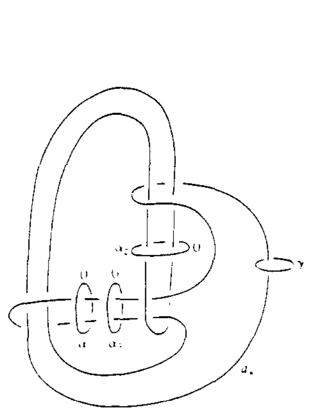
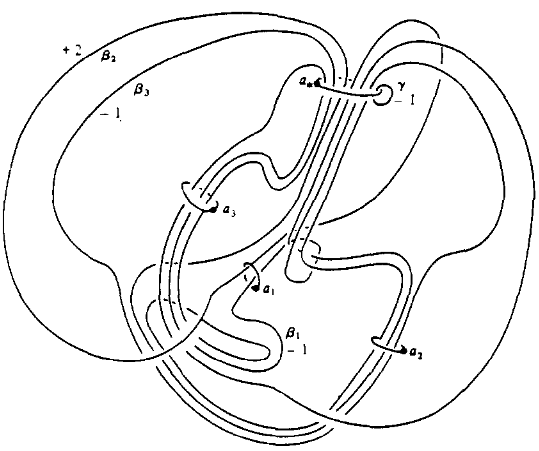


Fig. 2.



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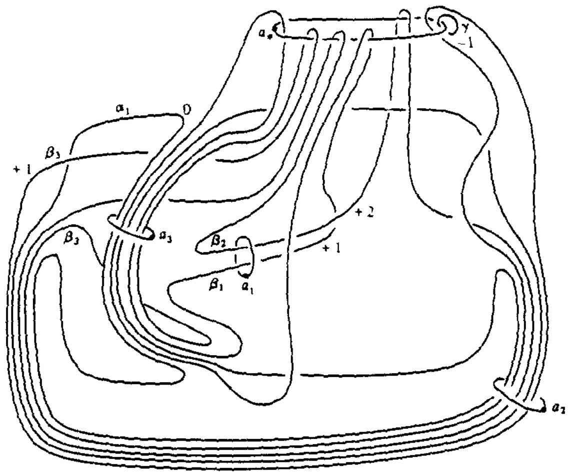

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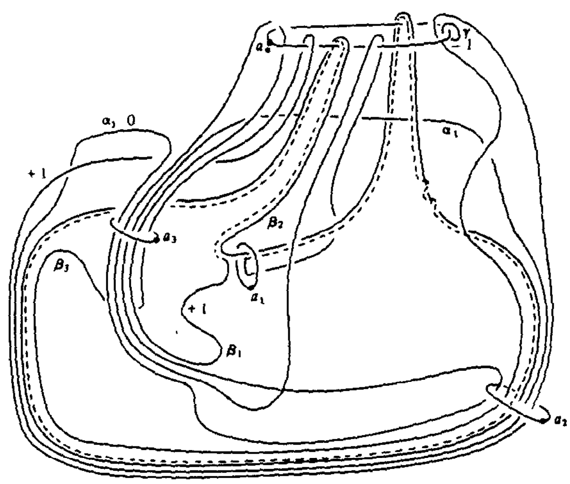

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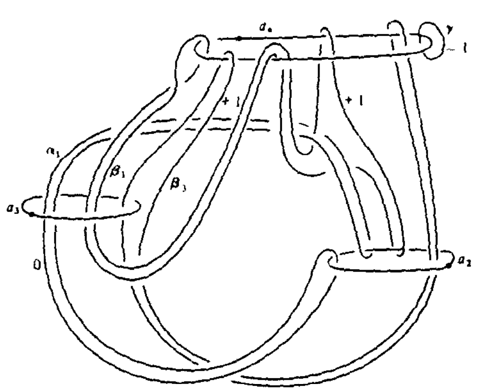

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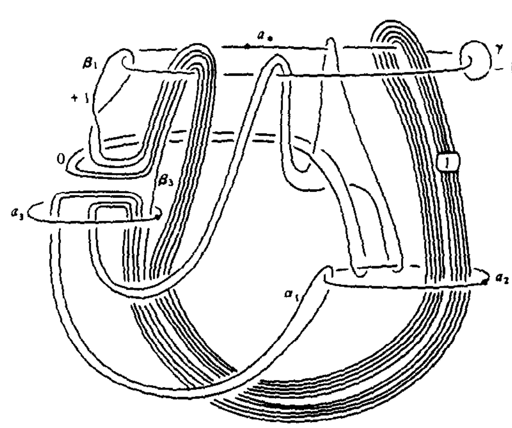

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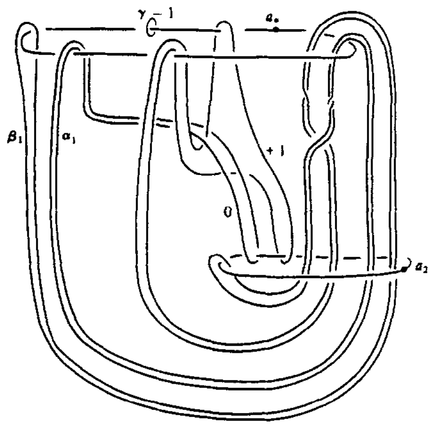

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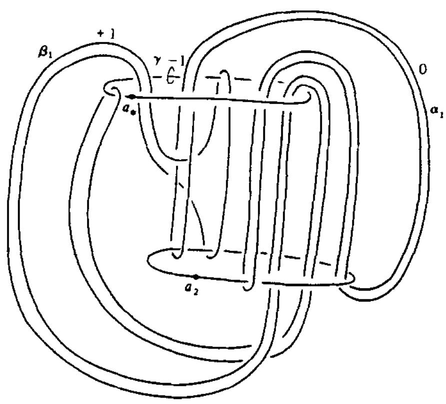

