

A FAKE 4-MANIFOLD

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In this paper we study 4-dimensional fake manifolds; mainly the fake $\mathbb{R}P^4$ which was constructed by Cappell and Shaneson [CS]. This is a smooth closed manifold Q^4 which is simple homotopy equivalent to $\mathbb{R}P^4$ but not diffeomorphic to $\mathbb{R}P^4$. From Q^4 we construct a new 4-manifold:

THEOREM 1: There exists a closed smooth manifold M^4 which is simple homotopy equivalent to $S^3 \tilde{\times} S^1 \# S^2 \times S^2$ but not diffeomorphic to it.

Here $S^3 \tilde{\times} S^1$ denotes the twisted S^3 bundle over S^1 . Figure 4.6 is a handlebody of M^4 . This handlebody is surprisingly simple, namely: $M^4 = B^3 \tilde{\times} S^1 \cup$ (two 2-handles) $\cup B^3 \tilde{\times} S^1$. From this, it easily follows that if

$M_0^4 = M^4 - \text{int}(B^3 \tilde{\times} S^1)$ then M_0^4 is a fake $B^3 \tilde{\times} S^1 \# S^2 \times S^2$ and furthermore:

COROLLARY: $M_0^4 \times I \approx (B^3 \tilde{\times} S^1 \# S^2 \times S^2) \times I$

where $\approx$ denotes a diffeomorphism. Along the way we prove that Q^4 is stably trivial.

THEOREM 2: $Q^4 \not\approx \mathbb{C}P^2 \approx \mathbb{R}P^4 \not\approx \mathbb{C}P^2$

This is interesting because the connected sum of Q^4 with arbitrarily many copies of $S^2 \times S^2$ is not diffeomorphic to the connected sum of $\mathbb{R}P^4$ with arbitrarily many copies of $S^2 \times S^2$ [CS].

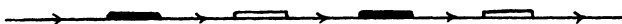
In Section 2 we prove a structure theorem for Q^4 similar to the 2-fold cover of Q^4 [AK₄], namely we demonstrate a properly imbedded 2-disk $\Delta^2 \subset D^2 \times \mathbb{R}P^2$ (in fact a ribbon disk) with $\partial \Delta^2 = S^1 \times (\text{a point})$ such that Q^4 is obtained by twisting $D^2 \times \mathbb{R}P^2$ along Δ^2 (Gluck construction) and taking a union with $B^3 \tilde{\times} S^1$. Figure 2.11 is the picture of Δ^2 . From this we obtain a solution to a problem of Cappell and Shaneson ([K₁], problem 4.14-B); namely removing the tubular neighborhood of the nontrivial circle in $D^2 \times \mathbb{R}P^2$ and replacing with a certain $(T^3 - B^3)$ -bundle over S^1 does not yield a fake $D^2 \times \mathbb{R}P^2$ but it gives a fake self homotopy equivalence of $D^2 \times \mathbb{R}P^2$. In [AK₃] the structure of the 2-fold cover $\tilde{Q}$ of Q was studied and it was shown that $\tilde{Q}$ is an invertible homotopy sphere (in particular it is homeomorphic to S^4), and $\tilde{Q}$ is obtained from S^4 by removing a tubular neighborhood of a knotted S^2 and sewing it back (Gluck construction). Therefore comparing this paper to

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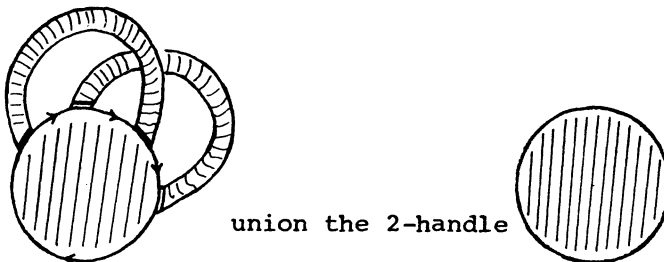
$[AK_3]$ and $[AK_4]$ at times could be useful. We would like to thank Larry Taylor for many helpful discussions on 4-manifold surgery. We also want to thank R. Kirby for a happy collaboration in $[AK_3]$ and $[AK_4]$ which led to this paper.

0. PRELIMINARIES

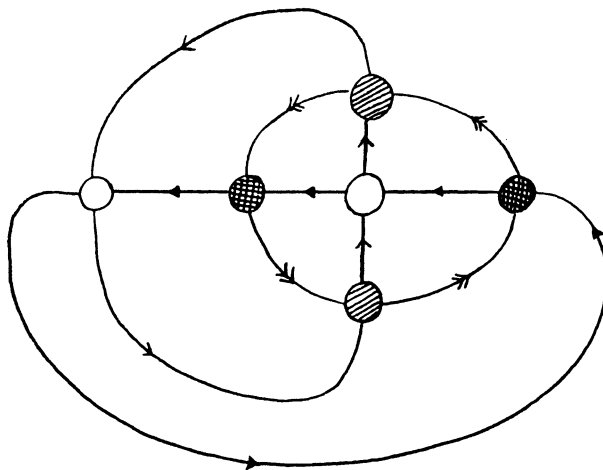
Throughout the paper we use $\approx$ to denote a diffeomorphism. In this section we discuss handlebodies of 4-manifolds. This presentation is similar to that of $[AK_1]$ and $[AK_2]$, except here 4-manifolds can be nonorientable. Recall that we can present any 2-manifold as a line (a local view of the boundary of the 0-handle) along with the attaching arcs of 1-handles and attaching circles of 2-handles. For example T^2 is



which is a shorthand for:



Similarly any 3-manifold can be represented by a plane (a local view of the boundary of the 0-handle) along with attaching discs of 1-handles and attaching circles of 2-handles and a 3-handle. This corresponds to the Heegaard presentation. For example the punctured 3-torus is:



For a given 4-manifold M^4 we draw the handlebody picture of M^4 in the similar way. Namely we will view M^4 from the boundary of the 0-handle ($=S^3$) and draw the attaching balls of the 1-handles and the attaching circles of the 2-handles in S^3 . We will not indicate three and four handles in our pictures.

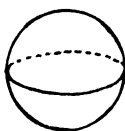
A pair of balls indicate an attaching $S^0 \times B^3$ of an oriented 1-handle. If we imagine coordinate axes in the centers of these balls the 1-handle identifies the boundaries of these balls by the map $(x, y, z) \rightsquigarrow (x, -y, z)$



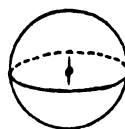
This is well defined because the axis, which is reflected, is the axis given by connecting the centers of these balls. In case of orientation reversing handles we put an arc on the centers of these balls which indicates the identification $(x, y, z) \rightsquigarrow (x, -y, -z)$



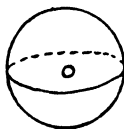
These arcs indicate the normal direction to the plane where the reflection is performed to the oriented 1-handle to get this handle. We can put one of the balls B_-^3 of the 1-handle at the point of ∞ , in which case we just draw the other ball B_+^3



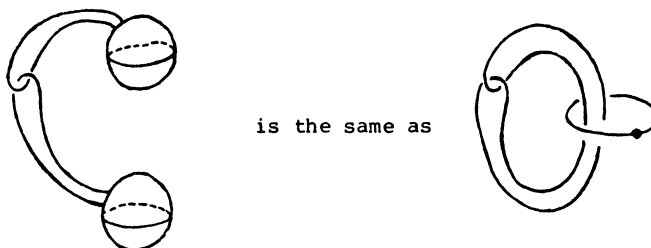
In the case of oriented 1-handle the boundary of B_+^3 is identified with the boundary of B_-^3 by identity (i.e. the radial map taking ∂B_+^3 to ∂B_-^3). In the case of nonoriented 1-handle we either draw B_+^3 as



which means ∂B_+^3 is first reflected across the plane perpendicular to the arc then identified with ∂B_-^3 by identity; or we draw:

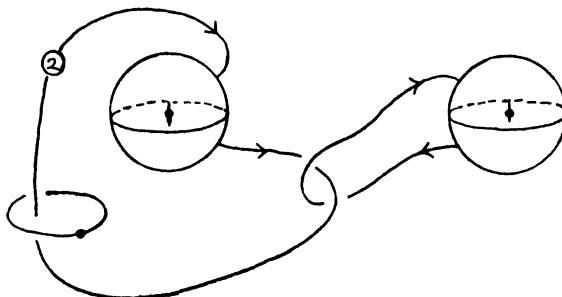


which means we first perform the antipodal map to ∂B_+^3 before identifying with ∂B_-^3 by identity. We also denote an oriented 1-handle by an unknotted circle with a dot on it (see [A] and also [AK₁]). The dotted circle means that we delete the thickened unknotted disc the unknot bounds in B^4 obtaining $S^1 \times B^3$. In other words anything that goes through the dotted circle is going over the 1-handle.

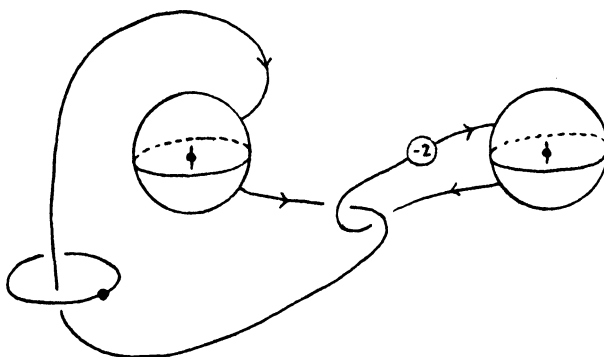


Replacing dot by a zero on the dotted circle corresponds surgering $S^1 \times B^3$ to $S^2 \times B^2$; and the vice-versa. We also use dotted ribbon knots which means that we delete the thickened ribbon disc from B^4 (also see [AK₂]). Since a ribbon knot may not bound a unique ribbon disc in B^4 we shade the particular ribbon to indicate the deleted ribbon disc.

If we don't specify the framing on the attaching knot of a two handle, it is the one coming from the normal vector field on the surface of the paper. If we put integers on the knot such as $\dots \text{---} \textcircled{n} \text{---} \dots$ it means that we add n -full twist to the above framing. This makes the framings well defined even in the presence of orientation reversing 1-handles. For example

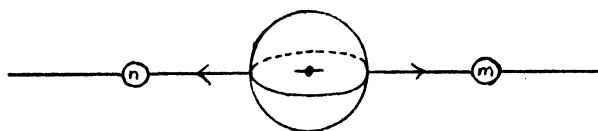


is the same as:



Because 2 twist becomes -2 twist having gone across the orientation reversing 1-handle.

Here is an example of a 4-manifold $M^4(n,m)$:



First of all by rotation of the ball B_+^3 360° around the y-axis we get $M(n,m) \approx M(n+1,m-1)$, and by transferring twists across the 1-handle we get $M(n,m) \approx M(n-m,0)$.

$$\text{Hence } M(n,m) \approx \begin{cases} M(0,0) & \text{if } n+m \text{ even} \\ M(1,0) & \text{if } n+m \text{ odd} \end{cases}$$

$M^4(0,0)$ is just $D^2 \times \mathbb{R}P^2$ because it is the 4-dimensional trivial thickening of the handlebody of $\mathbb{R}P^2$ which is

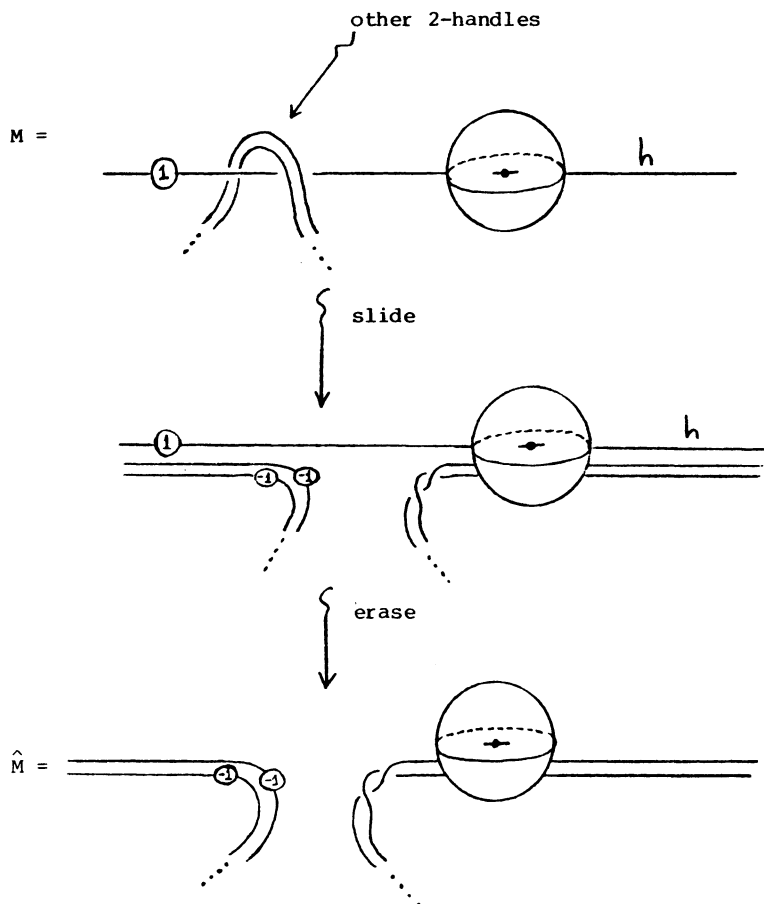


(the other attaching arc of the 1-handle is at ∞). Hence $M^4(1,0)$ is $D^2 \tilde{\times} \mathbb{R}P^2$ (the nontrivial D^2 -bundle over $\mathbb{R}P^2$) which is the nontrivial thickening of the handlebody of $\mathbb{R}P^2$.

Recall $\mathbb{R}P^4 = D^2 \tilde{\times} \mathbb{R}P^2 \cup B^3 \tilde{\times} S^1$ hence $\partial(D^2 \tilde{\times} \mathbb{R}P^2) \approx \partial(B^3 \tilde{\times} S^1)$. For a given 4-manifold M^4 containing $D^2 \tilde{\times} \mathbb{R}P^2$ we call the operation:

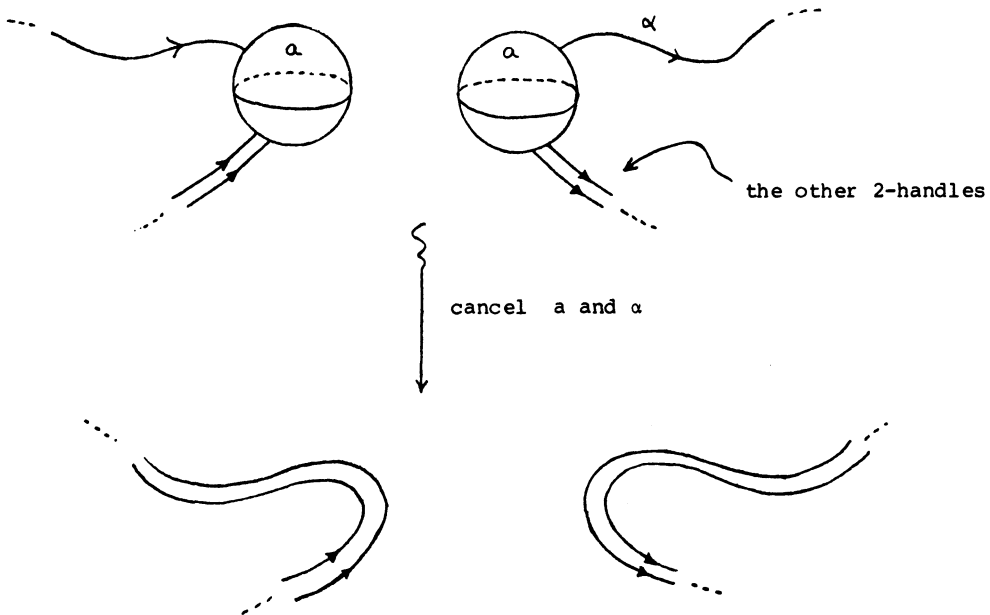
$$M^4 \rightsquigarrow \hat{M}^4 = (M - D^2 \tilde{\times} \mathbb{R}P^2) \cup_{\partial} B^3 \tilde{\times} S^1$$

blowing down the $\mathbb{R}P^2$. This is similar to the "blowing down $\mathbb{C}P^2$ " operation of $[K_2]$. In practice we perform this operation as follows: We slide the attaching circles of the other 2-handles over the 2-handle h of $D^2 \tilde{\times} \mathbb{R}P^2$ until they don't link h anymore and then we simply erase the two handle of $D^2 \tilde{\times} \mathbb{R}P^2$ as in the following figure:



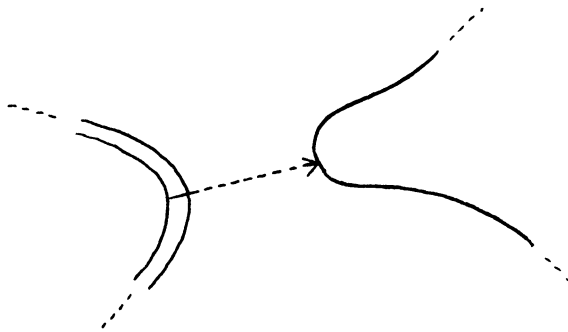
We leave the verification, that this process corresponds to the blowing down operation, as an exercise to the reader. We call the inverse of this operation blowing up an $\mathbb{R}P^2$.

If in a given 4-manifold an attaching circle of a two handle goes through an oriented 1-handle geometrically once we can cancel this pair of 1 and 2 handles by simply erasing them from the picture. The attaching circles of other two handles which go through the 1-handle has to be modified as follows (see $[AK_3]$)

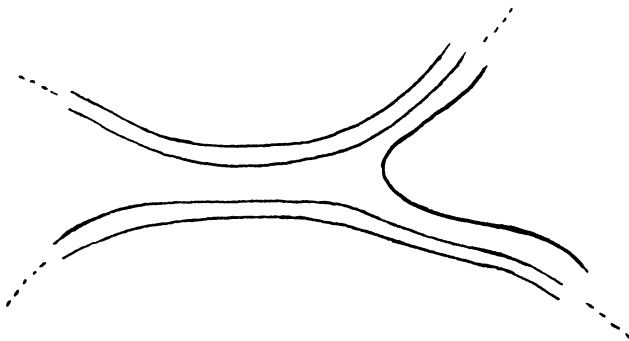


Also if we have a trivial 2-handle O^0 and a 3-handle attached onto this in the obvious way (i.e. along $S^2 \subset S^2 \times S^1 = \partial(O^0)$) we can cancel them by simply erasing O^0 and forgetting the 3-handle.

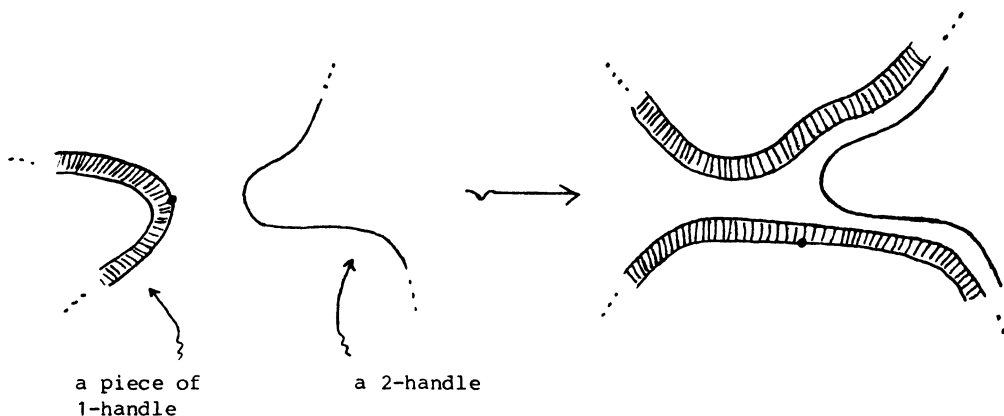
In our figures we use arrows such as



we ignored them, unless it indicated that we do a handle slide as shown by the arrow in which case it means slide two handles over each other i.e.

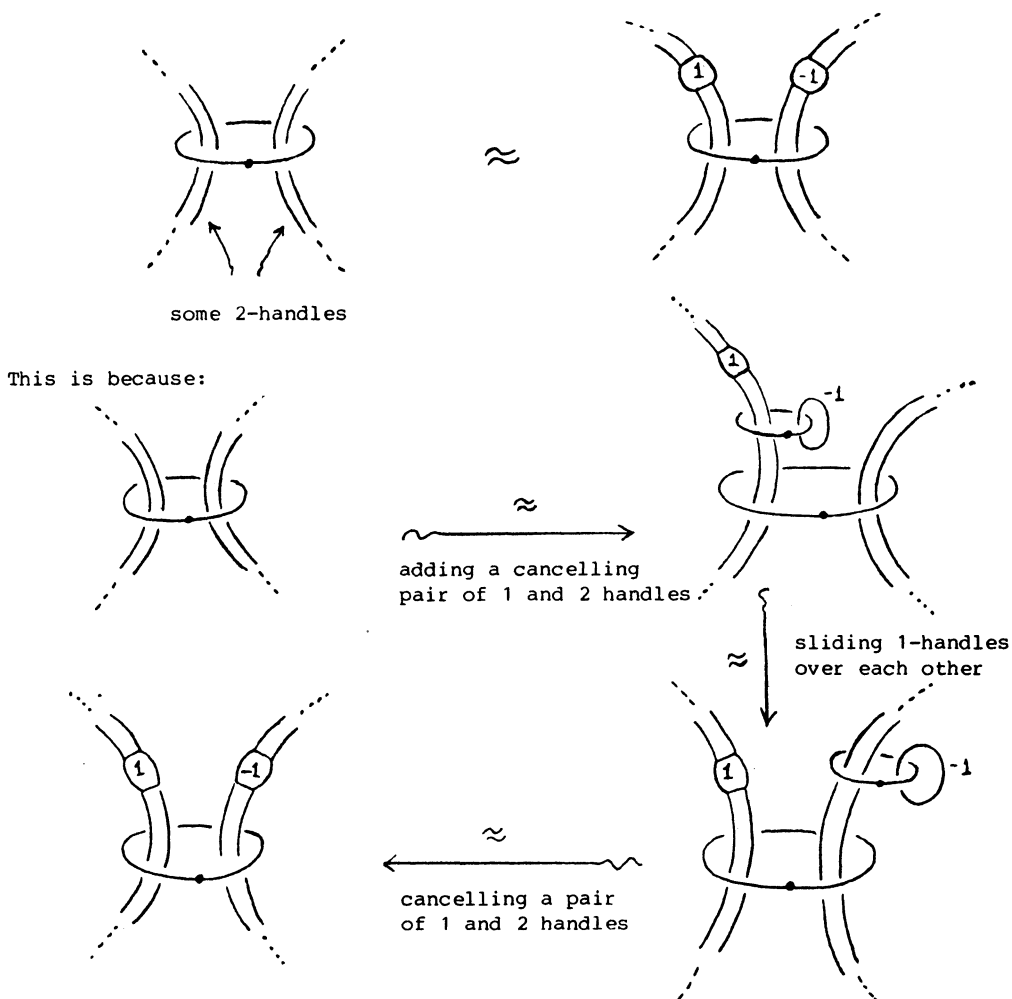


Sometimes 1-handles can be slid over 2-handles such as:



This is easily checked by reflecting on the definition of a dotted circle (B^4 minus a thickened disc this circle bounds).

Finally we will use (in Section 4) the following diffeomorphism



1. STRUCTURE OF Q^4

Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ then A induces an orientation reversing linear map

$A: \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Since A preserves the integral lattice in $\mathbb{R}^3$ it induces a self diffeomorphism of T^3 where $T^3 = \mathbb{R}^3/\mathbb{Z}^3$. Since $A(0) = 0$, A in fact induces a diffeomorphism $\tilde{A}: T_0^3 \rightarrow T_0^3$ where $T_0^3 = T^3 - \text{interior}(D^3)$ and D^3 is an imbedded disc in T^3 . Then since $\tilde{A}|_{\partial T_0^3}$ is an orientation reversing diffeomorphism of S^2 after a small isotopy we can assume that $\tilde{A}|_{\partial T_0^3}$ is the antipodal map of S^2 . Let $C = T_0^3 \times I / (x, 0) \sim (\tilde{A}(x), 1)$ be the mapping torus of $\tilde{A}$. Then clearly ∂C is the twisted S^2 bundle over S^1 which we denote by $S^2 \tilde{\times} S^1$. Let $D^2 \tilde{\times} \mathbb{R}P^2$ be the twisted D^2 bundle over $\mathbb{R}P^2$ (it is the tubular neighborhood of $\mathbb{R}P^2$ in $\mathbb{R}P^4$). Since $\partial(D^2 \tilde{\times} \mathbb{R}P^2) = S^2 \tilde{\times} S^1$ we can construct: $Q^4 = C \cup_{\partial} (D^2 \tilde{\times} \mathbb{R}P^2)$. This is the Cappell and Shaneson's construction of a fake $\mathbb{R}P^4$ [CS]. We draw a handlebody picture of Q^4 by the method of [AK₃]: Figure 1.1 is the handlebody picture of $T_0^3 \times I$. We isotop $\tilde{A}$ so that

- (i) $\tilde{A}$ is the antipodal map on the small ball centered at the origin.
- (ii) $\tilde{A}$ takes the 1-handles of T_0^3 to itself.