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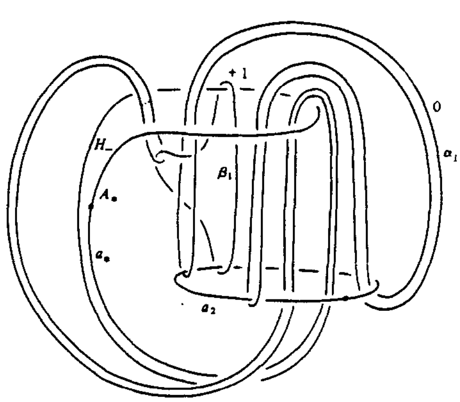

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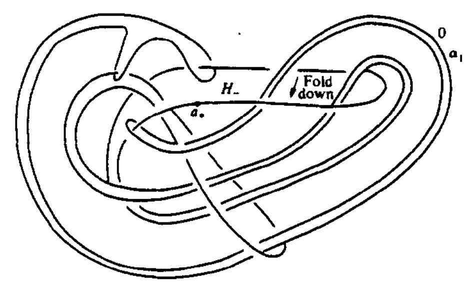


Fig. 11.

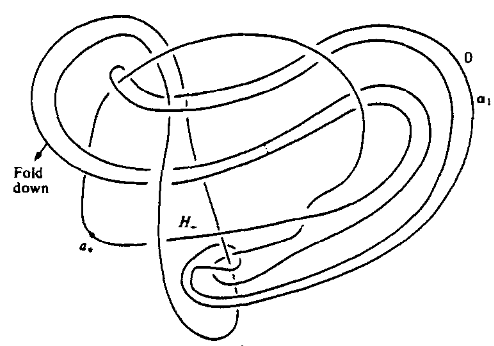


Fig. 12.

the ribbon move is indicated by a dotted arc which can be isotoped along the band to put it in a convenient position. Now cancel a_2 with β_1 by sliding over β_1 eight times, obtaining Fig. 11.

A sequence of isotopies (Figs. 12, 13 and 14) end with the 8_9 knot, a dotted line indicating the ribbon move, and a 2-handle. The reader can check that this is the same knot as in Fig. 2 with α_1 in the same position and the dotted arc giving the same ribbon move. The other ribbon is given by α_1. Figure 15 has a movie of K in S^4 and simultaneously a handlebody description of $S^4 - K$. We begin at the left with an unlink, and two 3-handles and a 4-handle (one 3-handle cancels the 4-handle) which are not drawn. The middle picture is a slice of K the

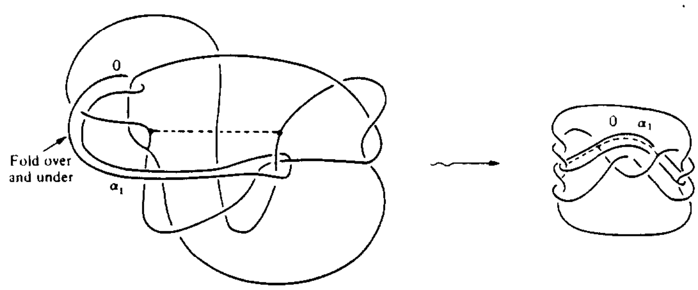
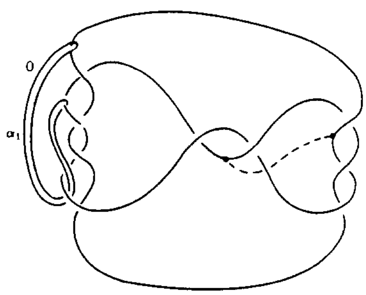


Fig. 13.



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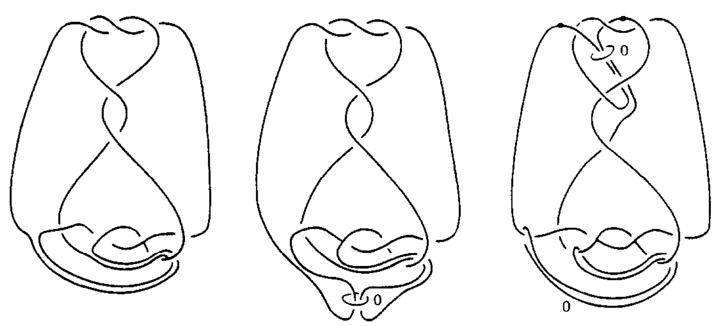


Fig. 15.

8_9 knot just after a ribbon move has occurred and a 2-handle has been added to $S^4 - K$. The left hand picture is the unlink again just after another ribbon move and another 2-handle has been added. Finally the two 1-handles of $S^4 - K$ are given by dotting the unlink (see also [5]).

Figure 16 is a more artistic way of drawing the 8_9 knot with its ribbon moves; this picture makes obvious an orientation reversing involution of the 8_9 knot which switches the ribbon moves (just rotated by π and reflect).

We redraw the left hand picture in Fig. 15 and add γ to get Fig. 17; together with a 3 and 4-handle it gives Σ^4. Cancel b_2 and γ to get Fig. 18. (This figure is actually $-\Sigma^4$).

Next, we add a cancelling 2–3 pair, with the 2-handle added to a circle parallel to the twist (Fig. 19). This circle must be trivial on the boundary, $S^1 \times S^2$, of the link in Fig. 18. To check this, draw a circle μ parallel to the "twist" circle, blowing up a -1 circle and changing both $+1$ framings to 0, slide μ over the -1 circle so that it links only the -1 circle, surger the 1-handle to a 2-handle by replacing the dot by a zero, and shrink a 2-handle as in Fig. 20. Then change the shrunk 2-handle to a 1-handle, slide the -1 2-handle over the 2-handle which cancels the 1-handle, and cancel. Isotopies and a few slides reduce to the simple link in Fig. 20

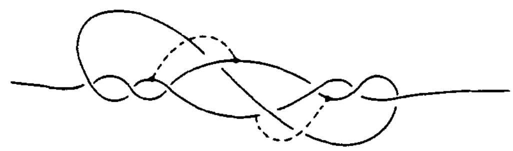
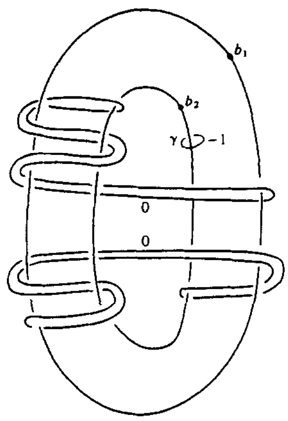


Fig. 16.



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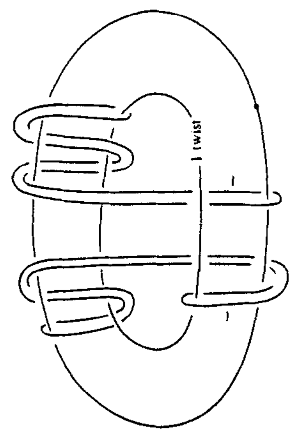

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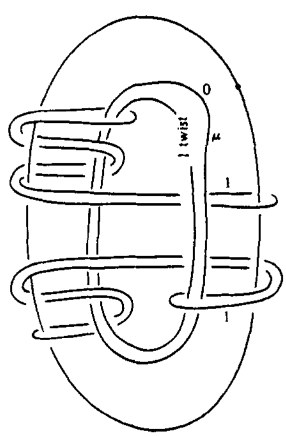

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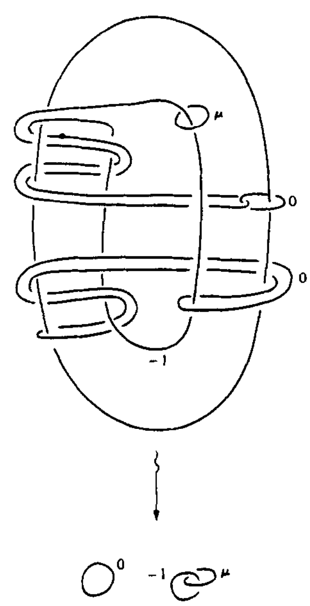
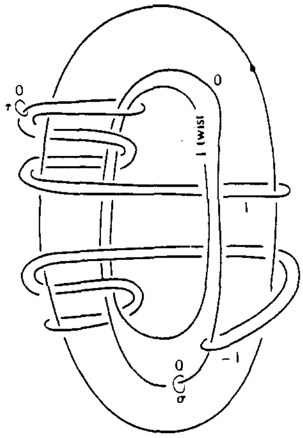


Fig. 20.

which shows that μ is trivial in the boundary so that we may add a cancelling 3-handle along the 2-sphere that μ defines.

Now slide a 2-handle over the 2-handle μ so that it becomes unlinked from the twist, and note that its framing changes to -1 (Fig. 21). The -1 handle cancels the 1-handle which leaves two 2-handles and two 3-handles. We want to turn this handlebody over to get two 1-handles and two 2-handles; this is done by drawing the dual 2-handles, σ and τ, changing the interior to the handlebody $0^c 0^c$ via handle slides while carrying along σ and τ, and then changing this unlink to two 1-handles (Figs. 21–28).



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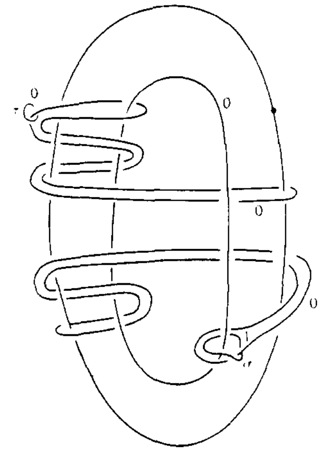

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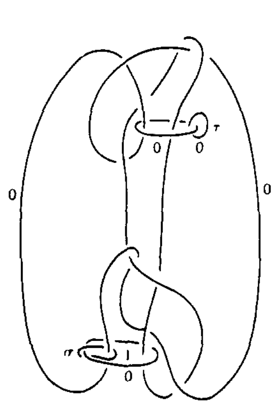

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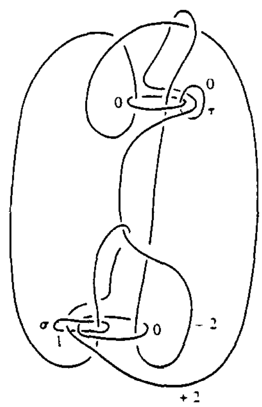

Fig. 24.

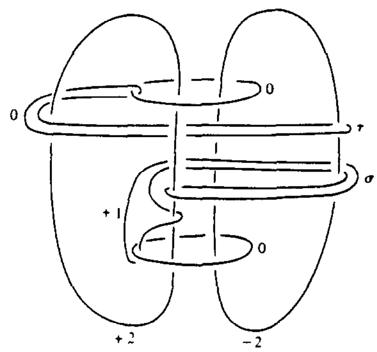

Fig. 25.