Figure 1.2 indicates this isotopy. The first picture in this figure is the images of the coordinates axis under $\tilde{A}$. Since the opposite faces of the cube is identified the coordinate axes are the cores of the 1-handles. So the isotopy moves the end points of the arcs to the centers of the sides of the cube (hence into the 1-handles).

So the handlebody of C is obtained from the handlebody of $T_0^3 \times I$ by identifying with $\tilde{A}$. This identification adds a $k+1$ handle to $T_0^3 \times I$ for every k handle of $T_0^3 \times I$. Hence we add one 1-handle three 2-handles and three 3-handles to get C . Figure 1.3 is the handlebody picture of C except the three handles are not drawn even though they are there. The new 1-handle is a nonoriented 1-handle (because of (i)) attached along the ball at the origin (as indicated in the figure) and the ball at ∞ (hence not seen in the figure).

Figure 1.4 is the same as Figure 1.3 except the two handles α_2, α_3 are not drawn, and the 1-handle a_1 is cancelled by the 2-handles which goes over a_1 once. We get Figure 1.5 by cancelling the 1-handle a_3 by the 2-handle which goes over a_3 once. Figure 1.6 is the same as Figure 1.5 except the 1-handle a_2 is indicated as a dotted circle. By further isotopies we get Figures 1.7 and 1.8. By rotating the B^3 at ∞ (where the one end of the 1-handle attached) by 180 degrees we get Figure 1.9. Hence the notation on the 1-handle in Figure 1.9 is changed. We claim that the boundary of the manifold in Figure 1.9 is $S^2 \tilde{\times} S^1 \neq S^2 \times S^1$. To see this surger the 1-handle (i.e. replace the dot with a zero), and then surger the two handle (i.e. put a dot on the attaching circle of this handle) as in Figure 1.10. A further isotopy gives

Figure 1.11. If we now cancel the new 1-handle with the obvious 2-handle (the one corresponding to the circle going through the 1-handle once) we get Figure 1.12. The boundary is obviously $S^2 \tilde{\times} S^1 \neq S^1 \times S^2$.

Now here comes an important point! Recall starting with Figure 1.4 we ignored to draw the 2-handles α_2, α_3 . If we draw these handles and carry them along the processes of Figure 1.4 through Figure 1.12 α_2, α_3 will end up being two unknotted circles in Figure 1.12 (check). This means that α_2, α_3 are attached to trivial circles on the boundary of the Figure 1.4 and therefore two of the three 3-handles of Figure 1.3 must be cancelling the 2-handles α_2, α_3 . In other words we are justified in ignoring α_2, α_3 from the picture along with two 3-handles. So Figure 1.9 along with one three handle is the picture of C^4 .

Q^4 is obtained by gluing $D^2 \tilde{\times} \mathbb{RP}^2$ to C^4 . Hence to get Q^4 we must add a 2-handle, a 3-handle and a 4-handle (upside down $D^2 \tilde{\times} \mathbb{RP}^2$) to C^4 along ∂C^4 . Since we don't draw 3 and 4-handles we only indicate the attaching circle of the 2-handle γ . γ is attached along the standard circle which goes twice around $S^2 \tilde{\times} S^1 = \partial C$ (see Section 0). In fact γ is attached as in Figure 1.13. To check this we apply the diffeomorphism $\partial(C^4) \approx S^2 \tilde{\times} S^1$ of Figures 1.9-1.12 to Figure 1.13; and we see that this diffeomorphism takes γ to the 'right' circle in Figure 1.12. The framing on γ is any odd number; so we assign +1 framing as indicated in the figure. Hence Figure 1.13 along with two 3-handles and a 4-handle is the handlebody of Q^4 .

By doing the indicated handle slides to Figures 1.13 and 1.14 we get Figures 1.14 and 1.15 respectively. Notice the 2-handle δ in Figure 1.15 goes over the 1-handle a_2 geometrically once (after an isotopy), hence it cancels it. After this cancellation we will have one 1-handle, two 2-handles (α_1 and γ) along with two 3-handles and a four handle left. We want to turn this handlebody upside down; i.e. we want to draw it as two 1-handles, two 2-handles and one 3-handle and a 4-handle. To do this we draw the dual 2-handles σ and τ , then we change the interior of the handlebody to $B^3 \tilde{\times} S^1 \neq B^3 \times S^1$ via surgeries and handle slides, while carrying σ and τ . Then $B^3 \tilde{\times} S^1 \neq B^3 \times S^1$ and the 2-handles σ, τ (and a three and a four handle) will be what we want.

To do this we go back to Figure 1.14 carrying along σ and τ . We then replace the dots on the handles α_1 and a_2 (i.e. surgery) and by an isotopy we get Figure 1.16 (this is similar to going from Figure 1.9 to Figure 1.11). After performing the obvious handle cancellation as in Figure 1.17 we arrive at Figure 1.18. Then by doing the indicated handle slides to this figure we get Figure 1.19. If we ignore σ, τ Figure 1.19 becomes just $D^2 \tilde{\times} \mathbb{RP}^2 \neq S^2 \times D^2$. Hence in order to change the interior to $B^3 \tilde{\times} S^1 \neq B^3 \times S^1$ we have to

- (1) Surger the imbedded S^2
- (2) Blow down the $\mathbb{R}P^2$

We surger S^2 by putting dot on the unknotted 2-handle as in Figure 1.19. We will blow down $\mathbb{R}P^2$ a little later (Figure 1.23 and Figure 1.24). By sliding σ over τ twice in the obvious way we get Figure 1.20. We continue to call the slid 2-handle by σ . By isotopies we get to Figures 1.21, 1.22. By a further isotopy (this time pulling the 1-handle around) we get Figure 1.23. By blowing down $\mathbb{R}P^2$ (i.e. γ) in Figure 1.23 we get Figure 1.24. After performing the indicated handle slides and pulling the 1-handle to the standard position we get Figure 1.25. After isotoping the ball at ∞ into the picture we get Figure 1.26 which is $Q^4 - \text{int}(B^3 \tilde{\times} S^1)$.

Figures 1.27 through 1.33 give even a simpler handlebody for $Q^4 - \text{int}(B^3 \tilde{\times} S^1)$. We go to Figure 1.27 from Figure 1.24 by an isotopy, then by the indicated handle slides we get Figure 1.28 and then Figure 1.29. After isotopies and the indicated handle slides we get Figures 1.30 through 1.33. Figure 1.33 is $Q^4 - \text{int}(B^3 \tilde{\times} S^1)$.

2. THE RIBBON IN $D^2 \times \mathbb{R}P^2$

Let $N^4 = Q^4 - \text{int}(B^3 \tilde{\times} S^1)$; N^4 is a fake $D^2 \tilde{\times} \mathbb{R}P^2$. Figure 1.14 with one 3-handle is the handlebody of N^4 . This is because Figure 1.14 along with two 3-handles and a 4-handle is Q^4 . Figure 2.1 is N^4 . This is because if we cancel b with $\S$ we get Figure 1.14 back.

Let $V^4 = N^4$ minus the handles $b \cup \S$, then $V^4 = C^4 \cup$ the 2-handle γ attached by 0-framing, but $D^2 \times \mathbb{R}P^2 = B^3 \tilde{\times} S^1 \cup$ the 2-handle γ attached by 0-framing (Section 0). Hence V^4 is obtained from $D^2 \times \mathbb{R}P^2$ by replacing a tubular neighborhood of the orientation reversing circle with C^4 . We will show that V^4 is diffeomorphic to $D^2 \times \mathbb{R}P^2$. This answers a question of Cappell and Shaneson ([K₁], problem 4.14-B).

To get V^4 we ignore b and $\S$ from Figure 2.1 and add one 3-handle. Then we do a handle slide (as indicated in Figure 2.1) to get Figure 2.2. By another handle slide we get Figure 2.3. By cancelling the obvious pair of handles from Figure 2.3 we get Figure 2.4 which is $D^2 \times \mathbb{R}P^2 \# S^2 \times D^2$. The three handle cancels $S^2 \times D^2$ (check) and we end up with $D^2 \times \mathbb{R}P^2$. Hence we have shown that $V^4 = D^2 \times \mathbb{R}P^2$.

Now we go back to N^4 ; i.e. we add back the handles $b, \S$ to Figure 2.1. If we carry along the handles $b, \S$ during the diffeomorphism $V^4 \approx D^2 \times \mathbb{R}P^2$ (as in Figures 2.1-2.4) we get Figures 2.5-2.8. Along the way we slide the 1-handle b over a 2-handle as indicated in Figure 2.5. By isotoping the B^3 at ∞ into the picture we get Figure 2.9. After a handle slide and an isotopy we get Figures 2.10 and 2.11. In Figure 2.11 the shaded ribbon disc is the ribbon 1-handle which $D^2 \times \mathbb{R}P^2$ is twisted along to get N^4 . Reader can verify that the 2-fold cover of Figure 2.11 gives the ribbon 2-sphere in S^4 which is

discussed in $[AK_4]$.

3. $Q^4 \# \mathbb{CP}^2 \approx \mathbb{RP}^4 \# \mathbb{CP}^2$

Recall Figure 1.19 after blowing down $\mathbb{RP}^2$ (i.e. γ) gives $N^4 = Q^4 - B^3 \tilde{\times} S^1$. Because in Section 1 we have seen that the blown down Figure 1.19 along with a 3-handle and a 4-handle gives Q^4 . To prove $Q^4 \# \mathbb{CP}^2 \approx \mathbb{RP}^4 \# \mathbb{CP}^2$ it suffices to show that $N^4 \# \mathbb{CP}^2 \approx (D^2 \tilde{\times} \mathbb{RP}^2) \# \mathbb{CP}^2$.

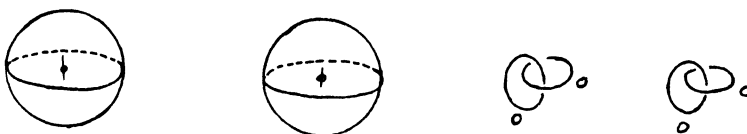
We claim the loop ρ in Figure 3.1 is the trivial loop on the boundary. This can be seen by going back to Figure 1.18 and sliding ρ over τ and then going back to Figure 1.15 and carrying ρ along. In Figure 1.15 ρ becomes the trivial dual circle to δ . Since σ and τ have zero framings we turn them into 1-handles; they then cancel α_1 and γ . After cancelling δ with α_2 ρ becomes an unknot in $\partial(B^3 \tilde{\times} S^1)$.

Hence if we add a 2-handle to Figure 3.1 along ρ with $+1$ framing it corresponds connected summing with $\mathbb{CP}^2$. We do this; and then by sliding σ over ρ we get Figure 3.2. An isotopy gives Figure 3.3. By a handle slide we obtain Figure 3.4. After cancelling the obvious pair of one and two handles we get Figure 3.5. We slide $+1$ framed handle over the 0 -framed handle it becomes free. Then we blow down $\mathbb{RP}^2$ and obtain Figure 3.6 which is $(D^2 \tilde{\times} \mathbb{RP}^2) \# \mathbb{CP}^2$ we are done.

4. A FAKE $S^3 \tilde{\times} S^1 \# S^2 \times S^2$

Recall Figure 1.24 is $N^4 = Q^4 - \text{int}(B^3 \tilde{\times} S^1)$. By performing only one of the indicated handle slides (the arrow pointing up) to Figure 1.24 we get Figure 4.1. By a diffeomorphism (see end of Section 0) we get Figure 4.2. By surgering Figure 4.2 (i.e. removing the dot) and then blowing down the obvious $\mathbb{RP}^2$ we get Figure 4.3. By isotopies we get Figures 4.4 and 4.5. By isotoping the 1-handle we get Figure 4.6 which we call M_0^4 . Since the surgery (removing the dot) to Figure 4.2 is performed to a null homotopic loop, it corresponds to taking connected sum with $S^2 \times S^2$. Therefore $N^4 \# S^2 \times S^2 = (D^2 \tilde{\times} \mathbb{RP}^2) \cup M_0^4$. Since $Q^4 \# S^2 \times S^2$ is fake [CS] so is $N \# S^2 \times S^2$. This implies that M_0^4 has to be a fake $B^3 \tilde{\times} S^1 \# S^2 \times S^2$, since any self-diffeomorphism of $S^2 \tilde{\times} S^1$ extends to $B^3 \tilde{\times} S^1 \# S^2 \times S^2$.

$\partial(M_0 \times I)$ is the double of M_0^4 . This is standard because it is obtained from Figure 4.6 by attaching two trivial (dual) 2-handles with 0 -framings (i.e. an unknotted circle for each 2-handle which links it geometrically once). By sliding the 2-handles of M_0^4 over the new 2-handles we get



The fact that $M_0^4 \times I \approx (B^3 \times S^1 \# S^2 \times S^2) \times I$ follows from 5-dimensional surgery exact sequence:

$$L_6(\mathbb{Z}, -) \rightarrow \mathcal{P}(X \times I, \partial) \rightarrow [X \times I/\partial; G/PL] \rightarrow L_5(\mathbb{Z}, -)$$

0

where $x = B^3 \tilde{x} S^1 \neq S^2 \times S^2$. The first map is zero map (check) and $[X \times I, \partial; G/PL] = 0$ so $\mathcal{Q}(X \times I, \partial) = 0$ and the claim follows (see [W]).

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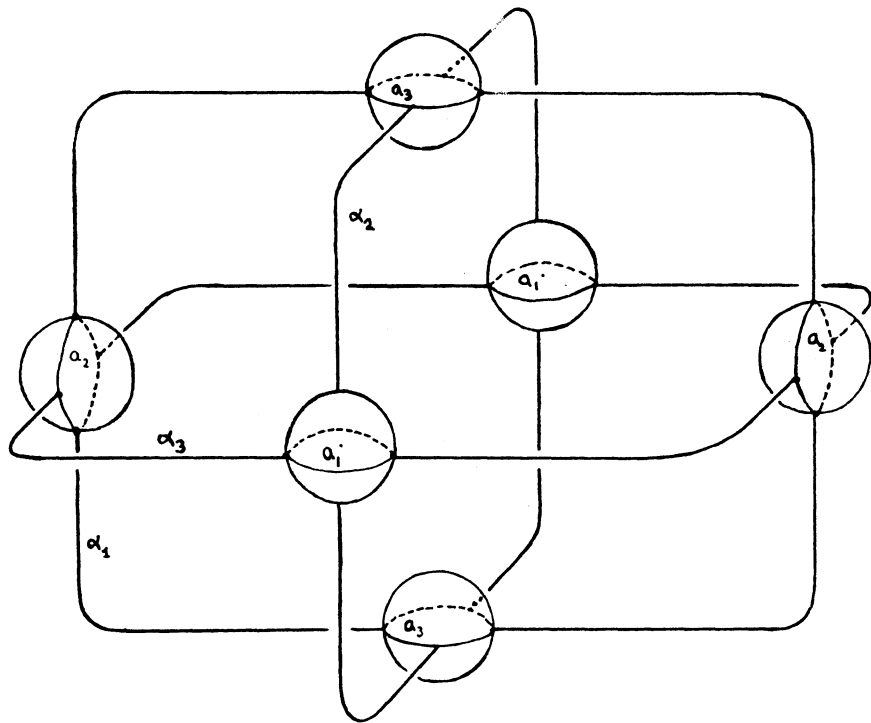


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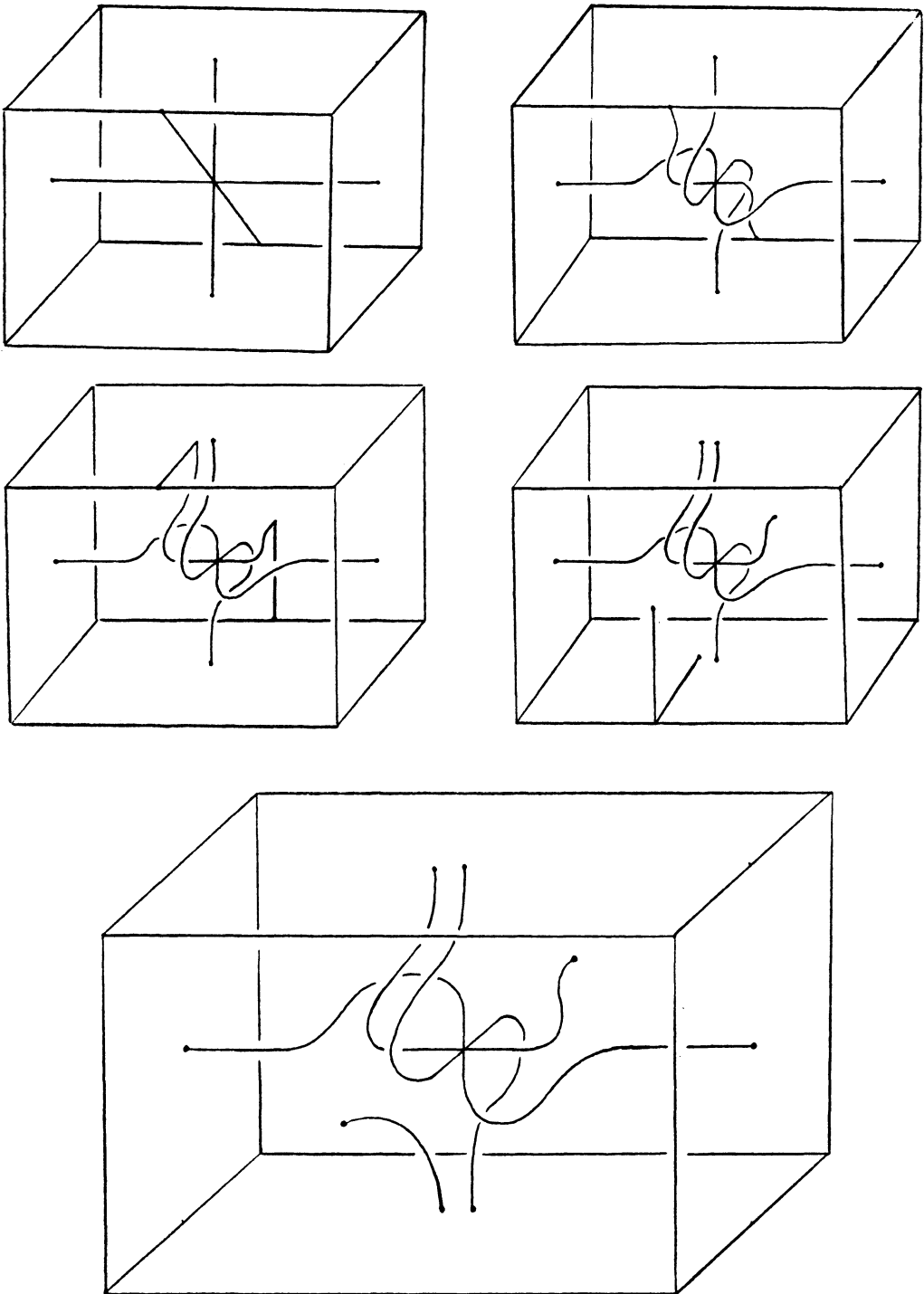


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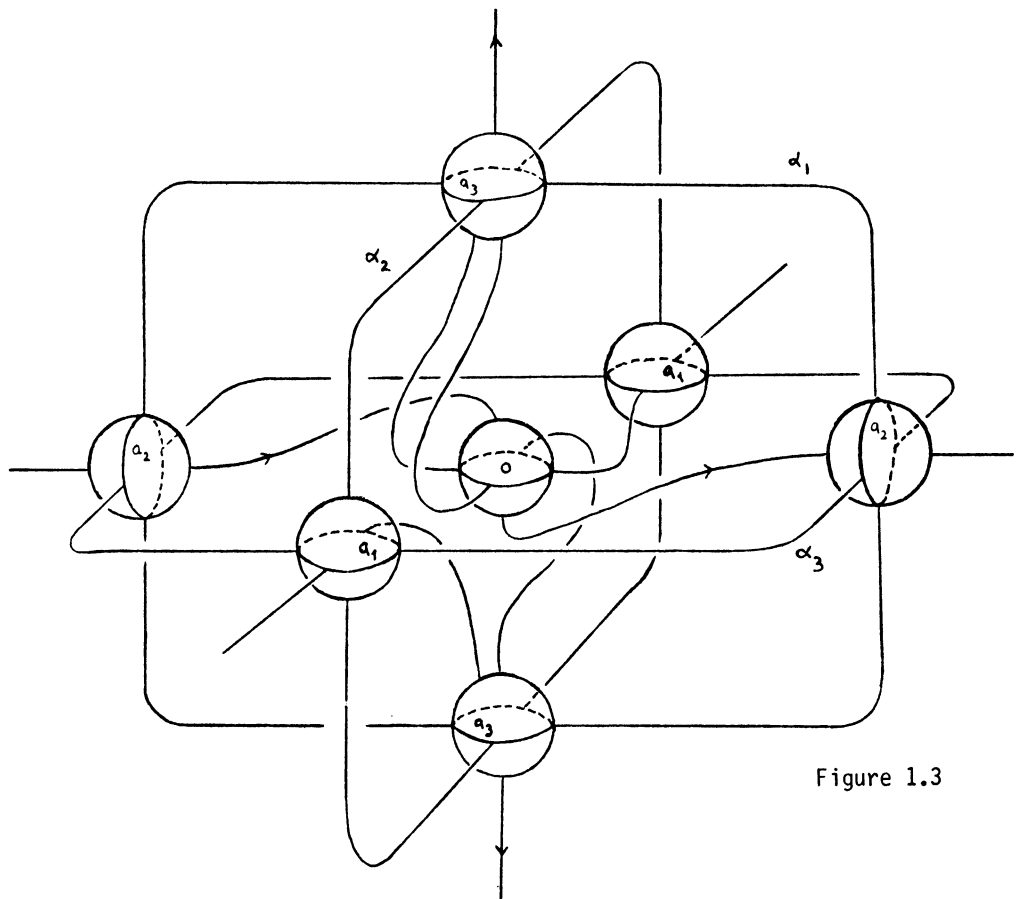


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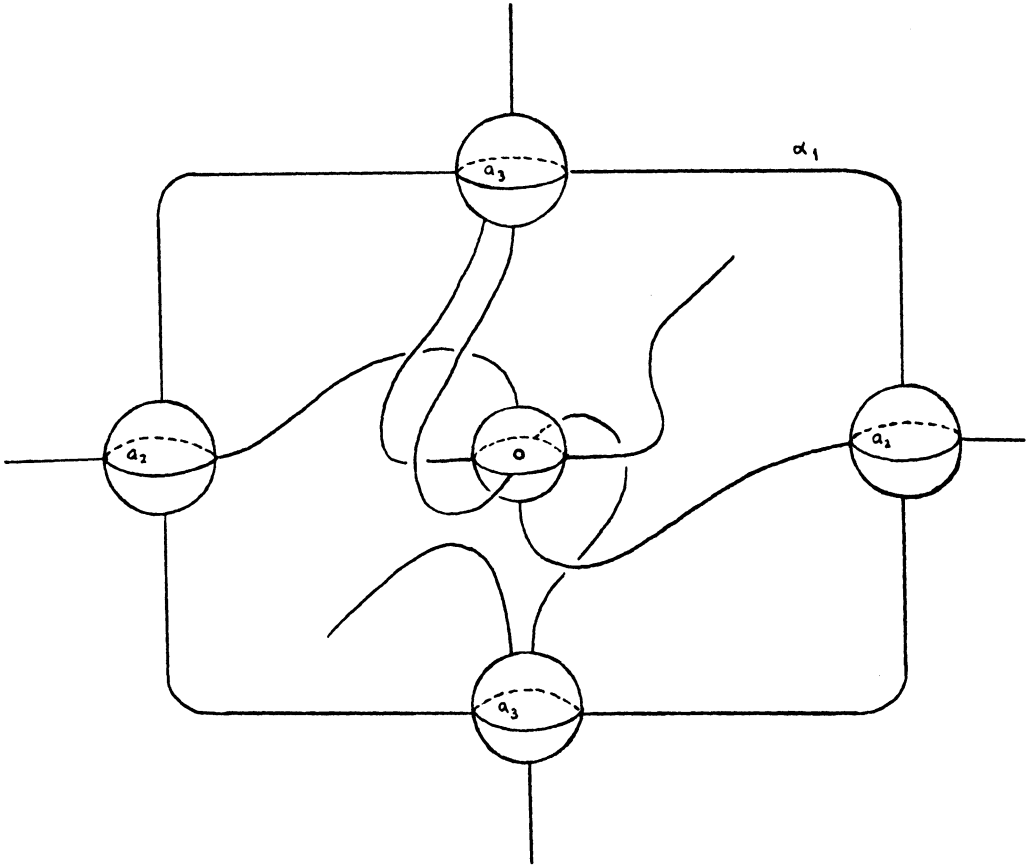


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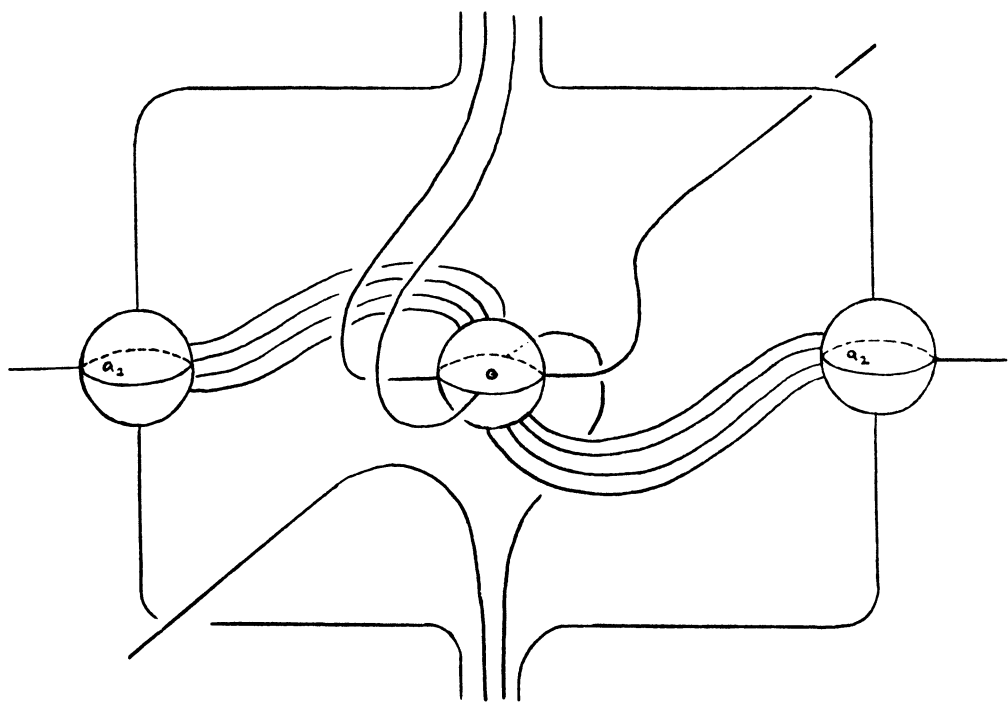


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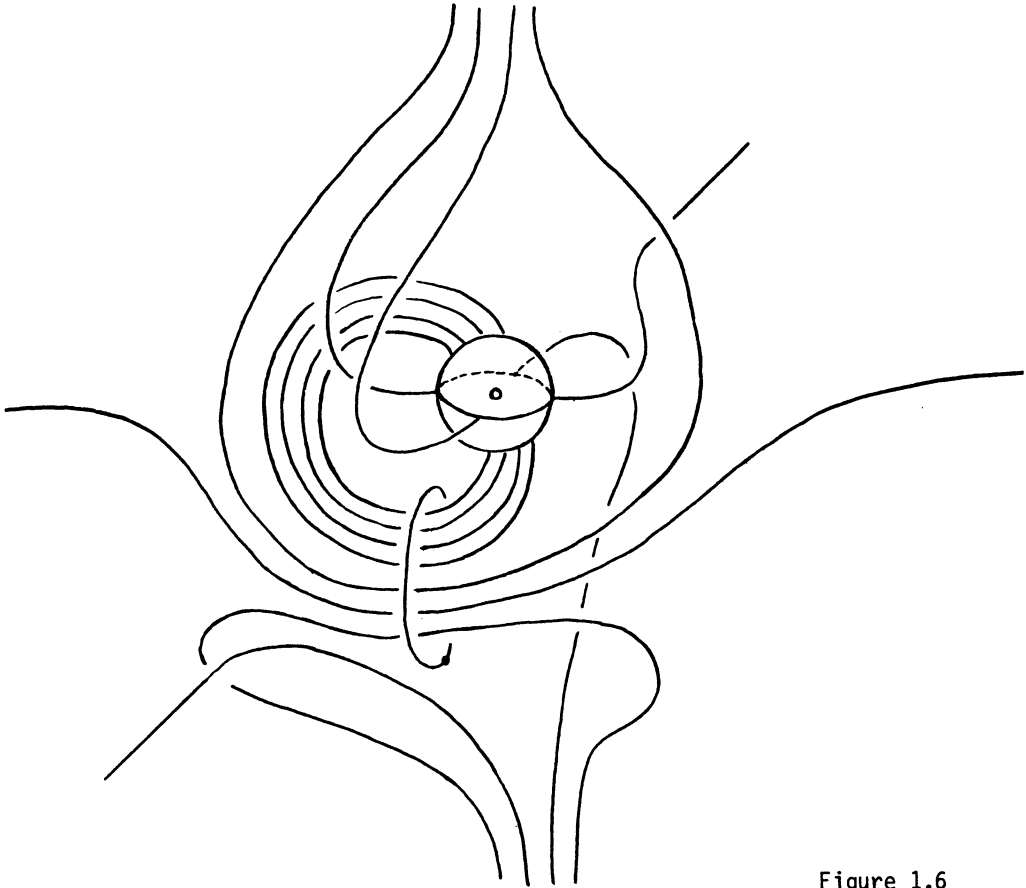


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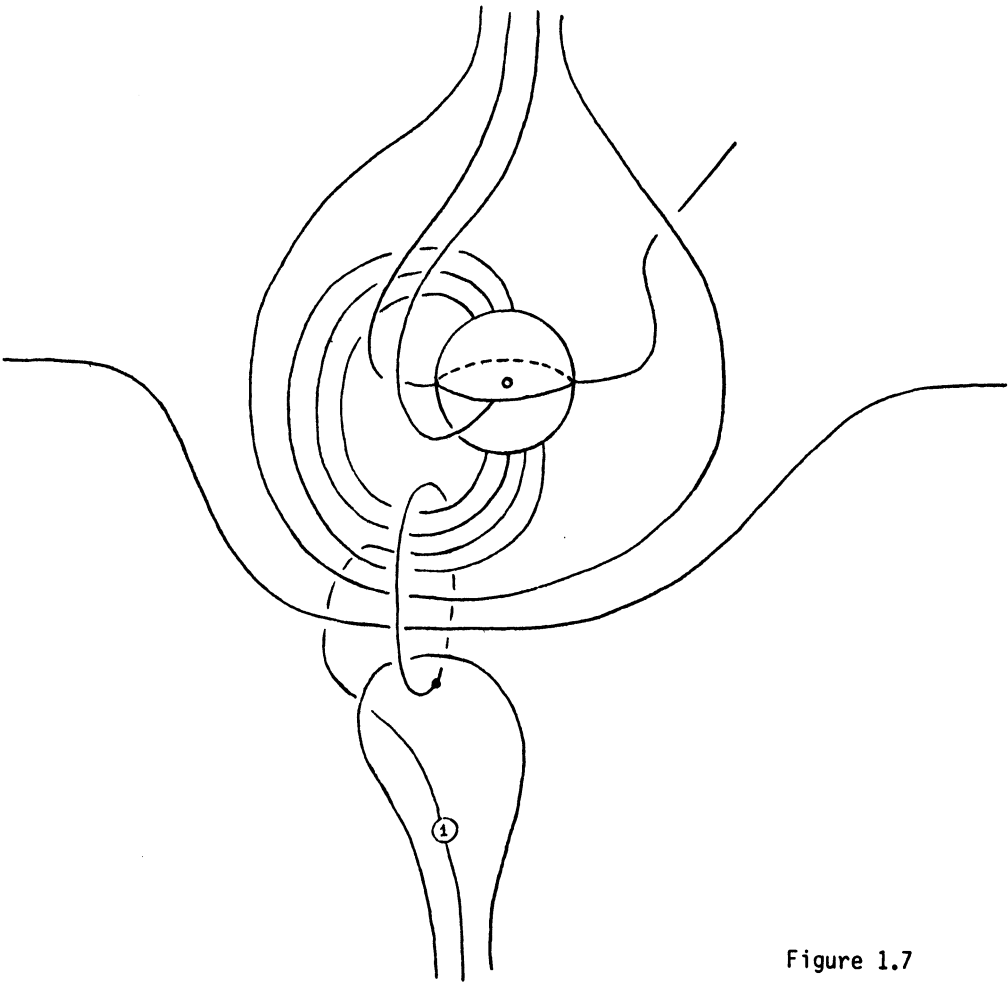


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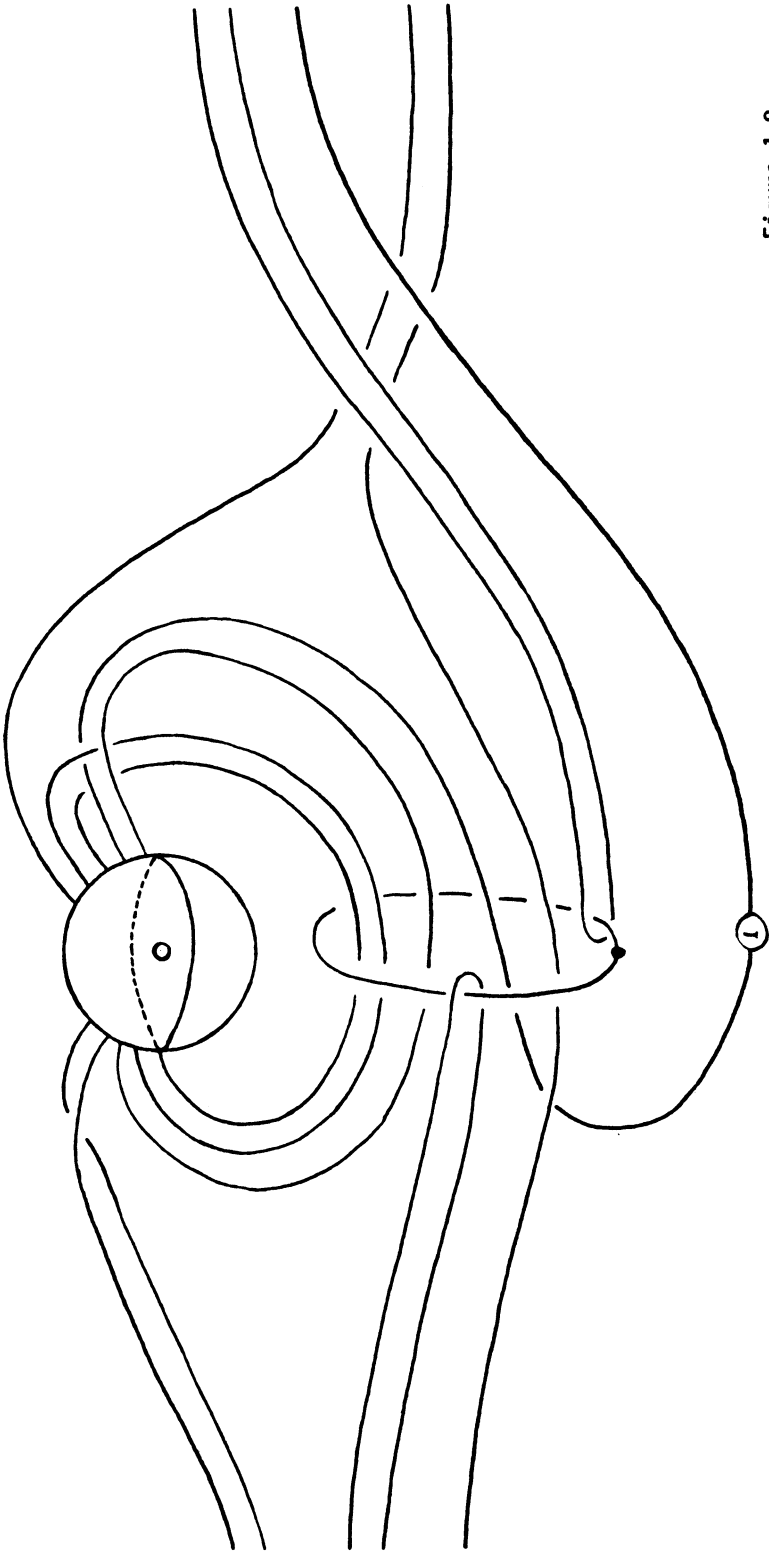


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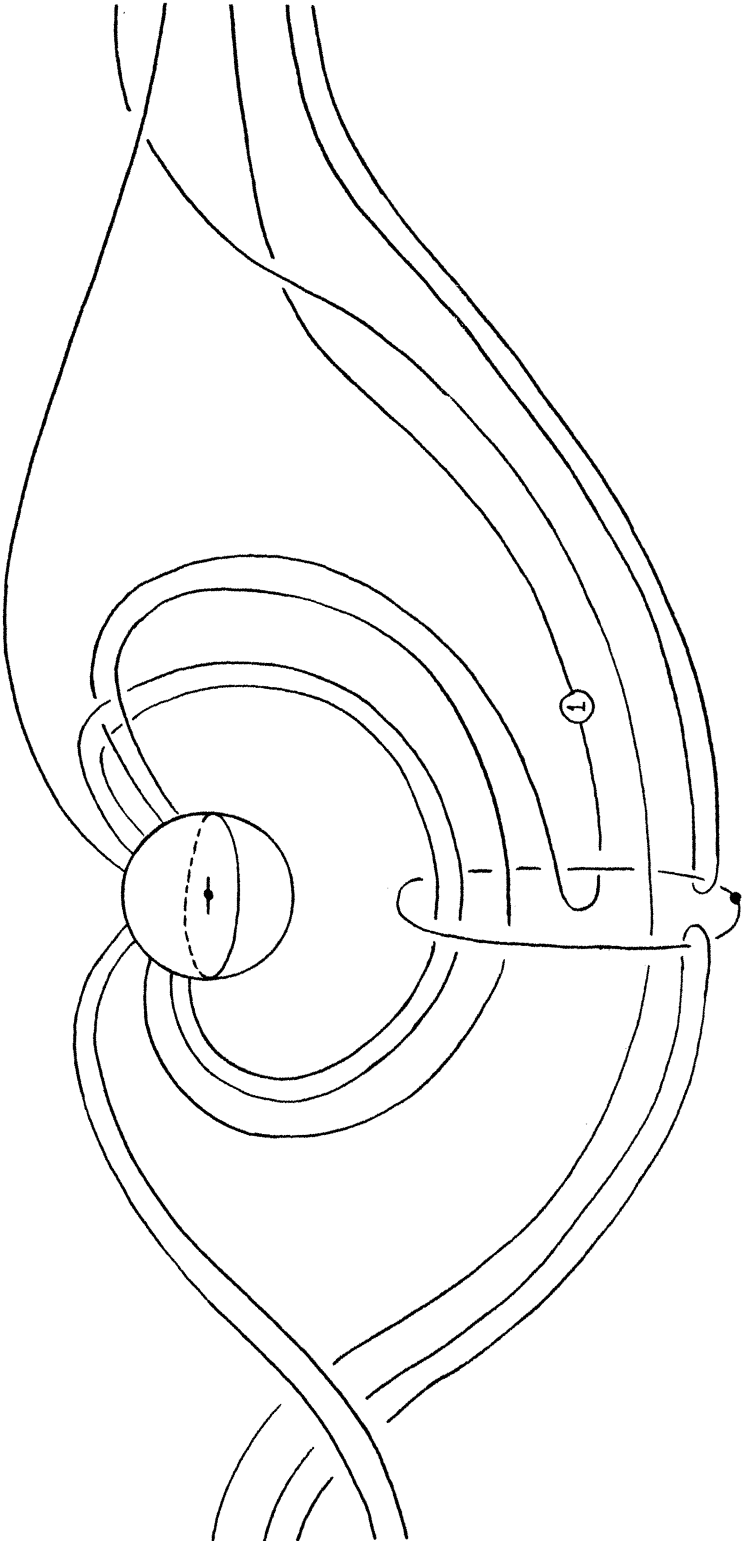


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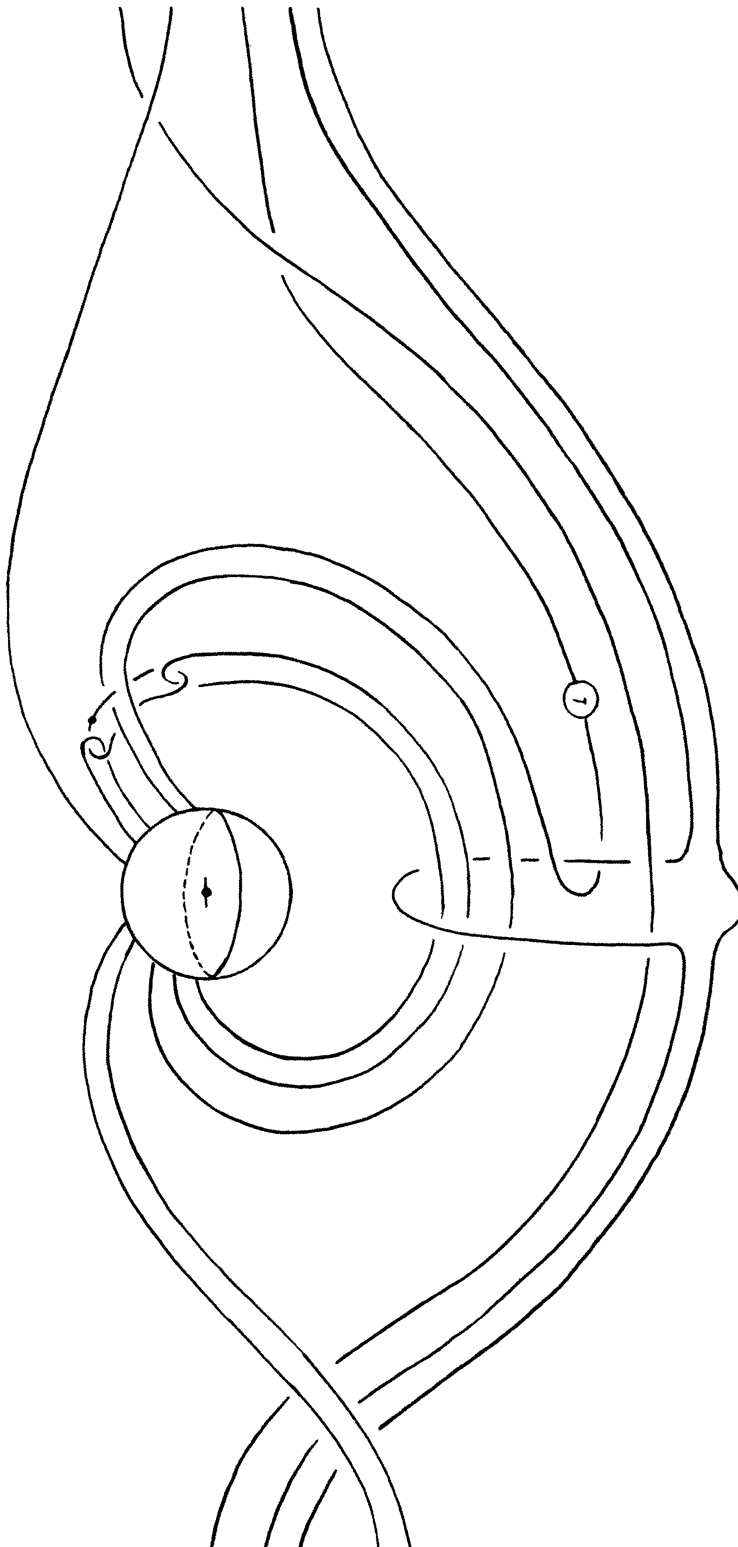


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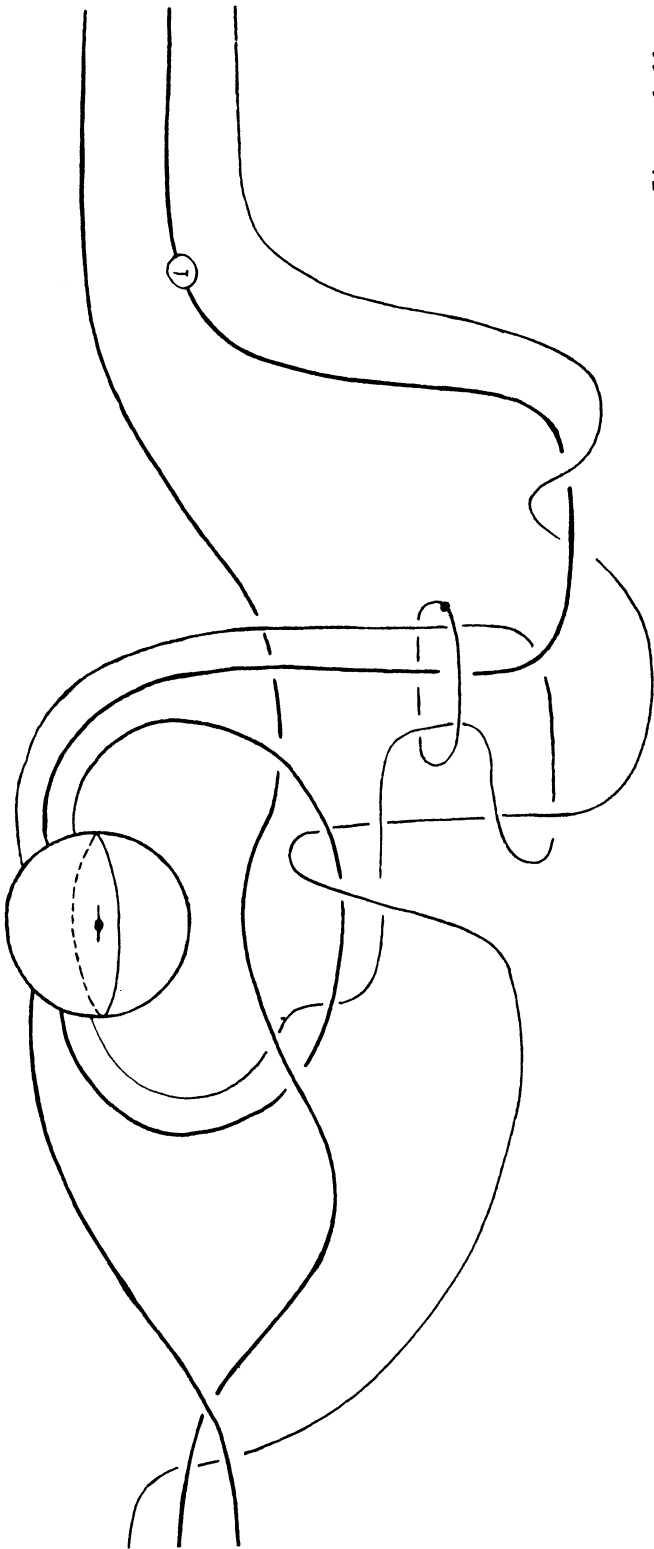


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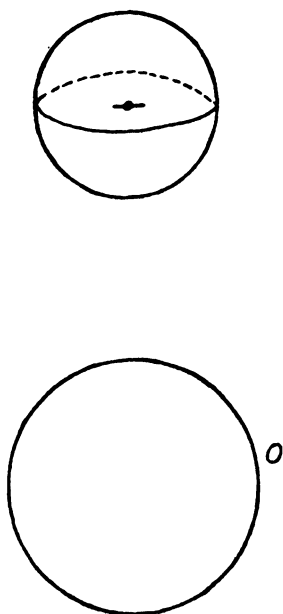