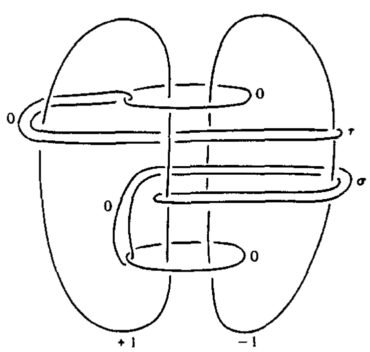

Fig. 26.

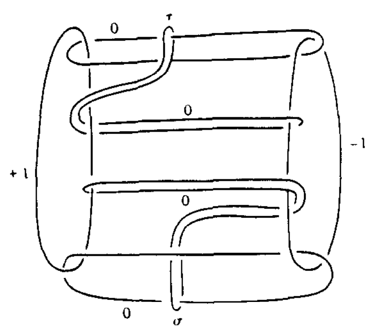

Fig. 27.

To get to Fig. 22, we blow up a -1 circle parallel to the twist, eliminating the twist and changing framings to zero; then slide the new -1 circle over the 0-framed 2-handle and blow it down changing framings as indicated. Surger the 1-handle and isotop to get Fig. 23. Two handle slides give Fig. 24. Another isotopy leads to Fig. 25. To remove the full right hand twist between the $+1$ and $+2$ curves, we blow up a -1 curve around them, slide it over the lower horizontal 0 curve and blow it down, obtaining Fig. 26 (note the framing changes). An isotopy gives Fig. 27 which is invariant under rotation by π followed by reflection.

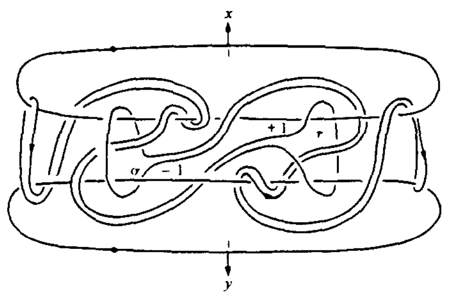

Fig. 28.

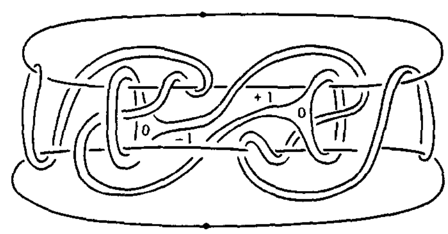

Fig. 29.

Now blow down both the $+1$ and -1 handles, isotop to Fig. 28 and surger the 0 curves to 1-handles. This figure is worth examiniation; it is a non-trivial picture of a homotopy 4-ball with boundary S^3, it has symmetry, and it gives an interesting presentation of the trivial group.

Labeling the 1-handles by x and y and starting at the arrows, we read off the relations

$$1 = yx^{-2}yxy^{-2}x(xy^{-1}x^{-1}) \qquad \text{and} \tag{1}$$

$$1 = yx^{-2}yxy^{-2}x(y^{-1}x^{-1}y) \qquad \text{so we deduce} \tag{2}$$

$$xy^{-1}x^{-1} = y^{-1}x^{-1}y \quad \text{or} \tag{3}$$

$$xyx = yxy. \tag{4}$$

Using (3) and (4) we see that (1) becomes

$$\begin{aligned} 1 &= yx^{-2}(yx)y^{-2}x^2(y^{-1}x^{-1}) \\ &= y(yx)y^{-4}(y^{-1}x^{-1})y^2 \\ &= x^{-5}(yx)x^{-1}y^3 \\ &= x^{-5}y^4 \end{aligned} \tag{5}$$

This presentation $\{x, y \mid xyx = yxy, x^5 = y^4\}$ is seen to be the trivial group as follows: $xyx = yxy$ implies that $y = (yx)^{-1}x(yx)$ so $y^5 = (yx)^{-1}x^5(yx) = (yx)^{-1}y^4(yx) = x^{-1}y^4x = x^{-1}x^5x = x^5 = y^4$ so $y = 1$ and $x = 1$. (Note that this proof works for the group $\{x, y \mid xyx = yxy, x^{n+1} = y^n\}$.)

We do not know whether this presentation, let along the original one with relations (1) and (2), can be trivialized by Andrews–Curtis moves (see [1], [10] Prob. 5.2); if so, then Σ^4 would be homeomorphic to S^4 by a standard argument [1].

We shall show that Σ^4 is homeomorphic to S^4 by adding two 2-handles to the handlebody of Fig. 28 in such a way that the new manifold is $S^2 \times B^2 \# S^2 \times B^2$; then two 3-handles and a 4-handle can be added to give S^4. Since the boundary of Fig. 28 is S^3, an application of the topological Schoenflies theorem gives that Fig. 28 is homeomorphic to B^4.

Add the 2-handles as in Fig. 29 and do the obvious two handle slides over the new 2-handles to get Fig. 30. Cancel the two 1-handles with the $+1$ and -1 handles to obtain $S^2 \times B^2 \# S^2 \times B^2$ in Fig. 31. This finishes a rather long series of moves which shows that Σ^4 is homeomorphic to S^4.