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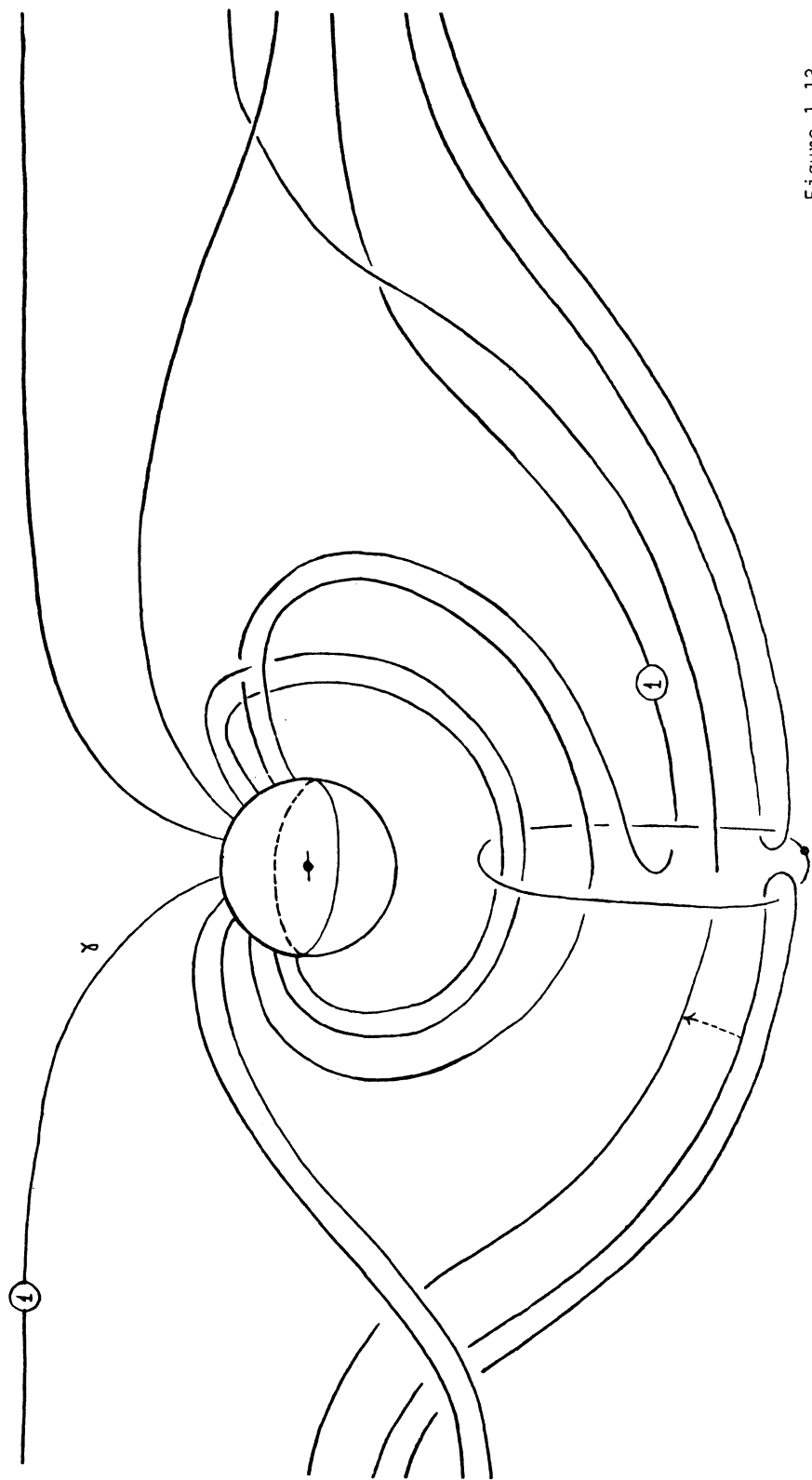


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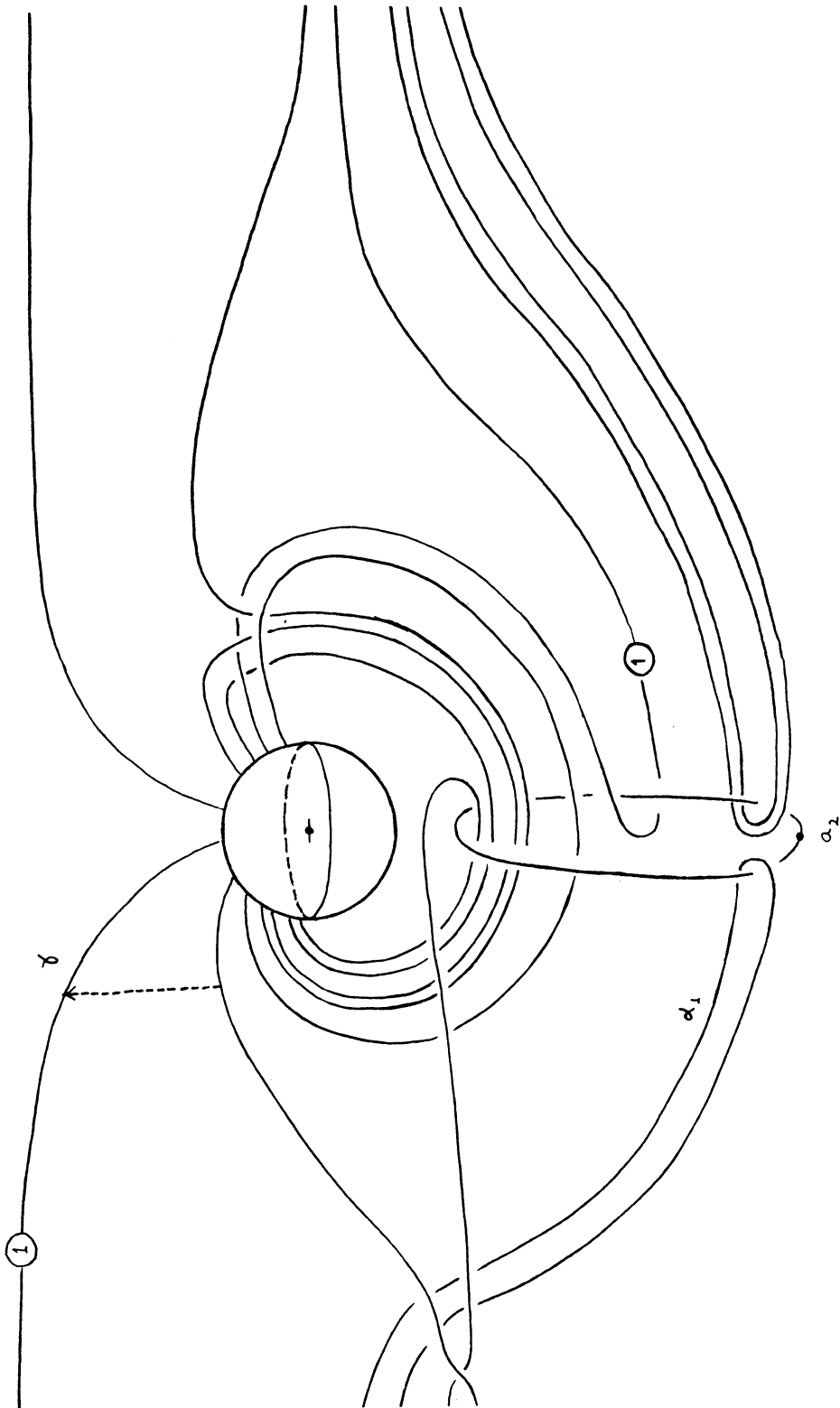


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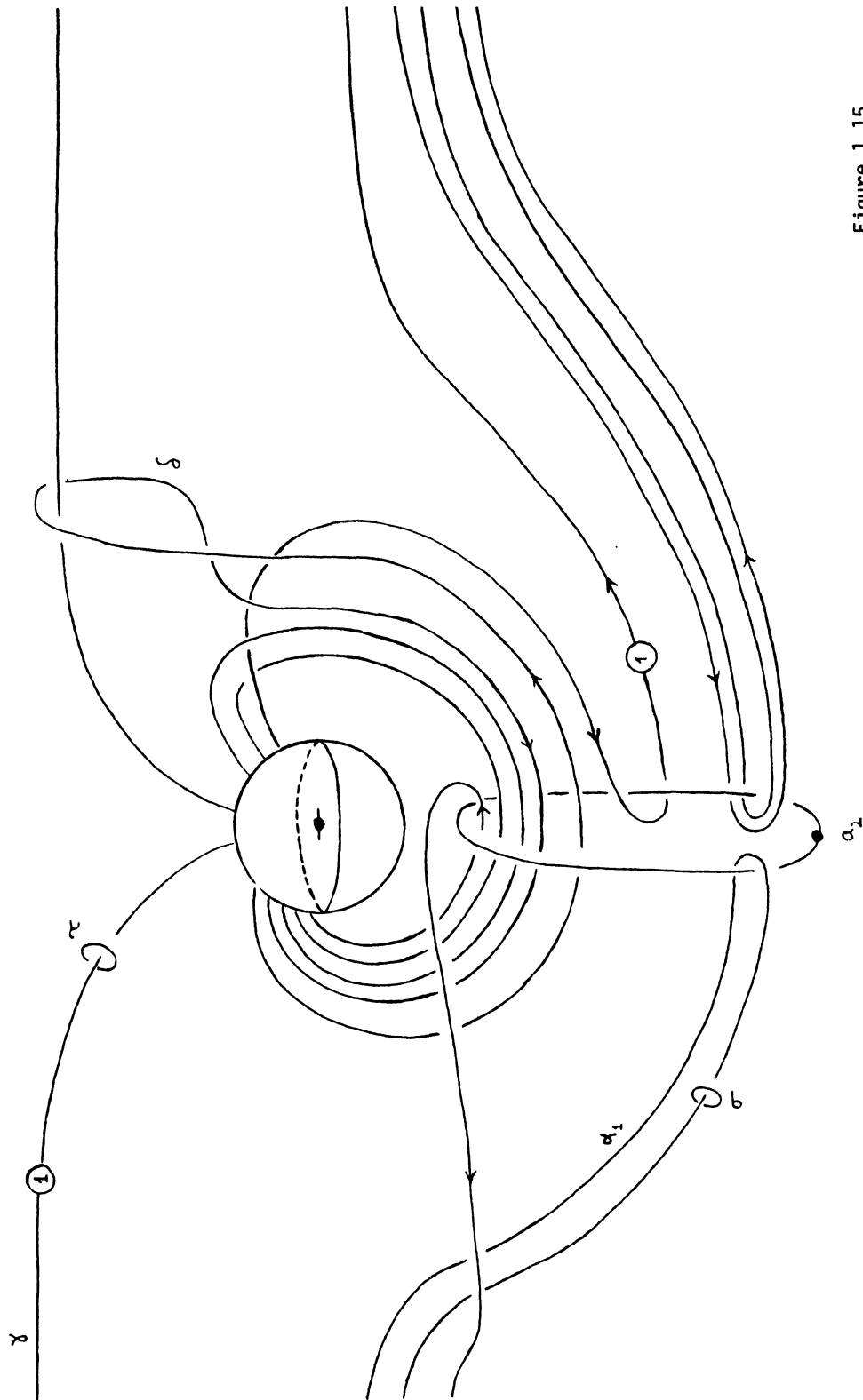


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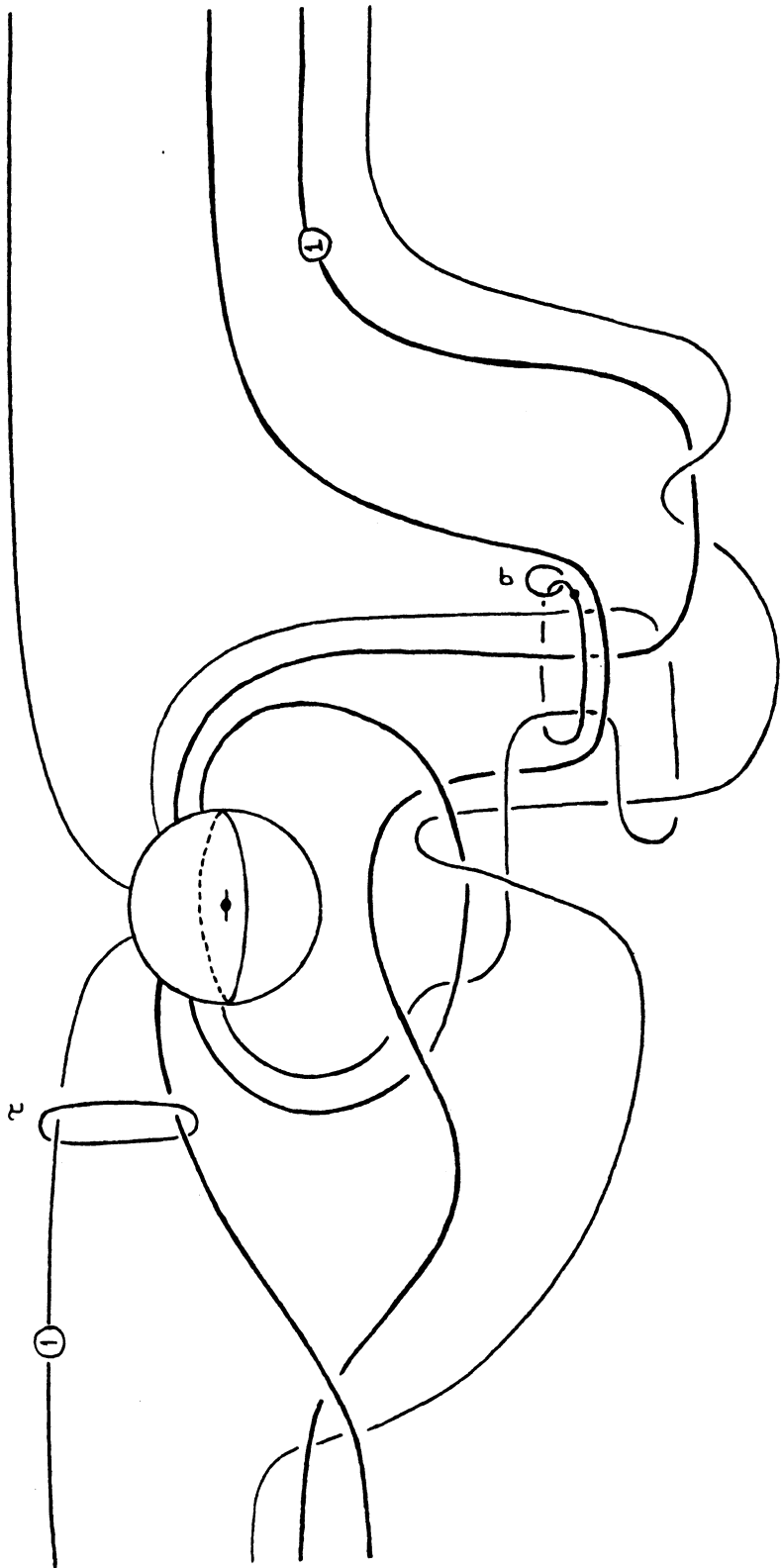


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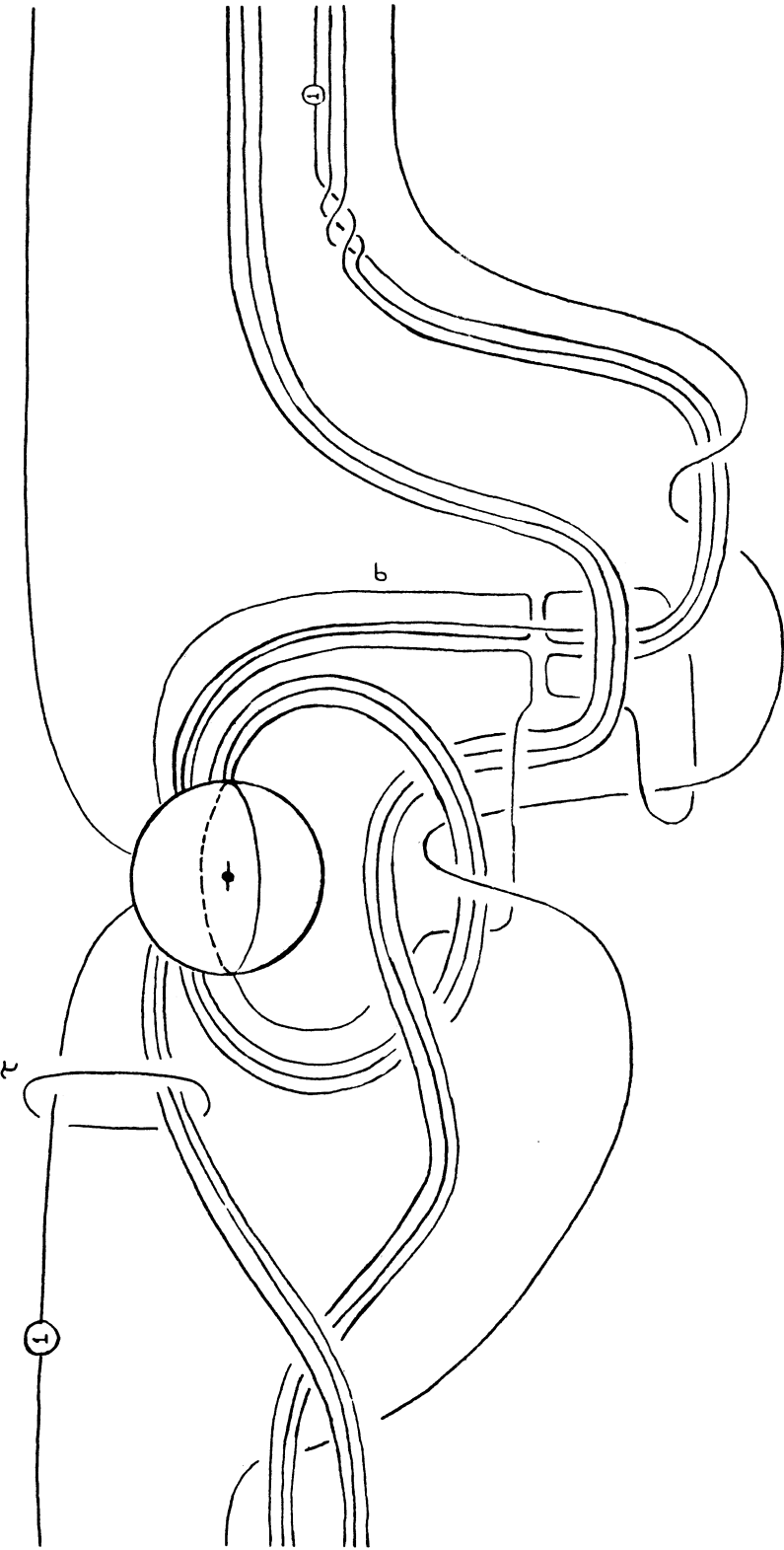


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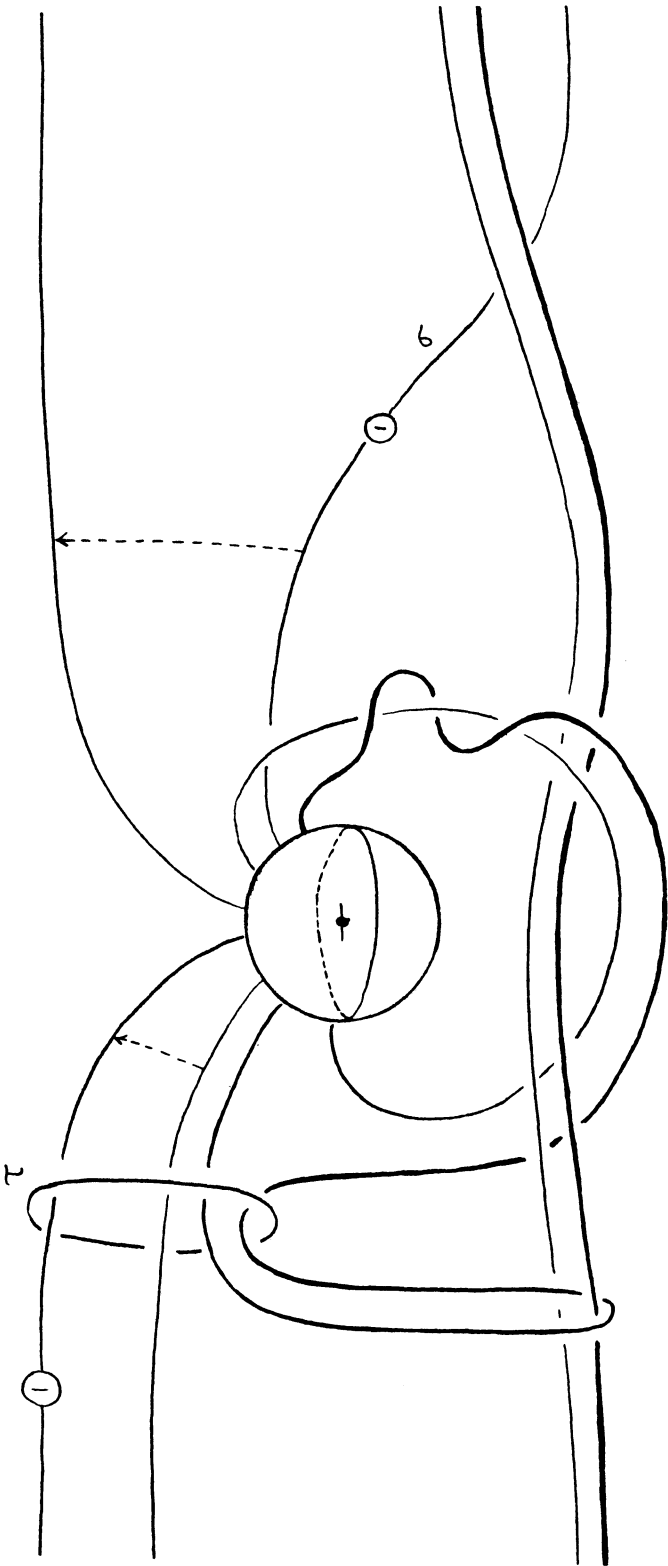