§3

It is interesting to have an example of a framed link L of two components, with framings and linking number all zero, whose 4-manifold W_L has boundary $S^1 \times S^2 \# S^1 \times S^2$ but is not obviously $S^2 \times B^2 \# S^2 \times B^2$. It is easy to construct nontrivial links with $\partial W_L = S^1 \times S^2 \# S^1 \times S^2$ by adding an unknot to a ribbon knot, e.g. Fig. 32; but in these cases W^4 is obviously $S^2 \times B^2 \# S^2 \times B^2$. If L has one component, then $\partial W^4 = S^1 \times S^2$ implies that L is slice [9] and may imply that L is the unknot.

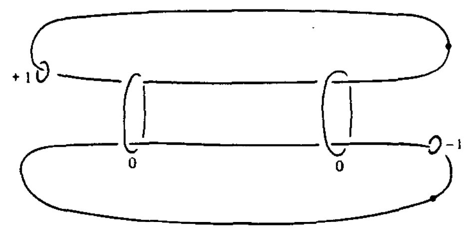

Fig. 30.

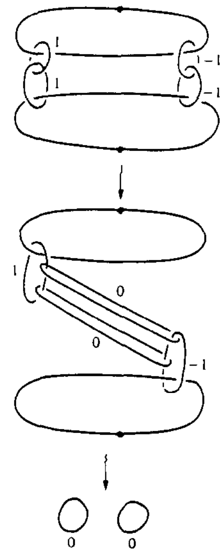

Fig. 31.

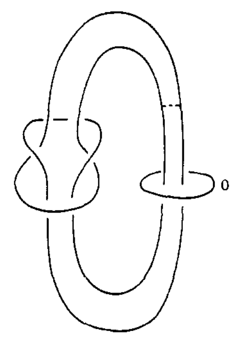

Fig. 32.

Our example begins with Fig. 9 and we cancel a_* and γ to get Fig. 33. We want to add a cancelling 2–3-handle pair with the 2-handle δ being attached to an unknotted circle parallel to the twist in Fig. 33. We need to know that δ represents a trivial circle with framing zero in the boundary of Fig. 33. To see this we construct a diffeomorphism of the boundary to $S^1 \times S^2 = \partial(0^\circ)$ and check that δ is trivial. The first step is to shrink α_1, change it to a 1-handle and change a_2 to a 2-handle (see Fig. 34). Cancel α_1 with β_1 by first sliding twice over β_1 and then erasing α_1 and β_1. Surprisingly, a_2 and δ become the unlink, as can be seen after a long isotopy.

So add the 2-handle δ and slide β_1 ovef δ so that the end of β_1 no longer goes through the twist, Fig. 35. Shrink β_1 (Fig. 36) and cancel a_2 and β_1 by sliding over β_1 eight times and then erasing a_2 and β_1. A further isotopy gives our example in Fig. 37. One can independently check that the boundary in Fig. 37 is $S^1 \times S^2 \# S^1 \times S^2$. First, blow up a -1 circle parallel to the twist, thus removing the twist; then slide the -1 circle over α_1 using a band connected sum along the dotted arc and isotop the -1 circle to the other end of α_1; finally blow down the -1 circle and check via isotopies and a few handle slides that one obtains the unlink.

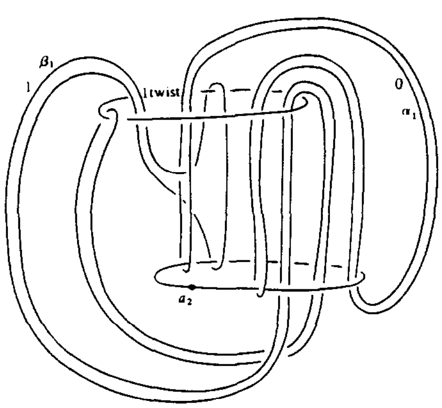

Fig. 33.

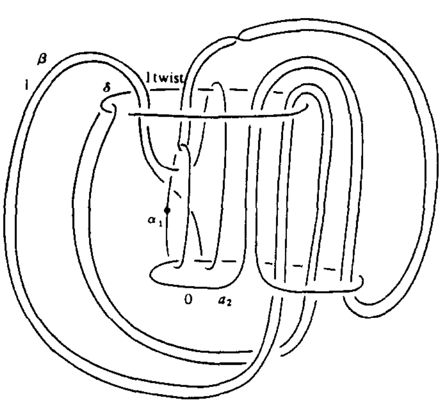

Fig. 34.

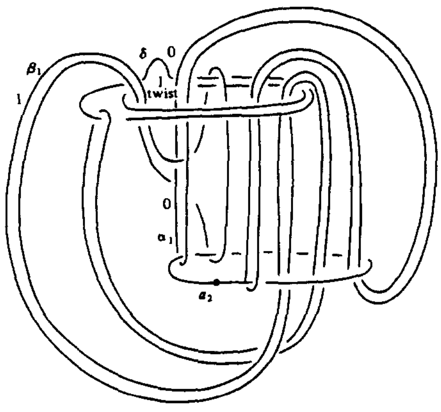

Fig. 35.

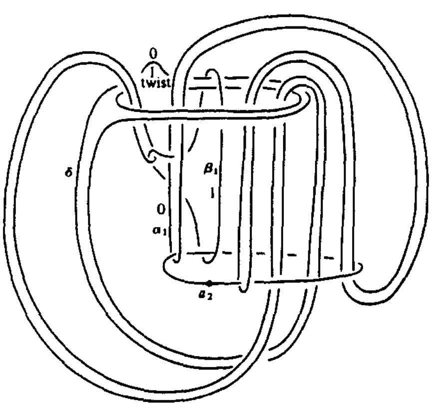

Fig. 36.