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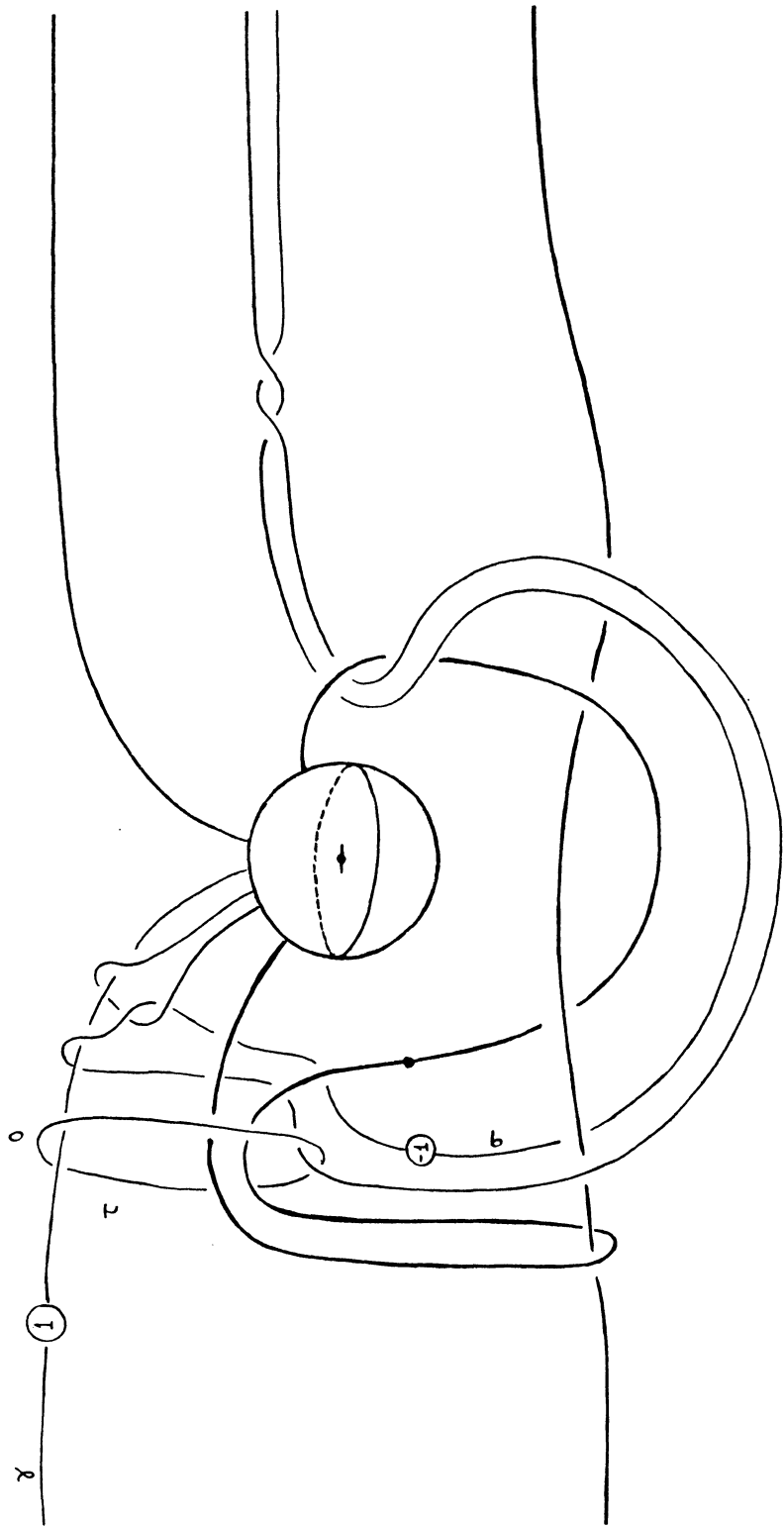
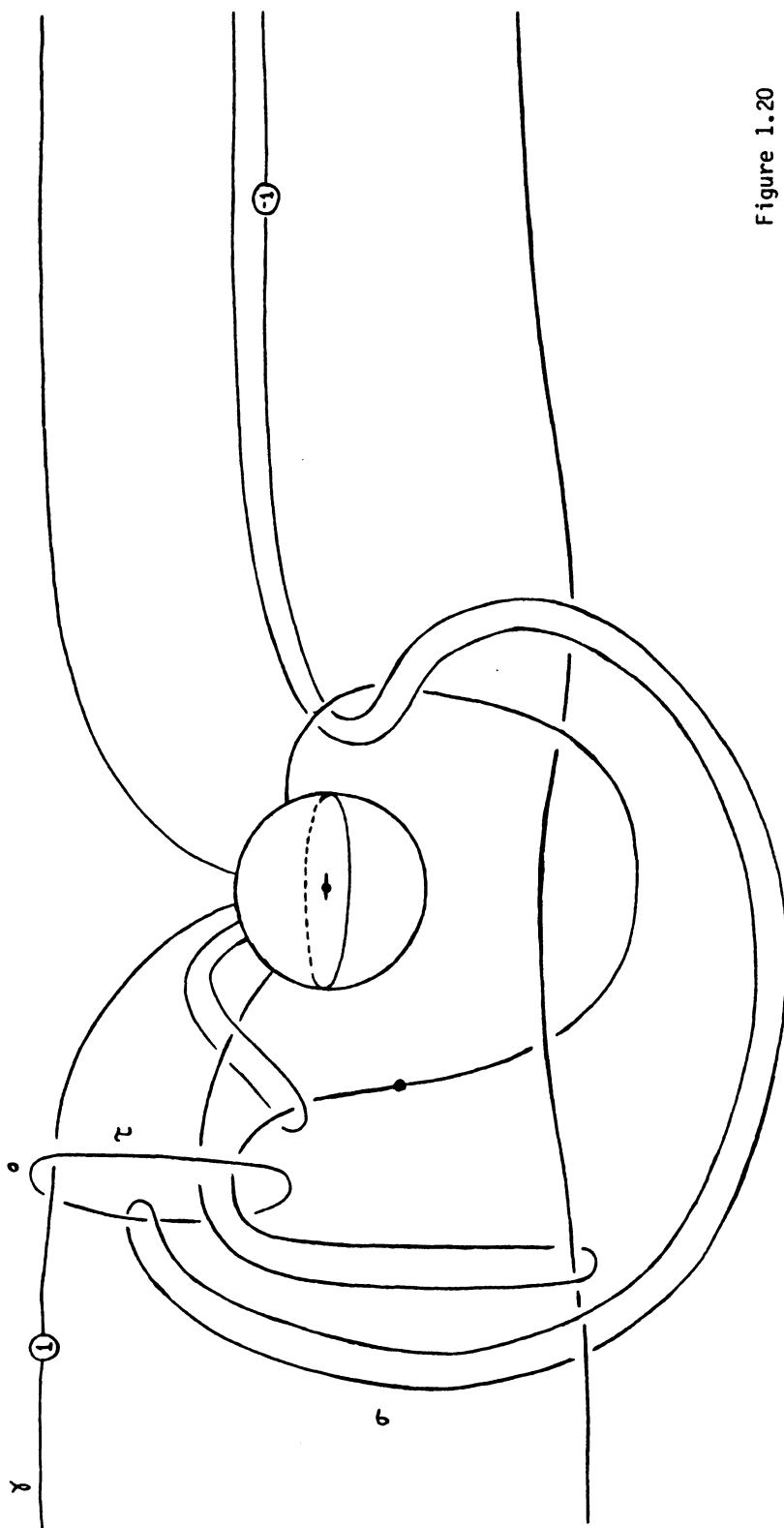


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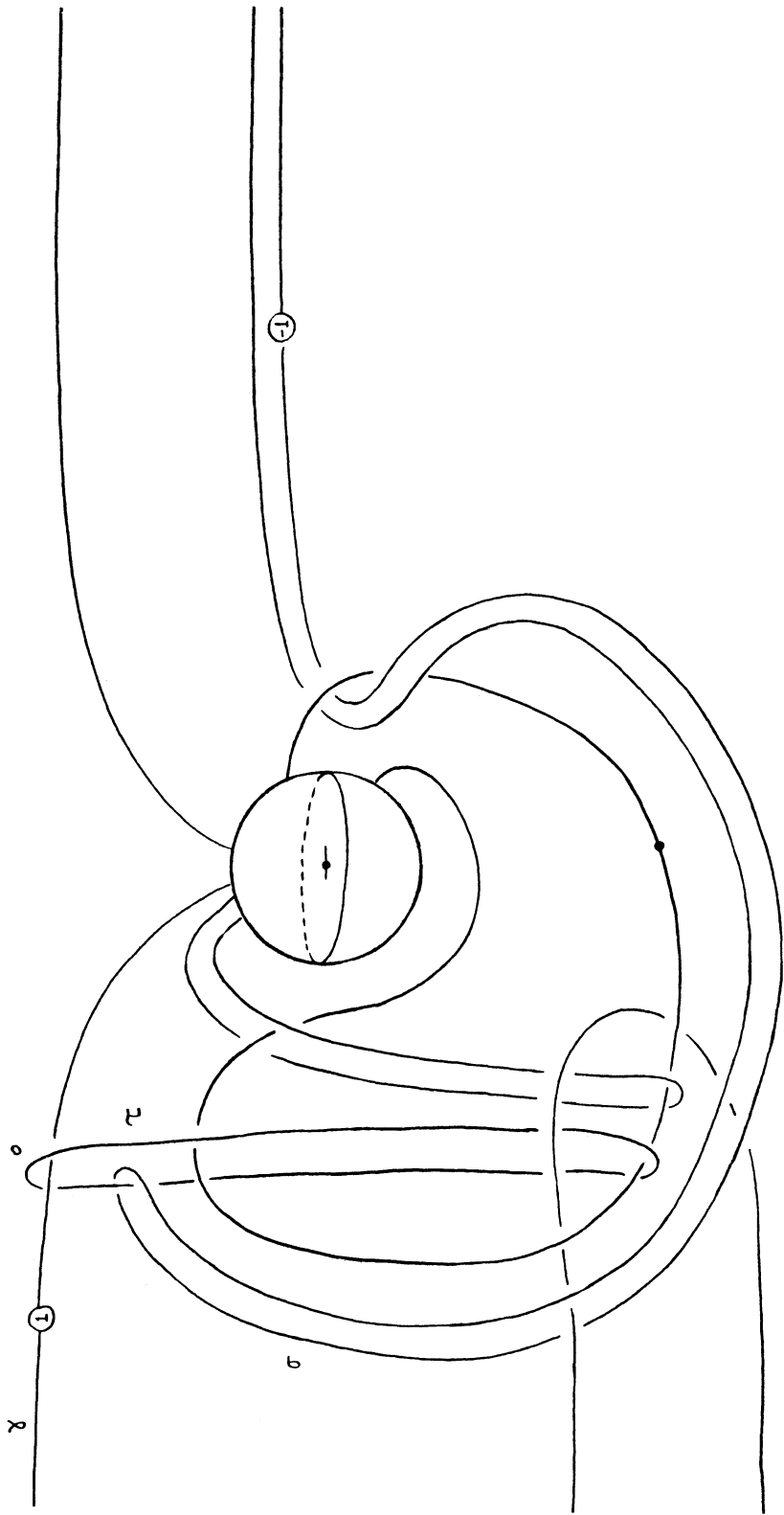


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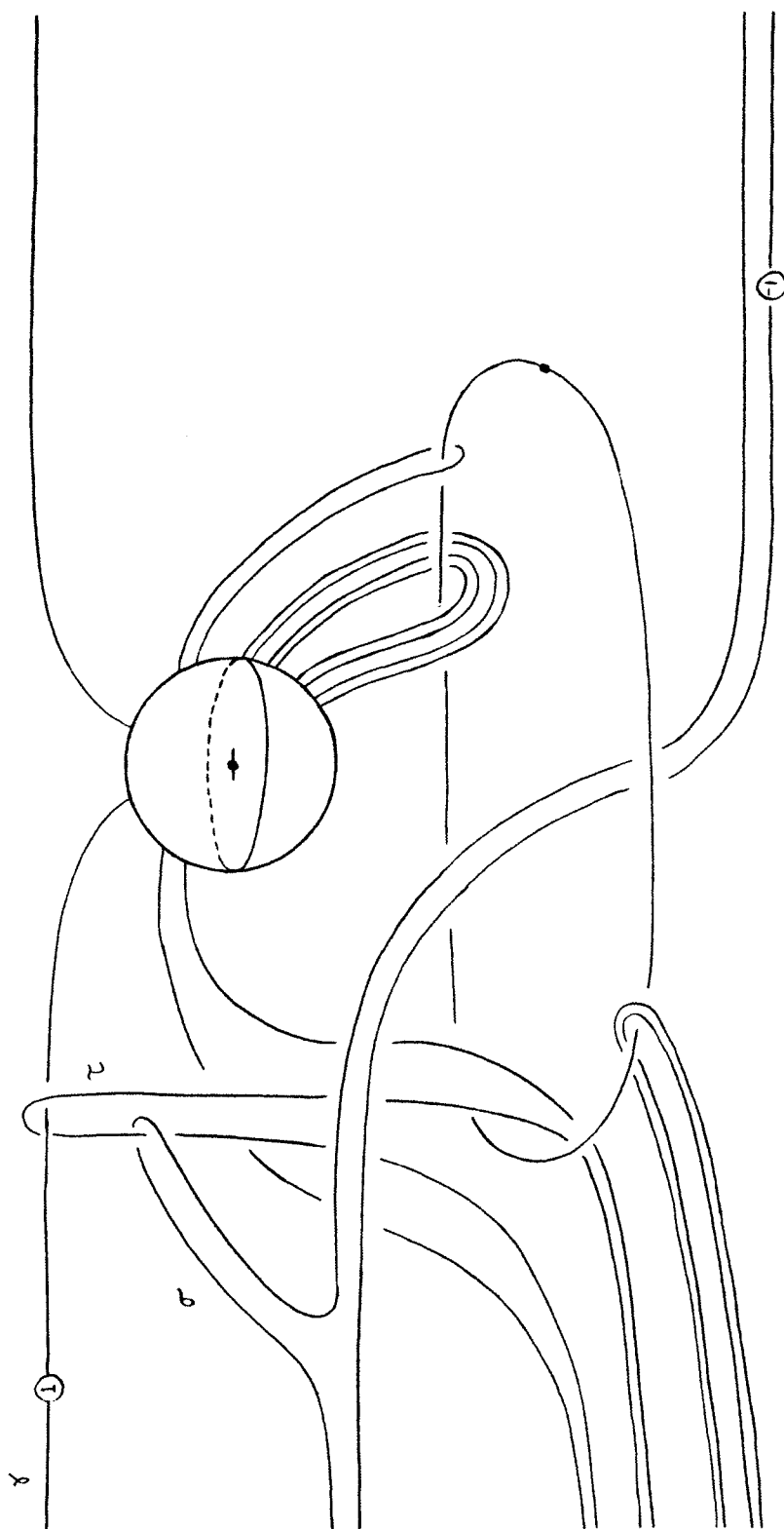


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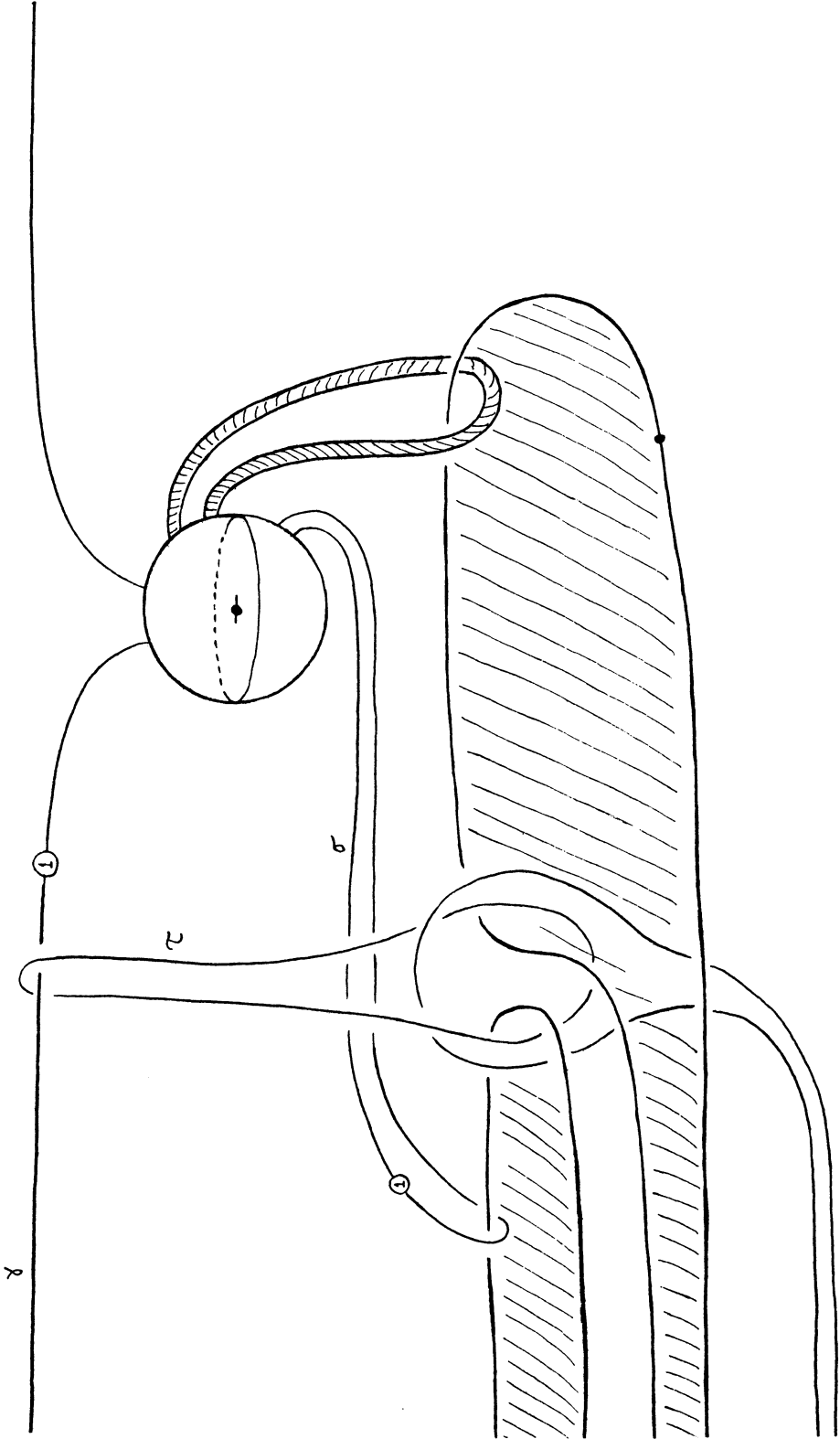


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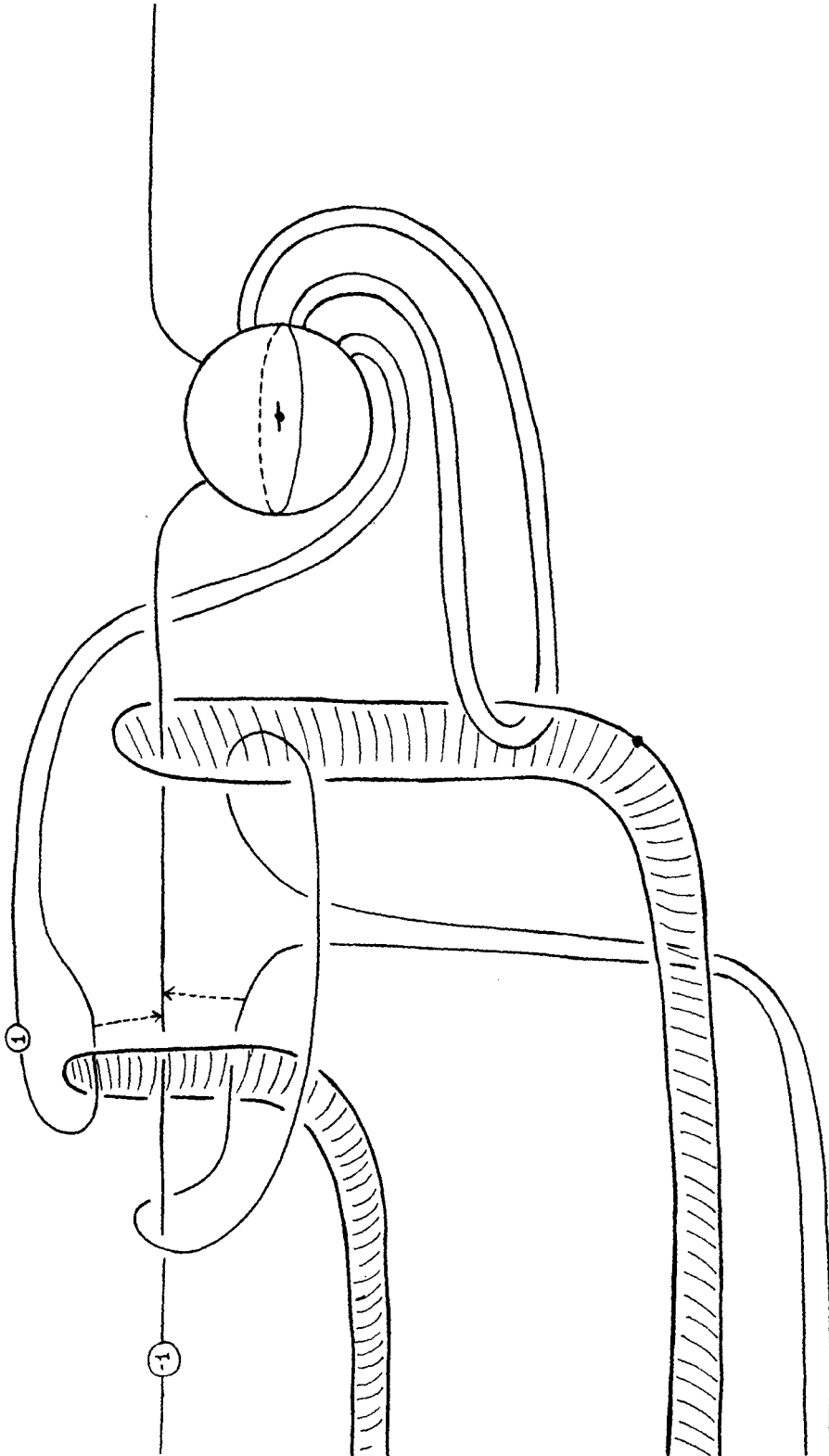


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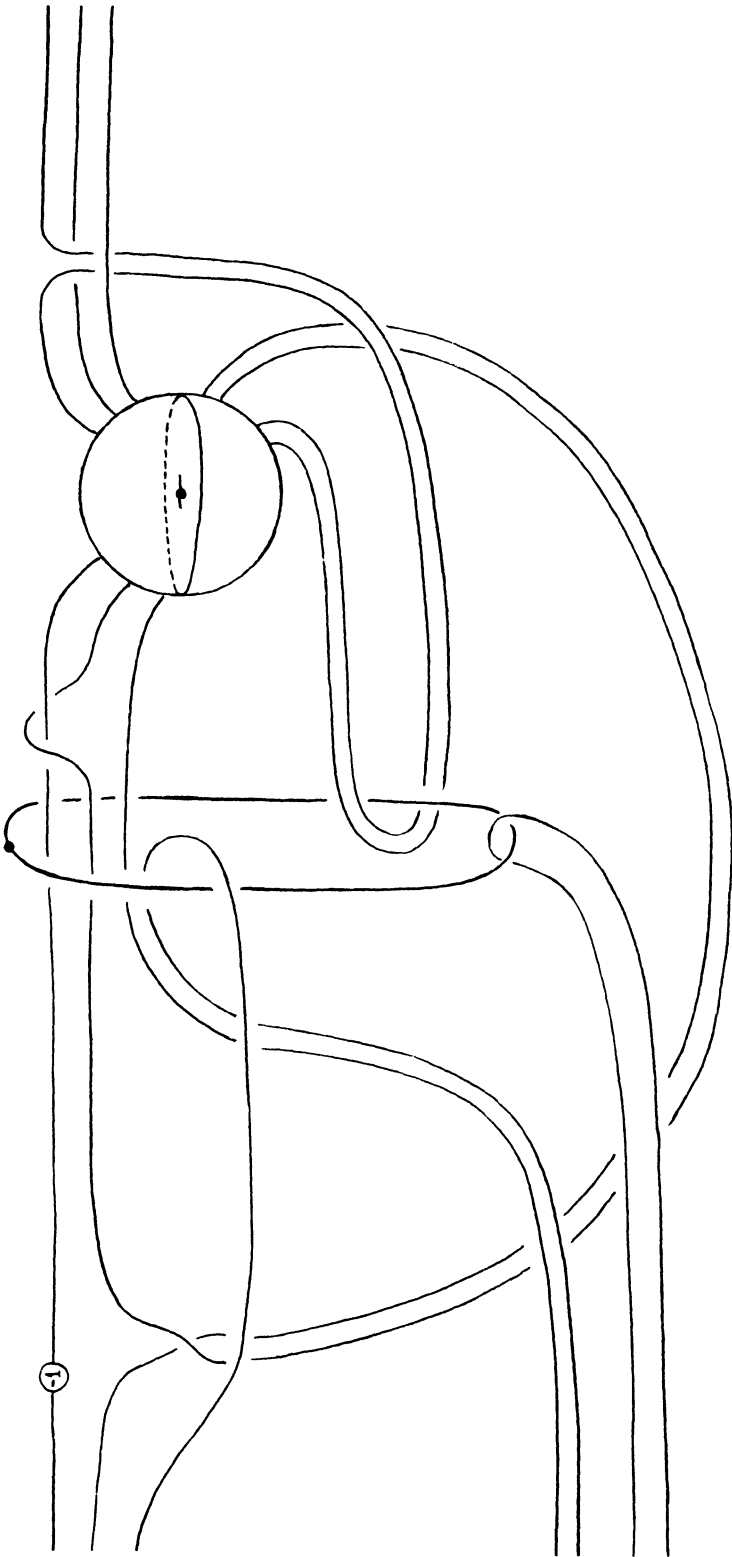


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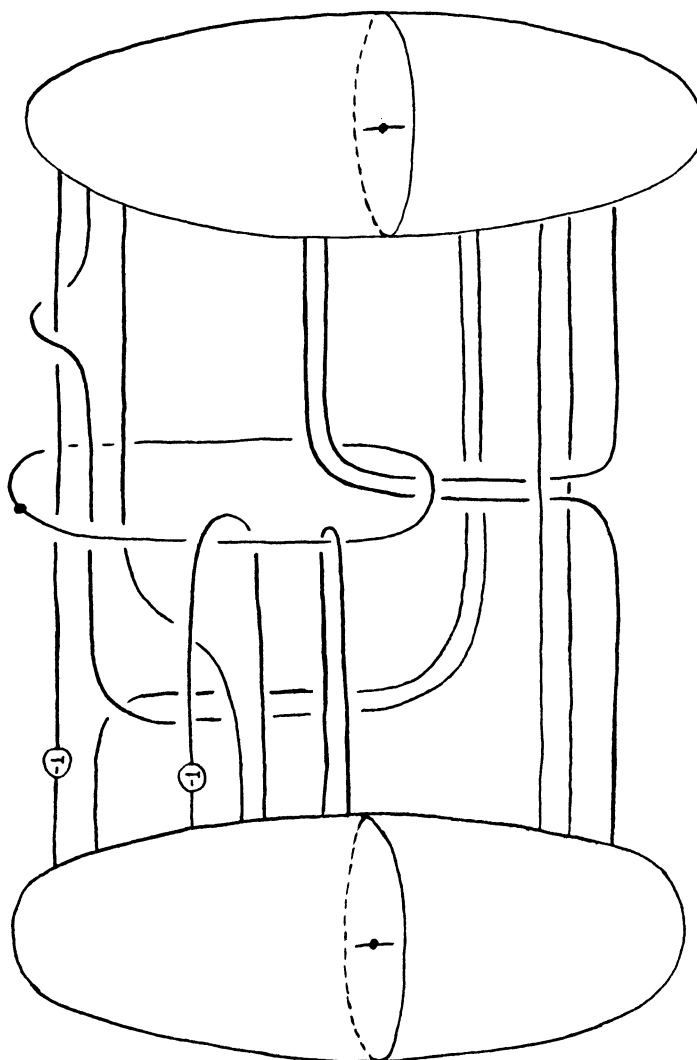