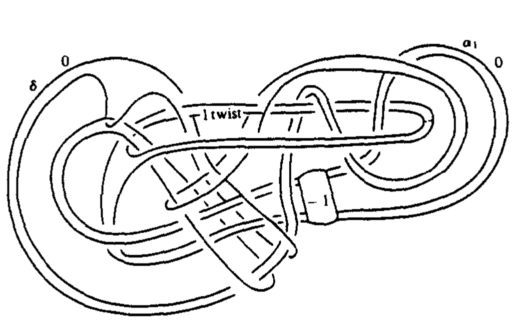

Fig. 37.

§4

Σ^4 would be diffeomorphic to S^4 if the smooth 4-dimensional Schoenflies conjecture could be proved. It may help in analyzing this conjecture to have a non-trivial example of an imbedded S^3 in S^4. To this end we draw (in Figs. S1–S11) a "movie" of $\partial(\Sigma^4\text{-4-handle}) = S^3$ in S^4. For simplicity we refer to this S^3 as $\partial\Sigma_0$.

Our starting point is Fig. 29 which is a framed link picture of S^4. In it we can see $\partial\Sigma_0$ as ∂(0-handle) with surgery on the 1-handles and the 2-handles with framing ± 1. We want to cancel the 1, 2 and 3-handles so that S^4 is constructed with only a 0 and 4-handle; as we cancel handles we keep track of $\partial\Sigma_0$ and also isotop it to a critical level imbedding. This means that if we think of S^4 as $(S^3 \times \mathbb{R}) \cup \{-\infty\} \cup \{+\infty\}$, then projection to $\mathbb{R}$ is a Morse function, f: $\partial\Sigma_0 \to \mathbb{R}$, when restricted to $\partial\Sigma_0$. Figures S1–S11 show $f^{-1}(t_i)$, for increasing i.

Assume that $S^3 \times (-\infty, 0] \cup \{-\infty\}$ is the 0-handle of S^4. Then except for the surgeries, $\partial\Sigma_0 \cup S^3 \times 0$. We begin with Fig. S4 which shows the boundary of $\{S^3$ with two $S^0 \times B^3$'s and two $S^1 \times B^2$'s removed$\}$. This 2-manifold can be thought of as four S^2's (the boundaries of the two $S^0 \times B^3$'s) and twenty two tubes connecting them; the tubes correspond to the boundaries of the attaching maps of the two handles. The interior of $\{S^3$ with two $S^0 \times B^3$'s and two $S^1 \times B^2$'s removed$\}$ has been pushed down into $S^3 \times (-\infty, 0)$ and is drawn in Figs. S1–S3. First, the surface in Fig. S4 is "unknotted" by an isotopy as in Fig. S3; next the obvious holes are filled in as we pass sixteen critical points of index 1 to reach Fig. S2; then we unknot again by an isotopy, fill in three holes to get a 2-sphere; (Fig. S1) and finally cap off *outside.*

To proceed upwards from Fig. S4, we must see what happens to $\partial\Sigma_0$ as the 2-handles slide as in Figs. 29, 30. The reader should work through the lower dimensional case in which F is the boundary of $B^3 \cup$ 2-handle, a second 2-handle is added to B^3, and the first 2-handle is slid over the second. To begin with, push F into B^3 except for the $S^1 \times S^0$ which remains in ∂B^3 (see Fig. 38). Then, using a collar $S^2 \times [0, 1]$ on ∂B^3, we see the $S^1 \times S^0$ get pushed around the

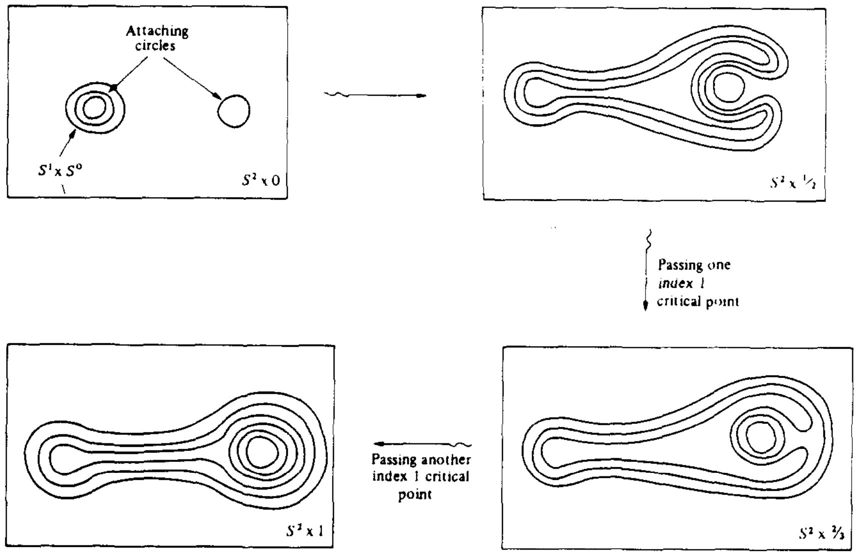

Fig. 38.