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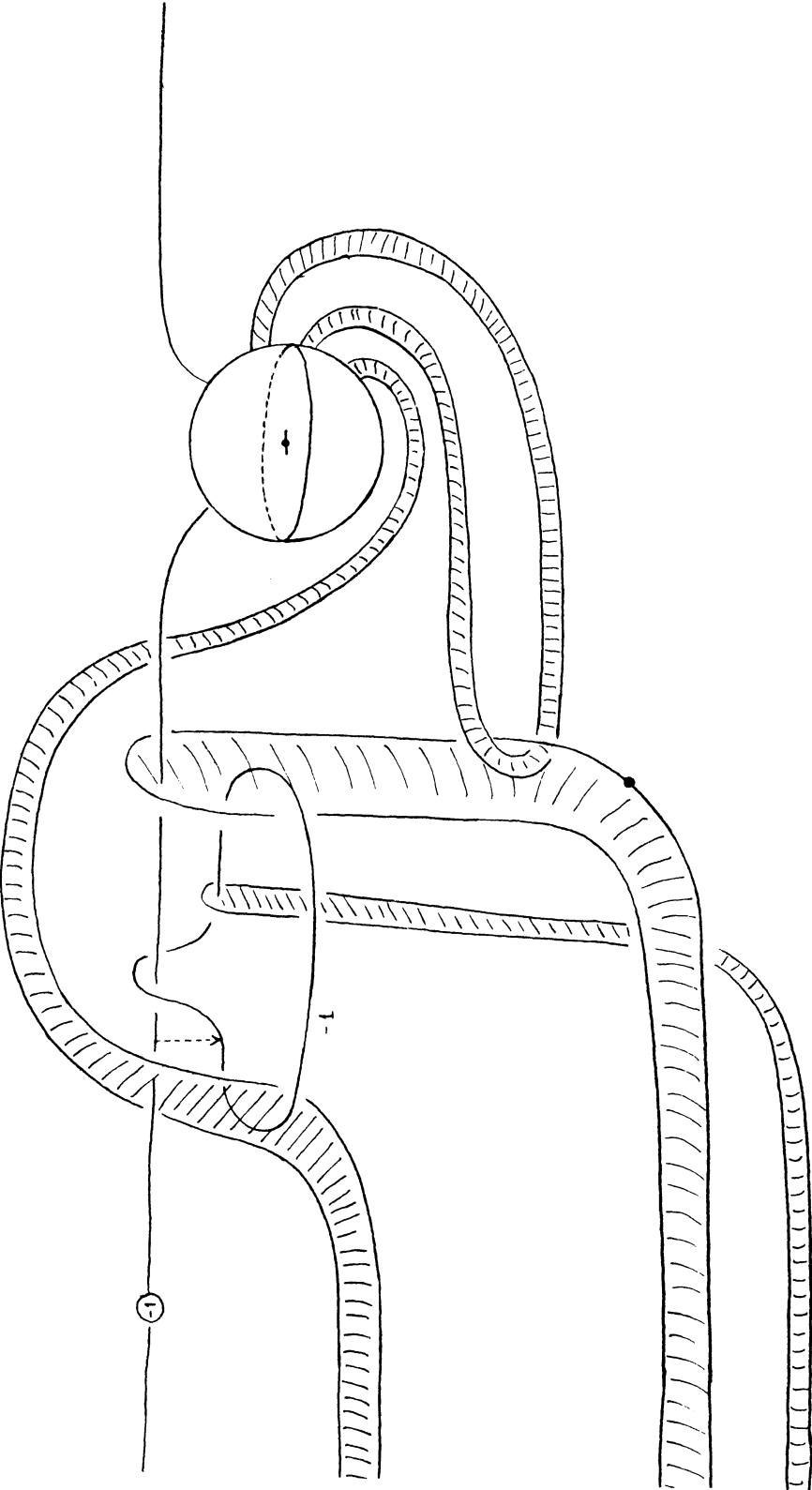


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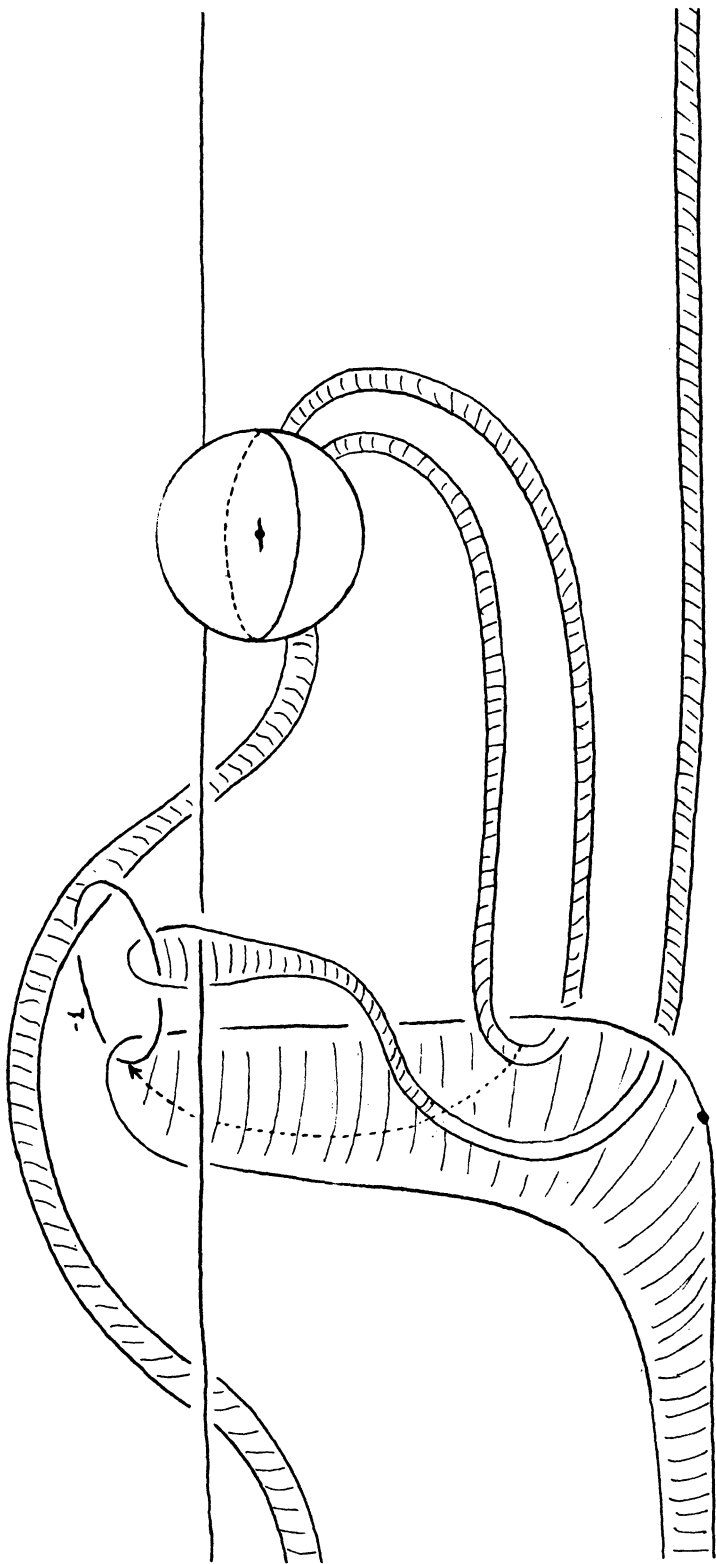


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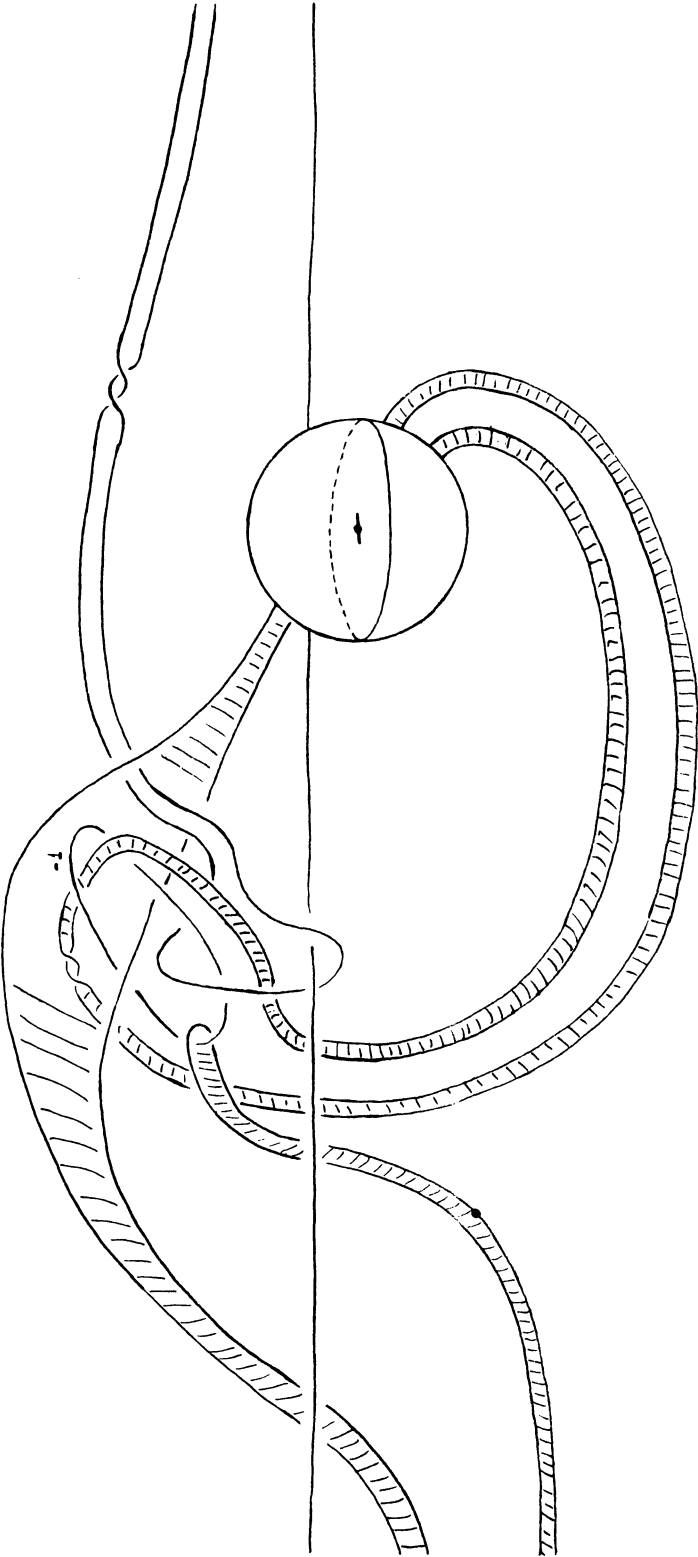


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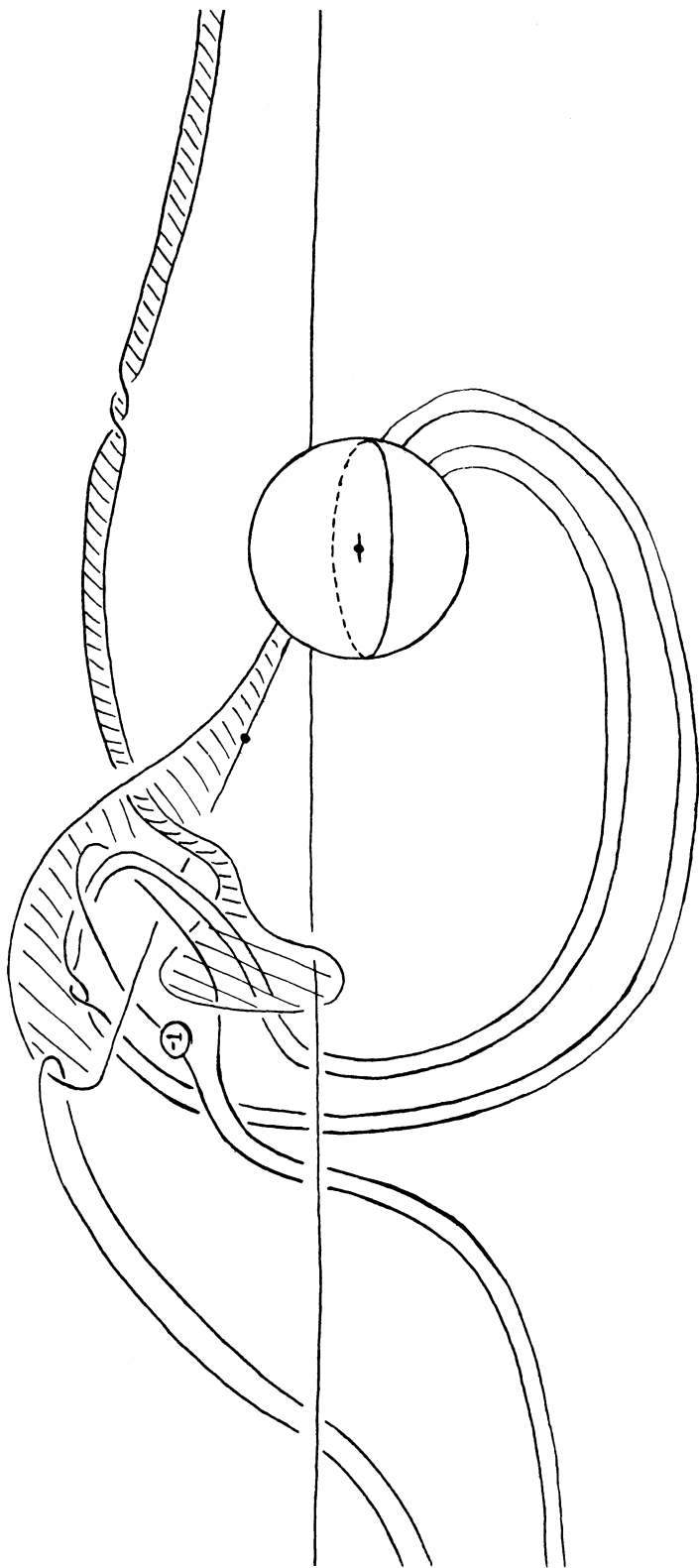


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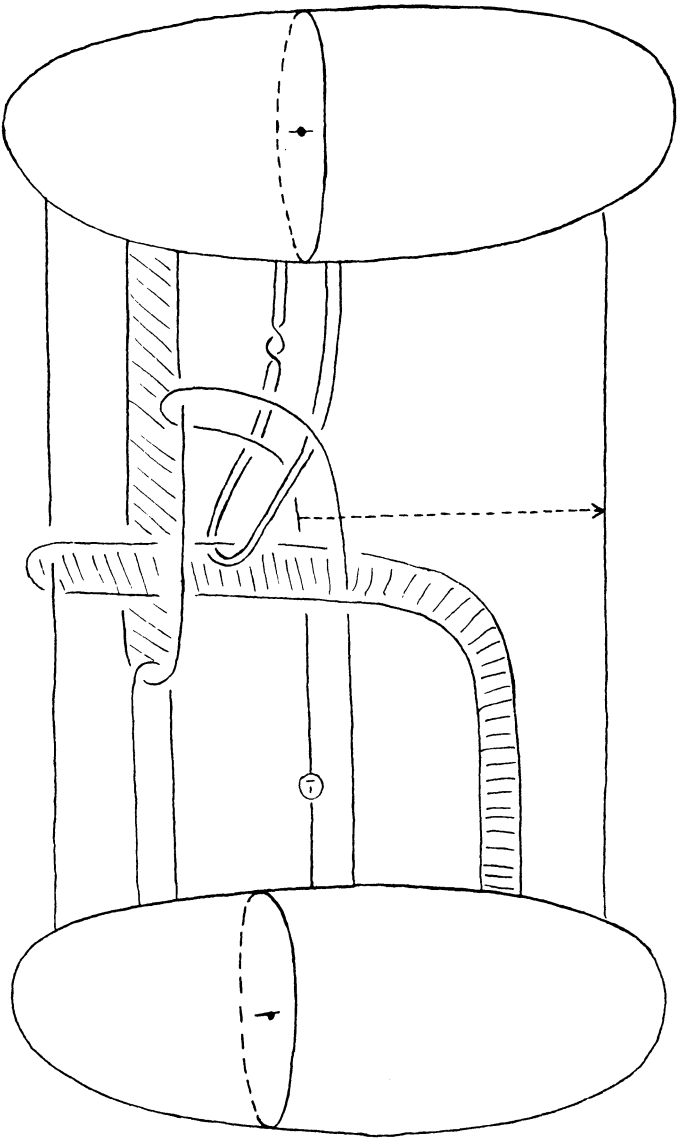


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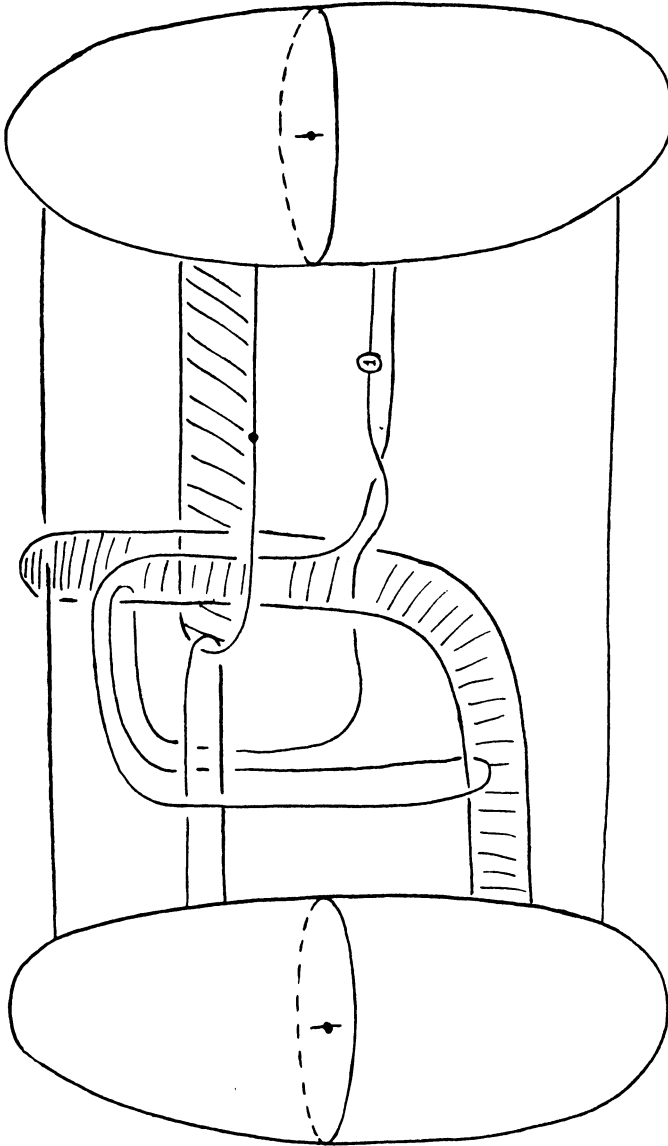


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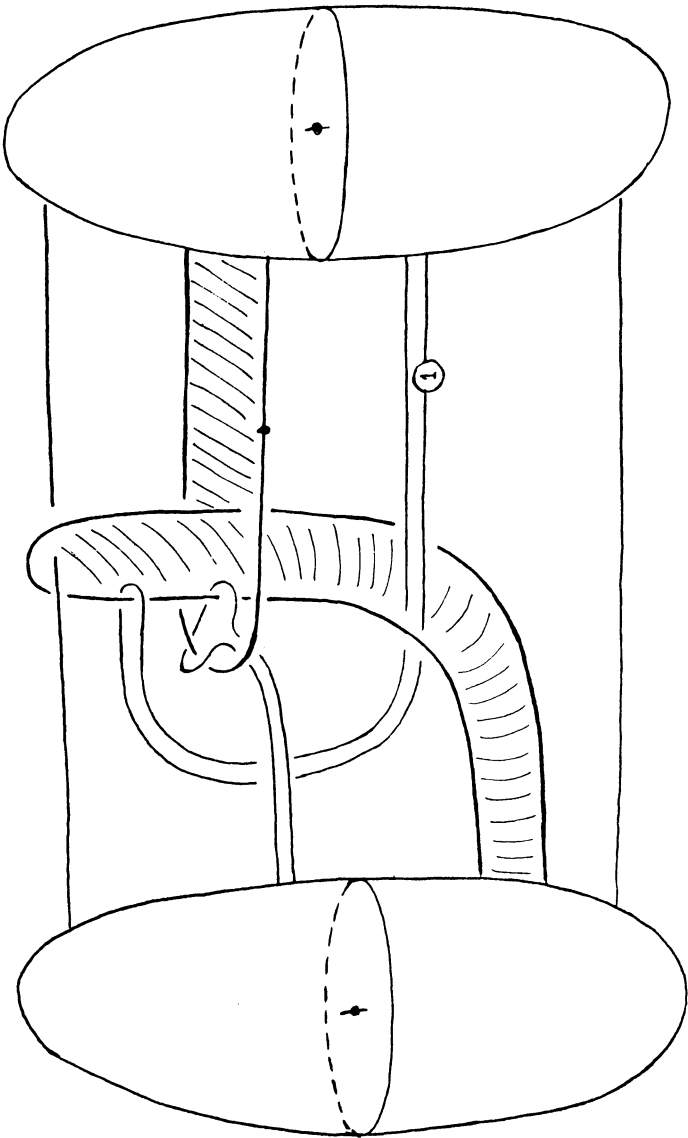


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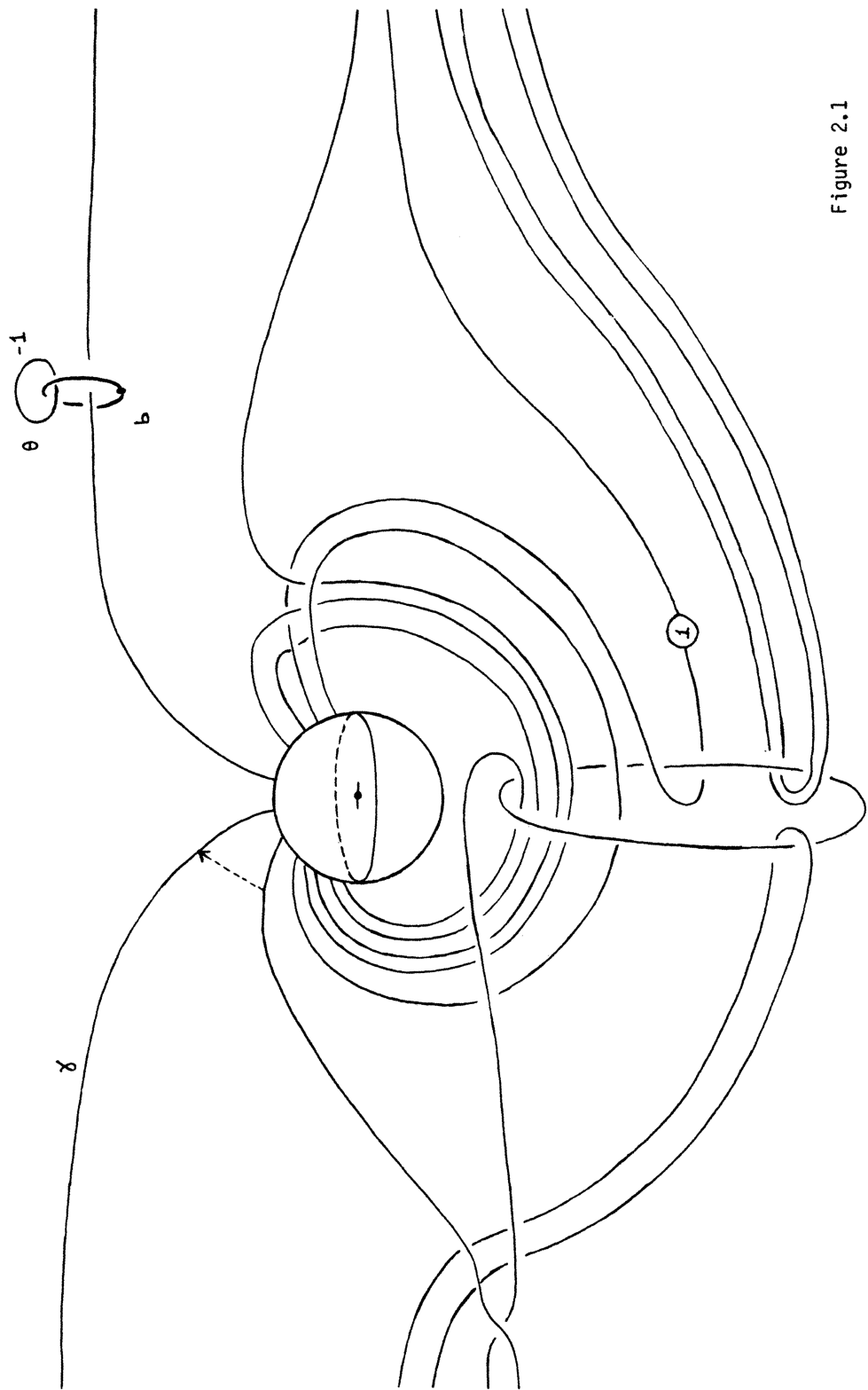


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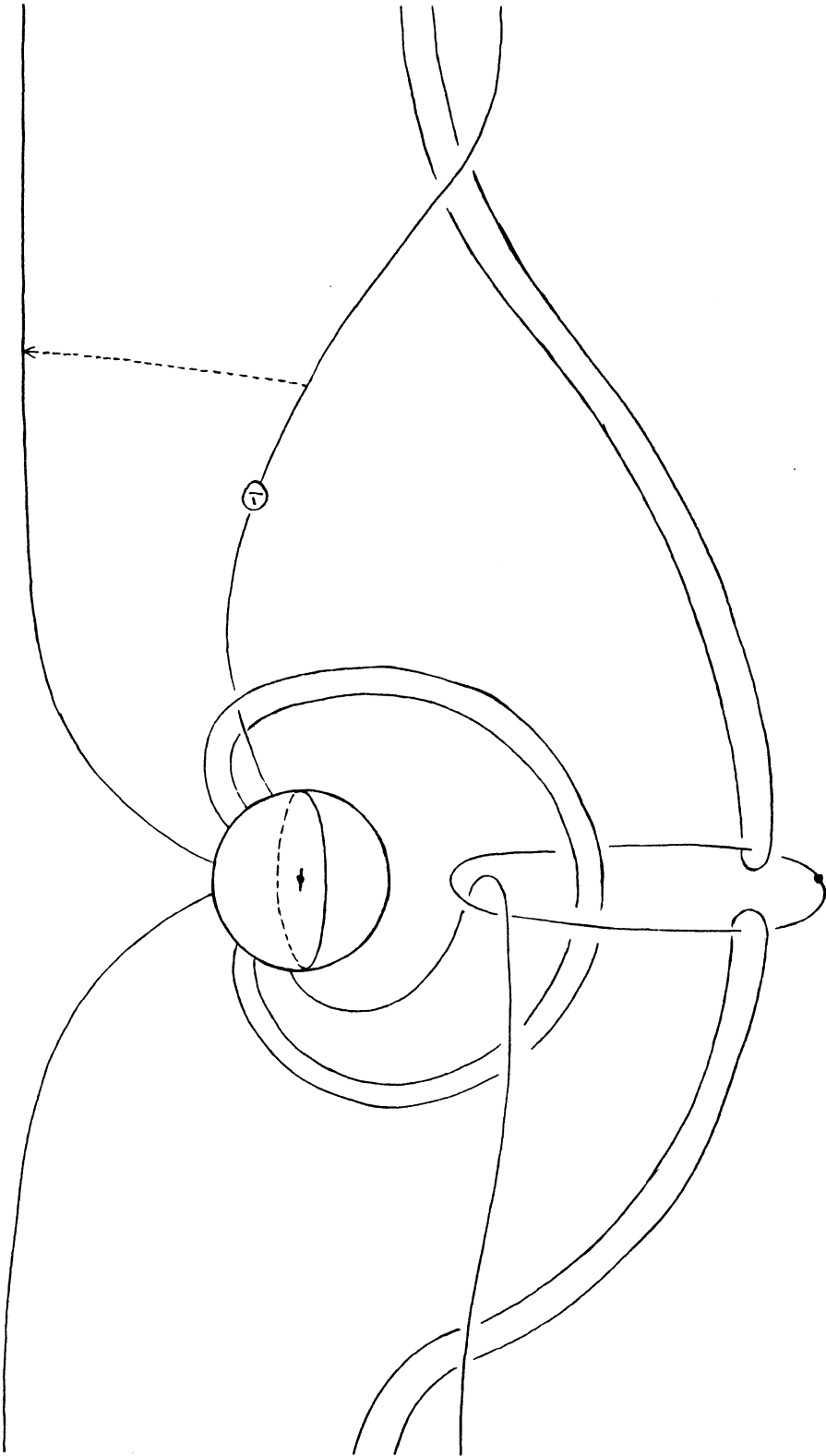


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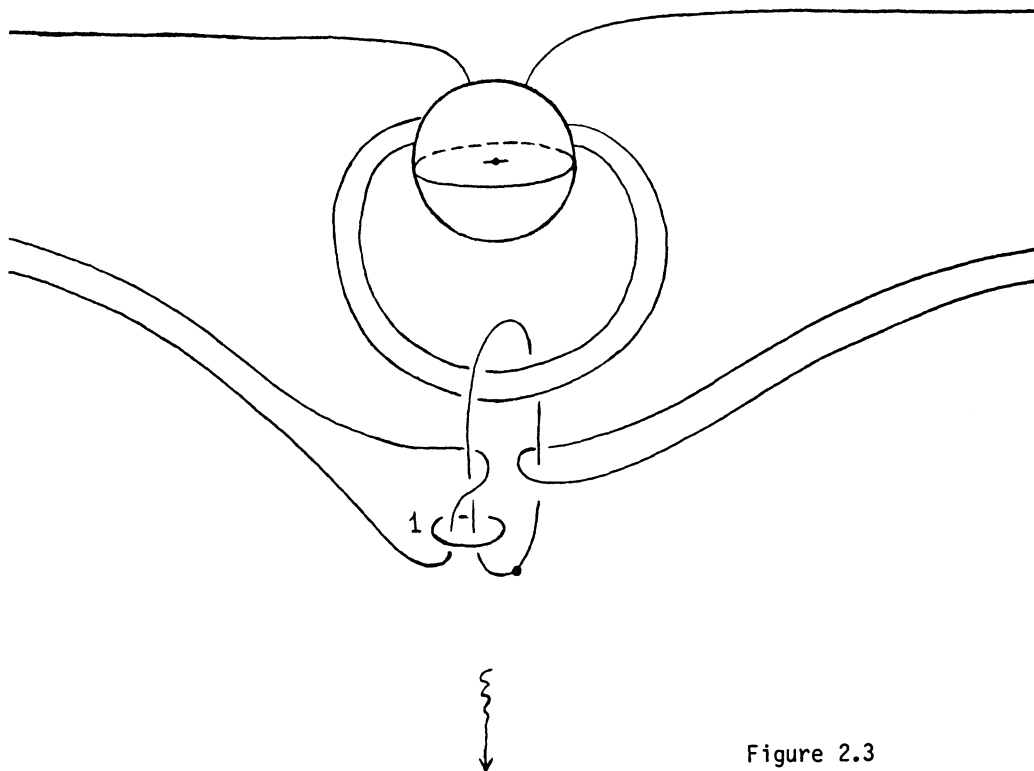


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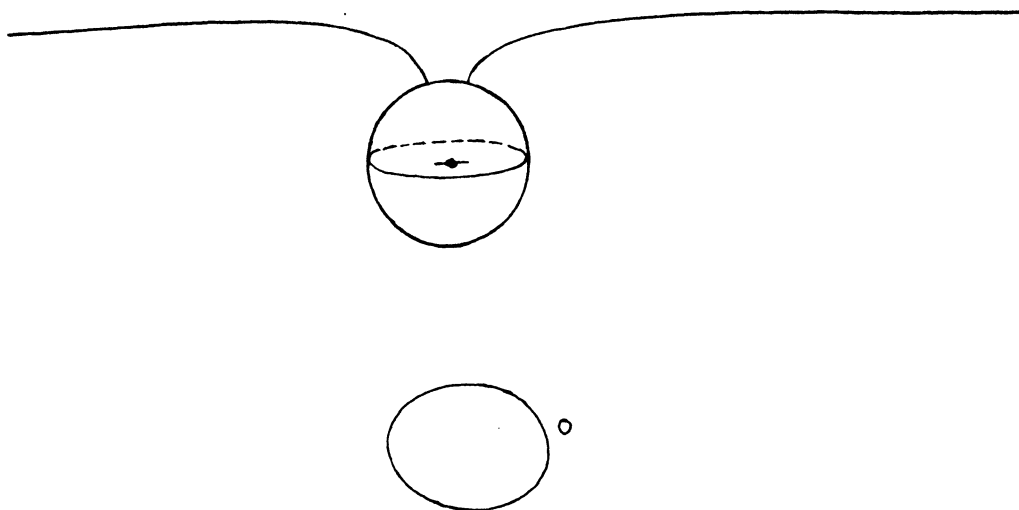
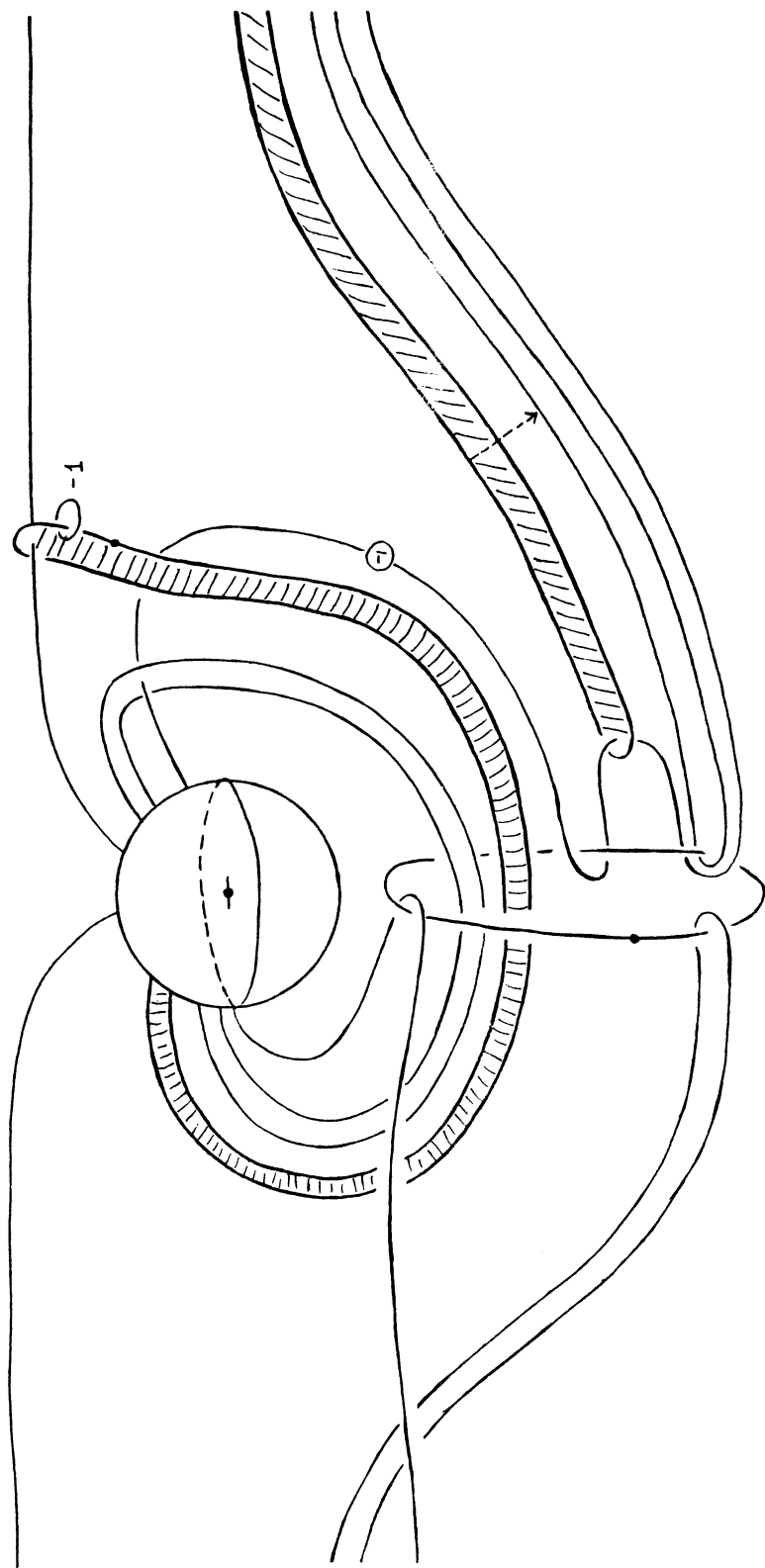


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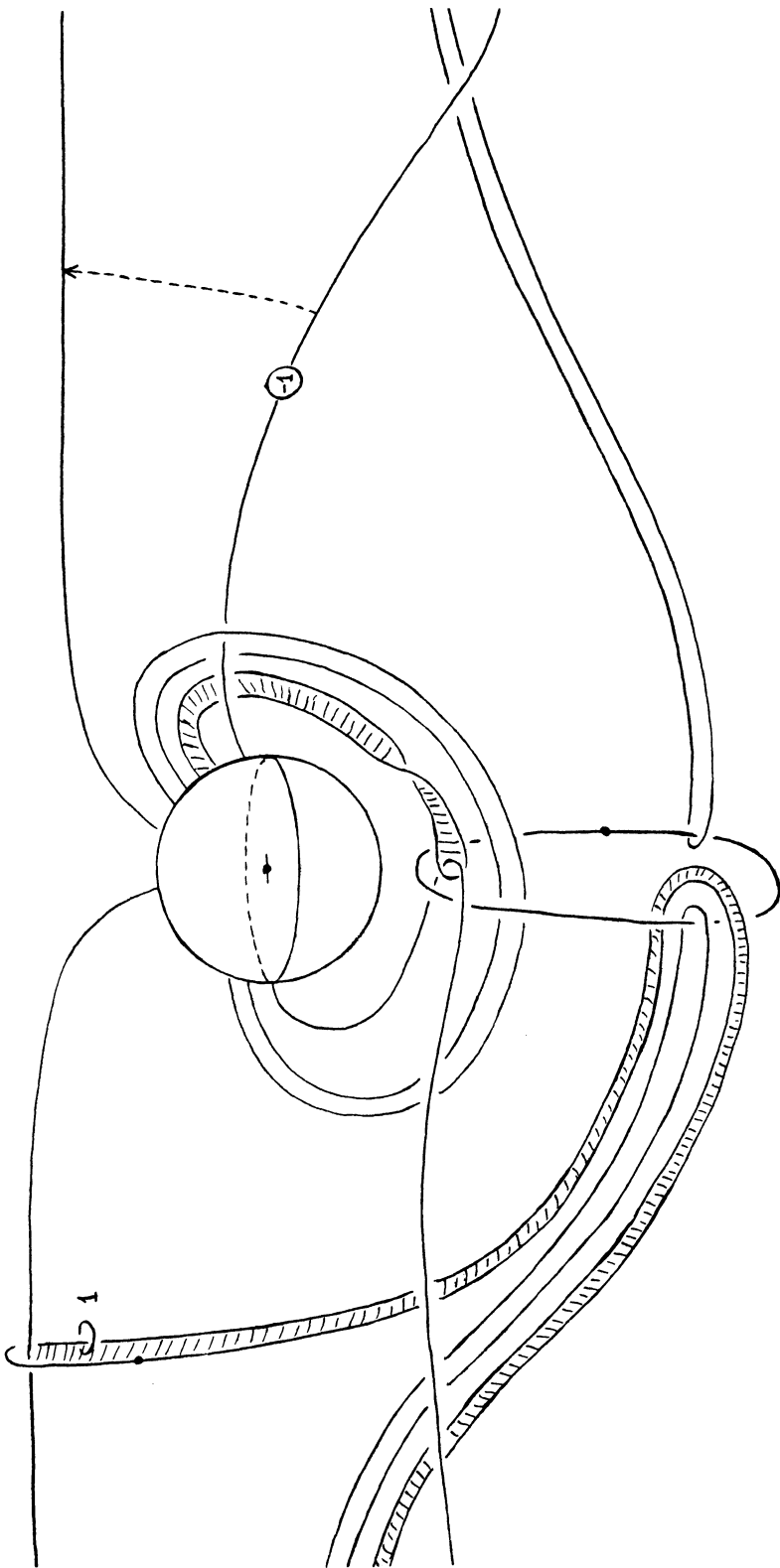


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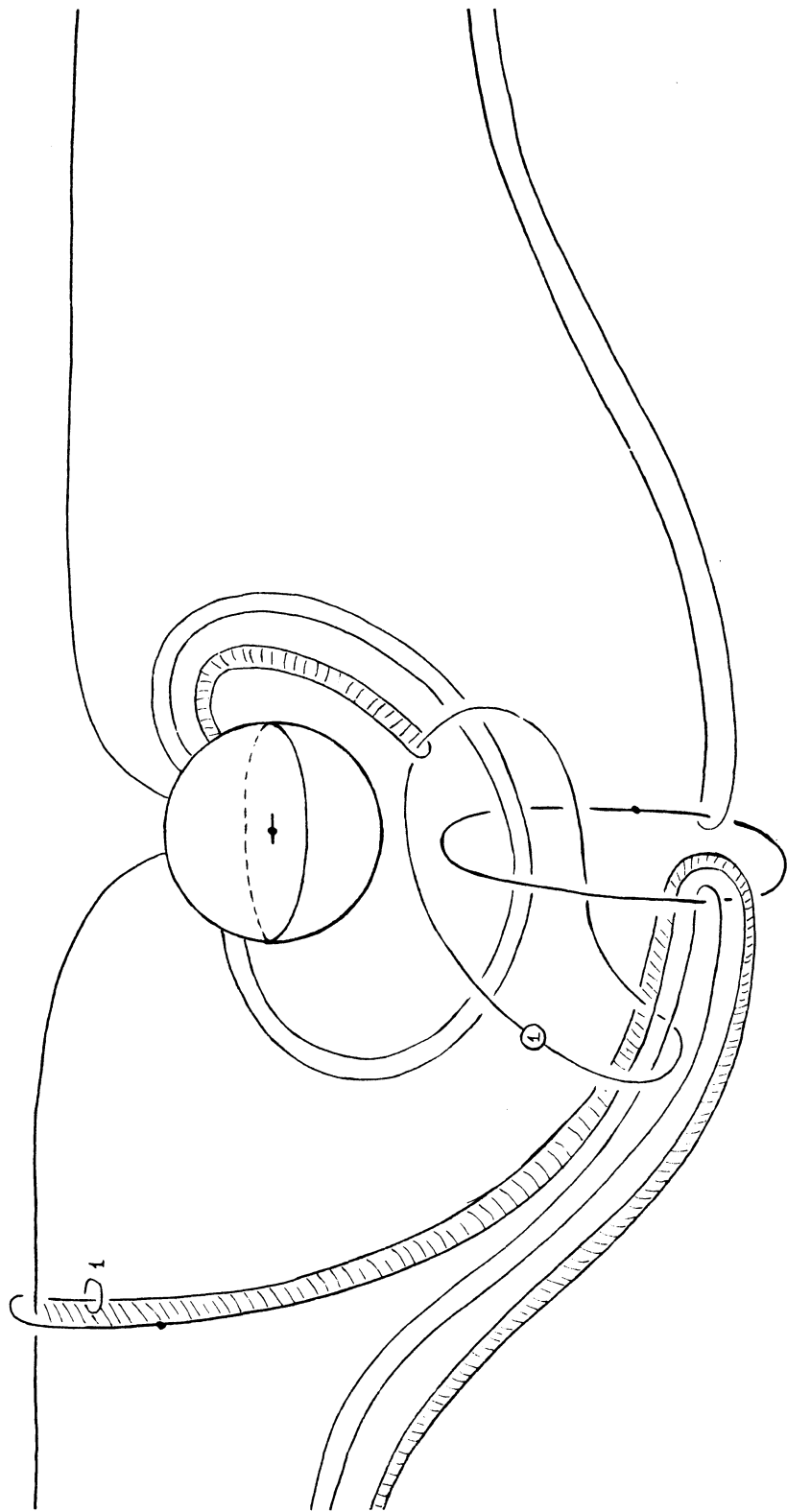


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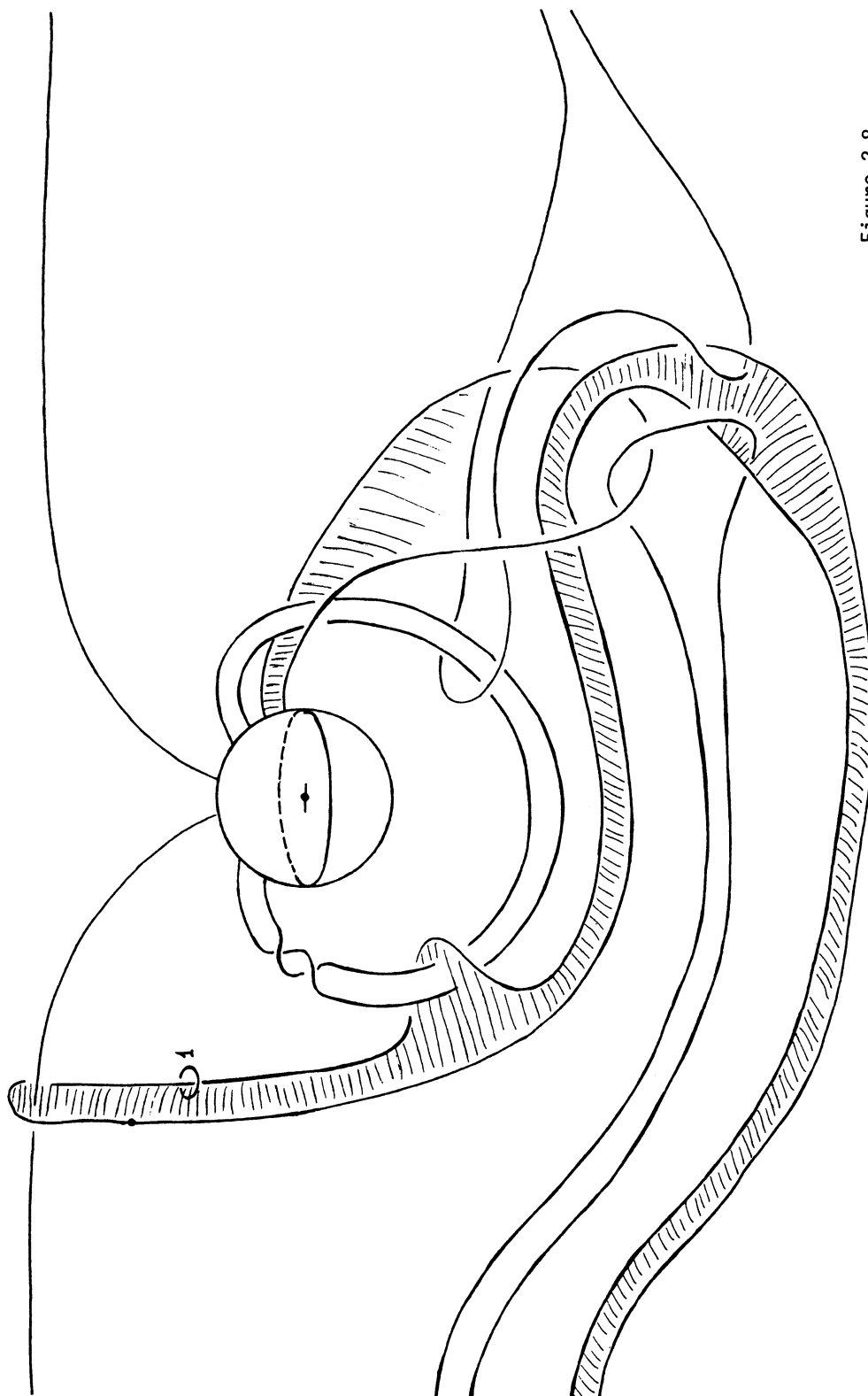