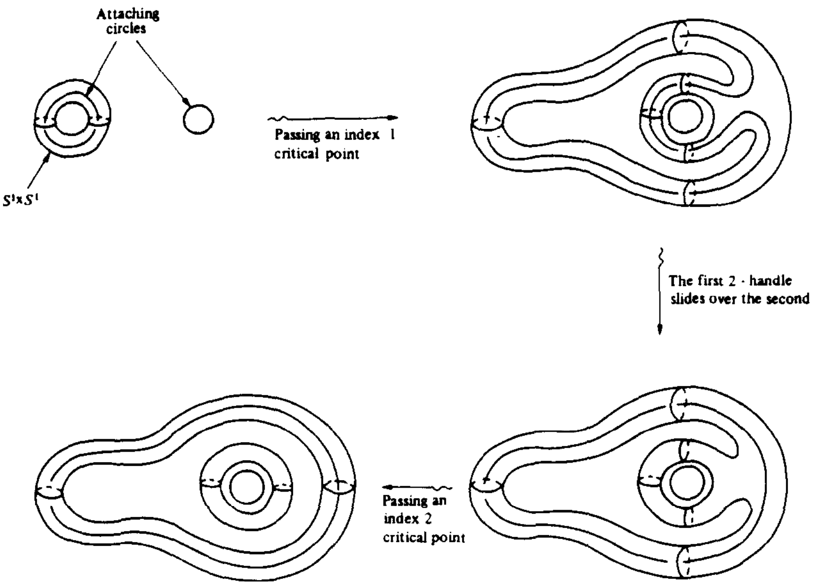

Fig. 39.

attaching circle of the second 2-handle as we simultaneously push the first 2-handle and F over the second 2-handle (Fig. 38). In Fig. 39 we draw the same case but in dimension 4. Notice that we freely add collars to B^3 (or B^4) when pushing F (or $\partial\Sigma_0$) around, and similarly we fatten the 2-handles when convenient.

To get from Fig. S4 to Fig. S6 we must push $\partial\Sigma_0$ as we slide two 2-handles; the picture is more complicated than Fig. 39 because there are 1-handles present and the 2-handles go over them. Figure S5 shows part of the movie between Figs. S4 and S6. We see what happens as the -1 2-handle (not drawn since it lies inside the obvious tube) slides over the drawn 2-handle and then begins to isotop off the 1-handle.

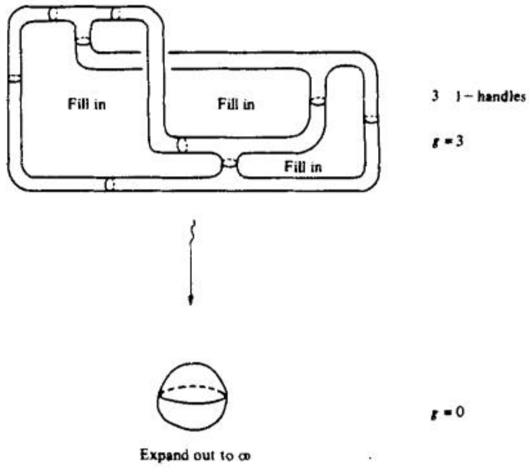

Fig. S1

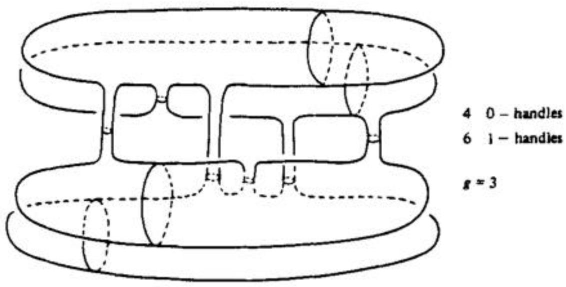

Fig. S2.

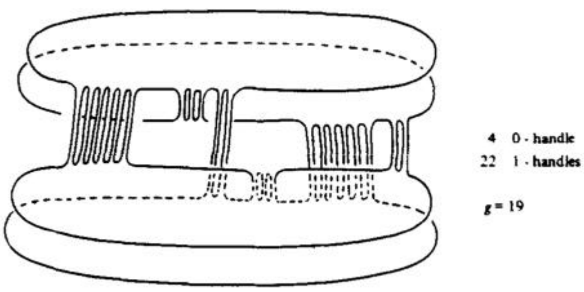

Fig. S3.

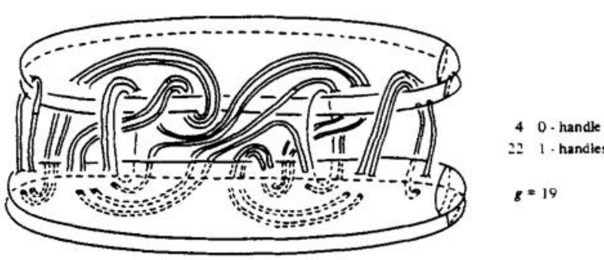

Fig. S4.

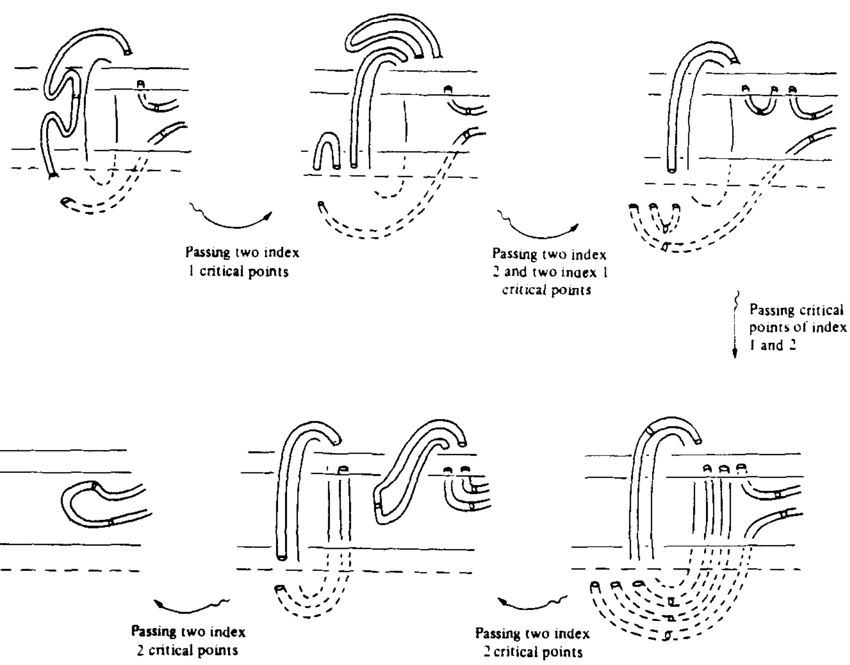

Fig. S5.

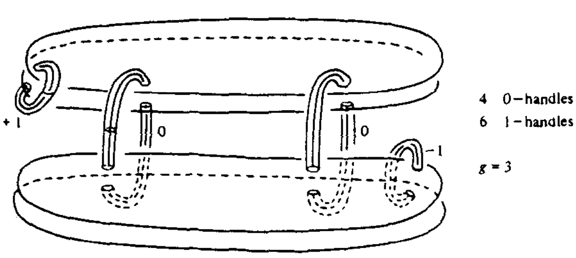

Fig. S6.

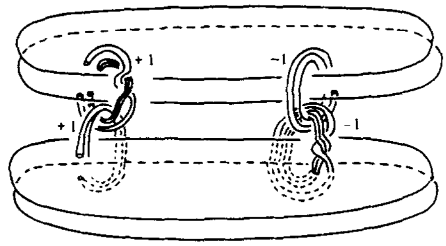

Fig. S7.

The same technique is used in the handle slides going from Fig. S6 to S8. In Fig. S8, the tori parallel to the 0-framed circles are isotoped away from the rest. To see what happens when the 1 and 2-handles are cancelled, it is enough to examine the case in dimension 3; this is left to the reader. From Fig. S9 to Fig. S10, we pass four critical points of index 2. Finally in

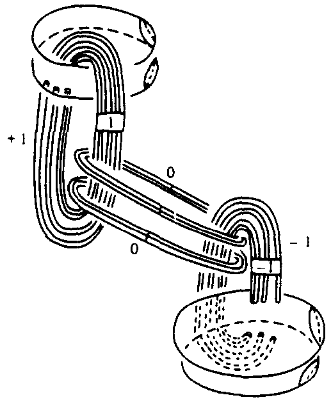

Fig. S8.

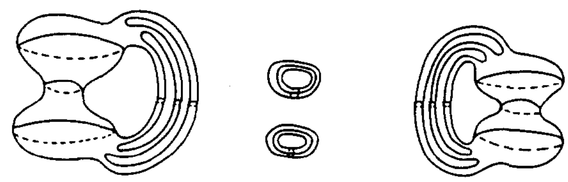

Fig. S9.

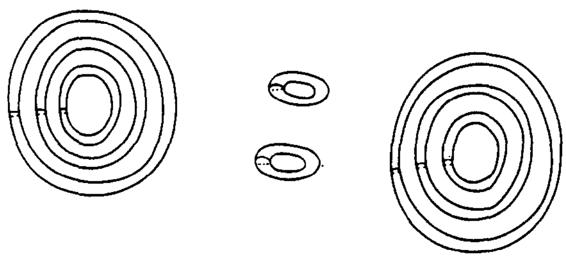

Fig. S10.

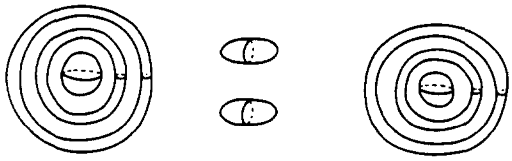

Fig. S11.

Fig. S10 we cap off each torus by passing a critical point of index 2 and then index 3, as indicated in Fig. S11.

This description of $\partial\Sigma_0$ has one critical point of index 0, 51 of index 1, 58 of index 2 and 8 of index 3. Many of these can be cancelled, but we have not tried to simplify the movie; it is simply the translation of the handlebody picture. We humbly offer $\partial\Sigma_0$ to the devotees of the Schoenflies conjecture.

REFERENCES

1. J. J. ANDREWS and M. L. CURTIS: Free groups and handlebodies, *PAMS* **16** (1965), 192–195.
2. I. AITCHISON and J. H. RUBENSTIEN: Letter, June 1979.
3. I. AITCHISON: Involutions and handle decompositions of 4-manifolds, M.Sc. thesis, U. Melbourne, 1979.
4. S. AKBULUT and R. KIRBY: Exotic involutions of S^4, *Topology* **18** (1979), 75–81.
5. S. AKBULUT and R. KIRBY: Branched covers of surfaces in 4-manifolds, *Math. Ann.* **252** (1980), 111–131.
6. S. AKBULUT and R. KIRBY: Mazur Manifolds, *Michigan Math. J.* **26** (1979), 259–284.
7. S. CAPPELL and J. SHANESON: Some new four-manifolds, *Ann. Math.* **104** (1976), 61–72.
8. R. FINTUSHEL and R. STERN: An exotic free action on S^4, *Ann. Math.* **113** (1981), 357–365.

9. R. Kirby and P. Melvin: Slice knots and Property R, *Inv. Math.* **45** (1978), 57–59.
10. R. Kirby: Problems in low dimensional manifold theory, *Proc. Sym. Pure Math.* **32** (1978), 273–312.
11. D. Rolfsen: *Knots and Links*, Publish or Perish, Berkeley, California.
12. M. Scharlemann: The four-dimensional Schoenthes conjecture is true for genus two imbeddings, *Topology* **23** (1984), 211–218.

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