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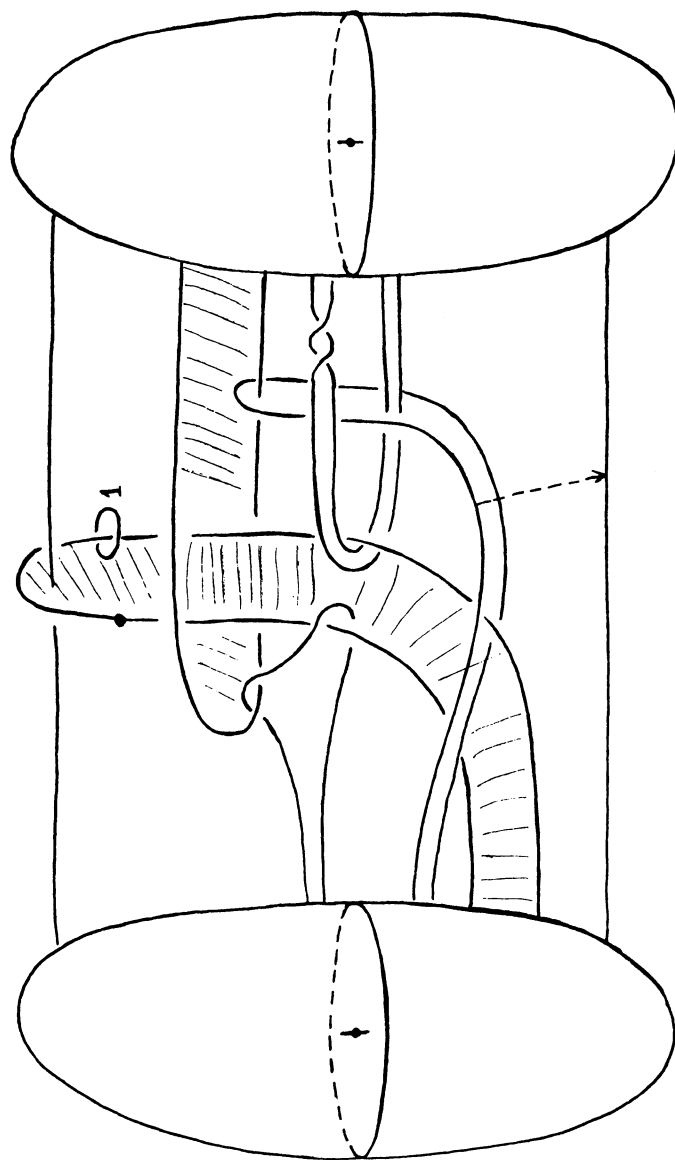


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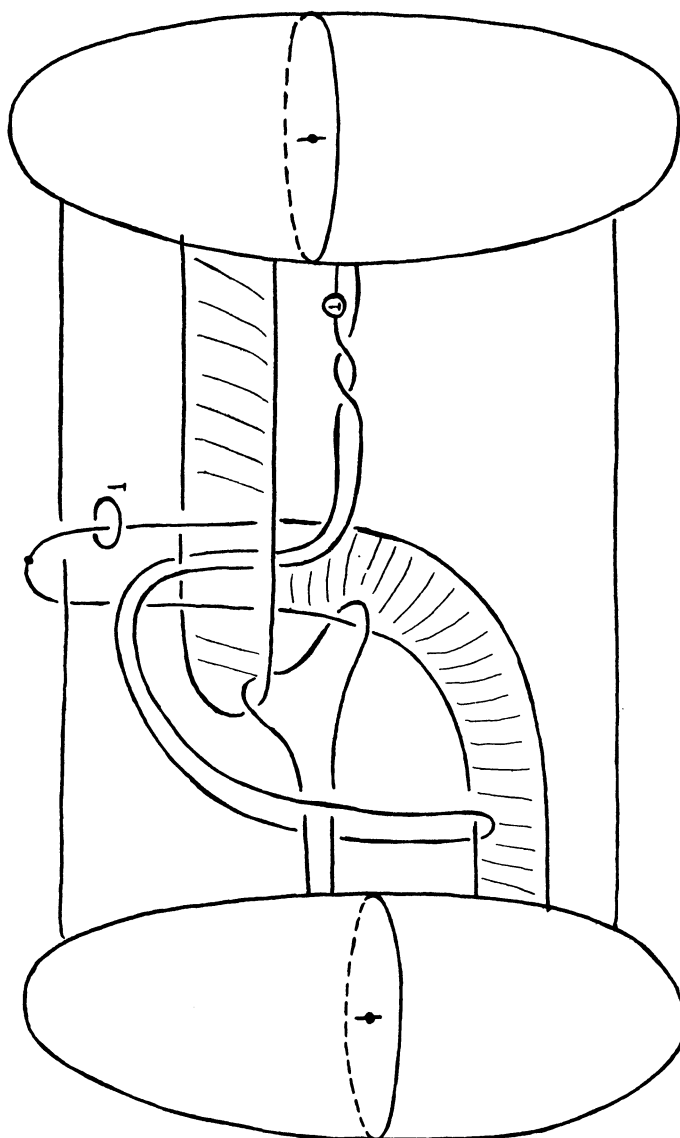


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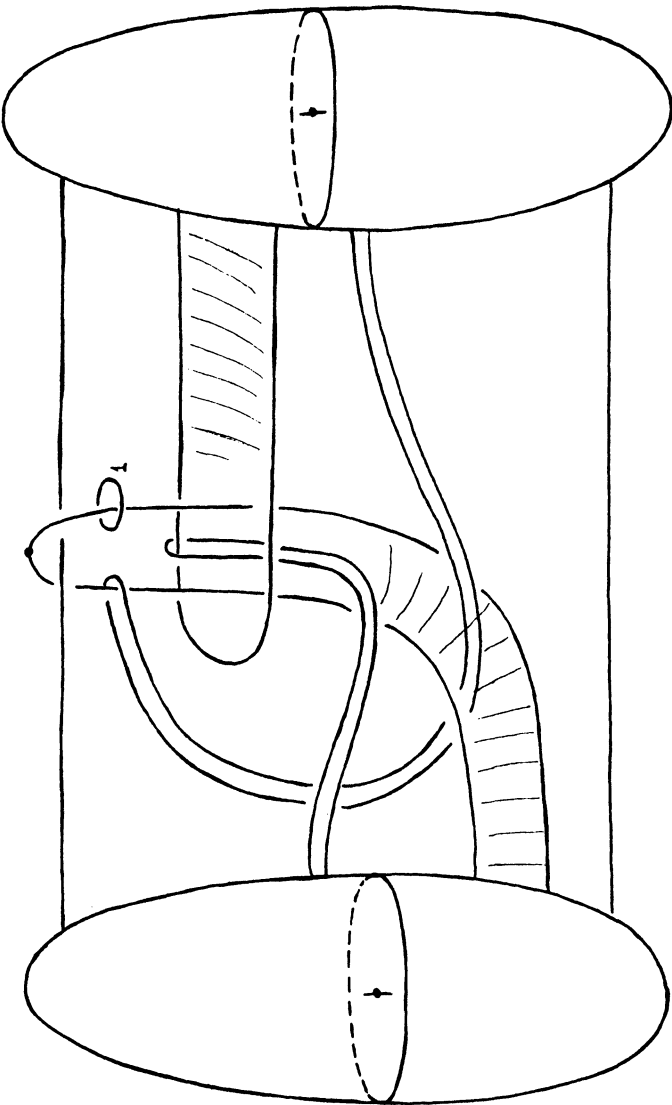


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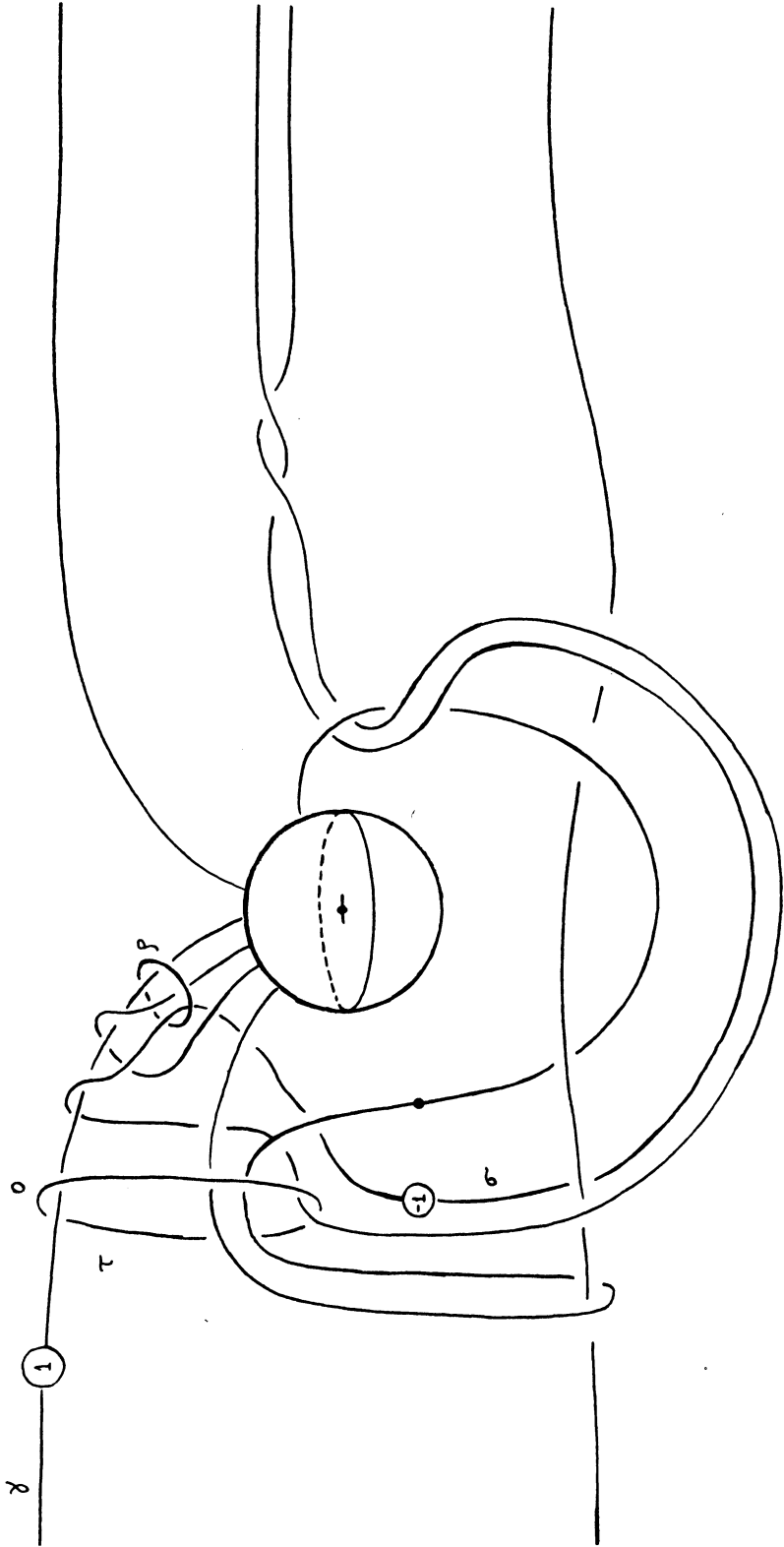


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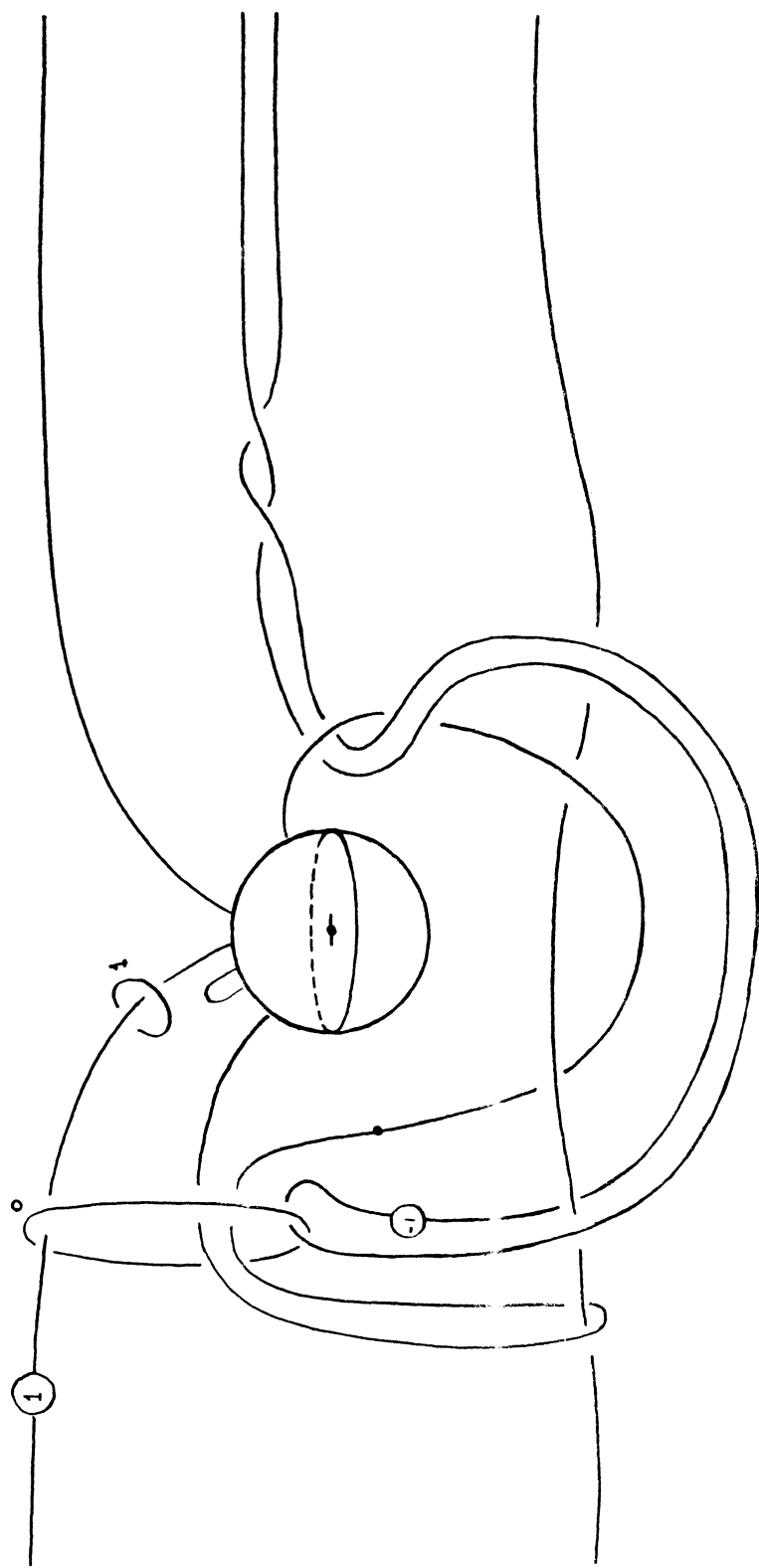


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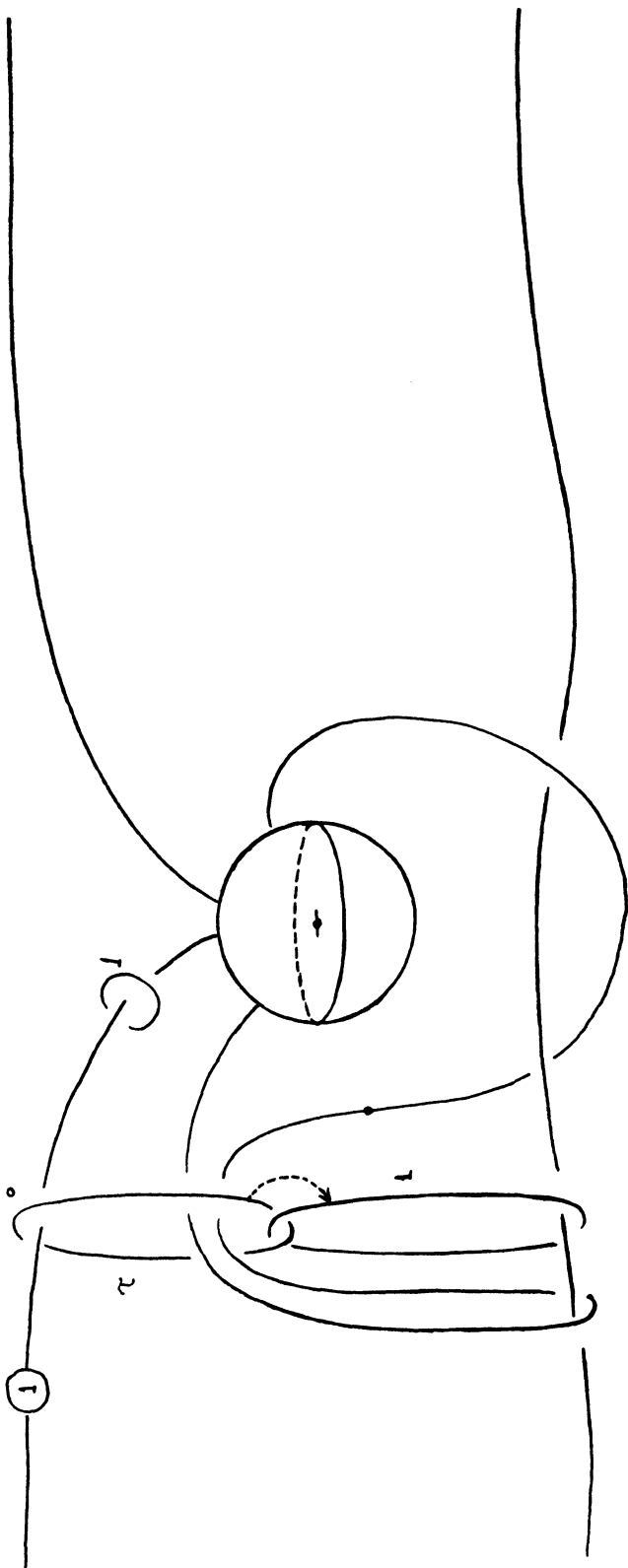


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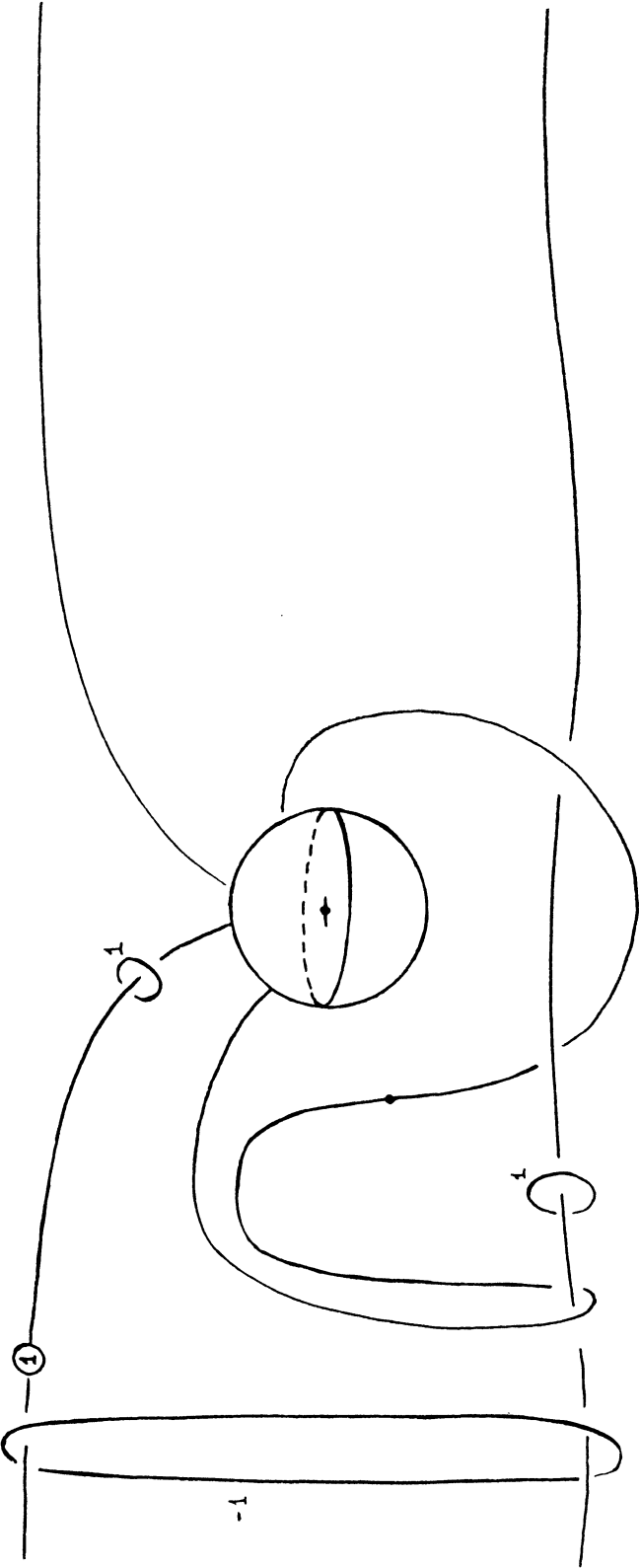


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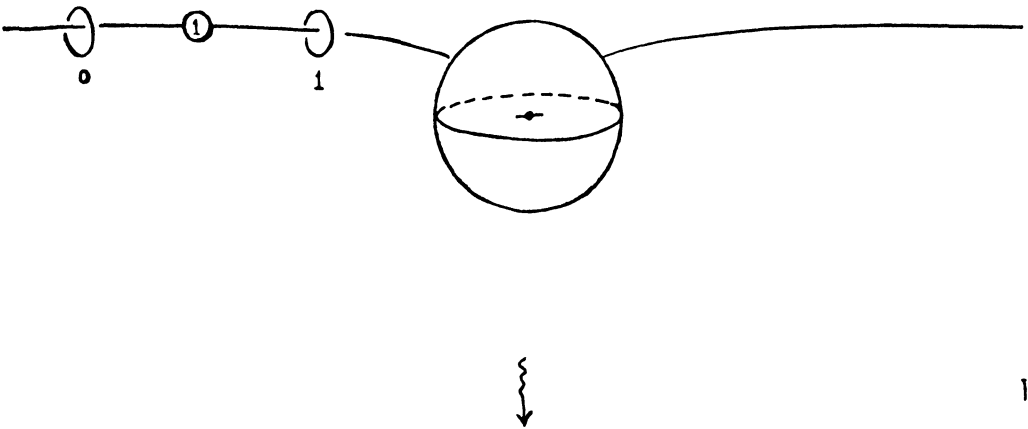


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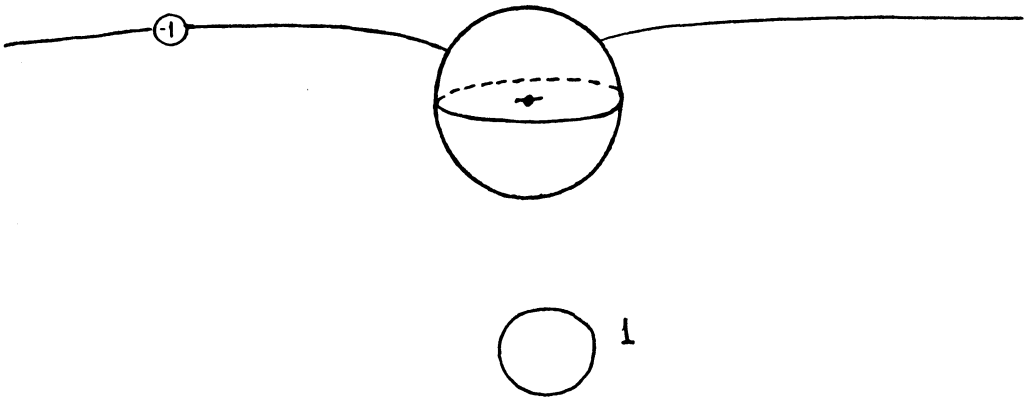


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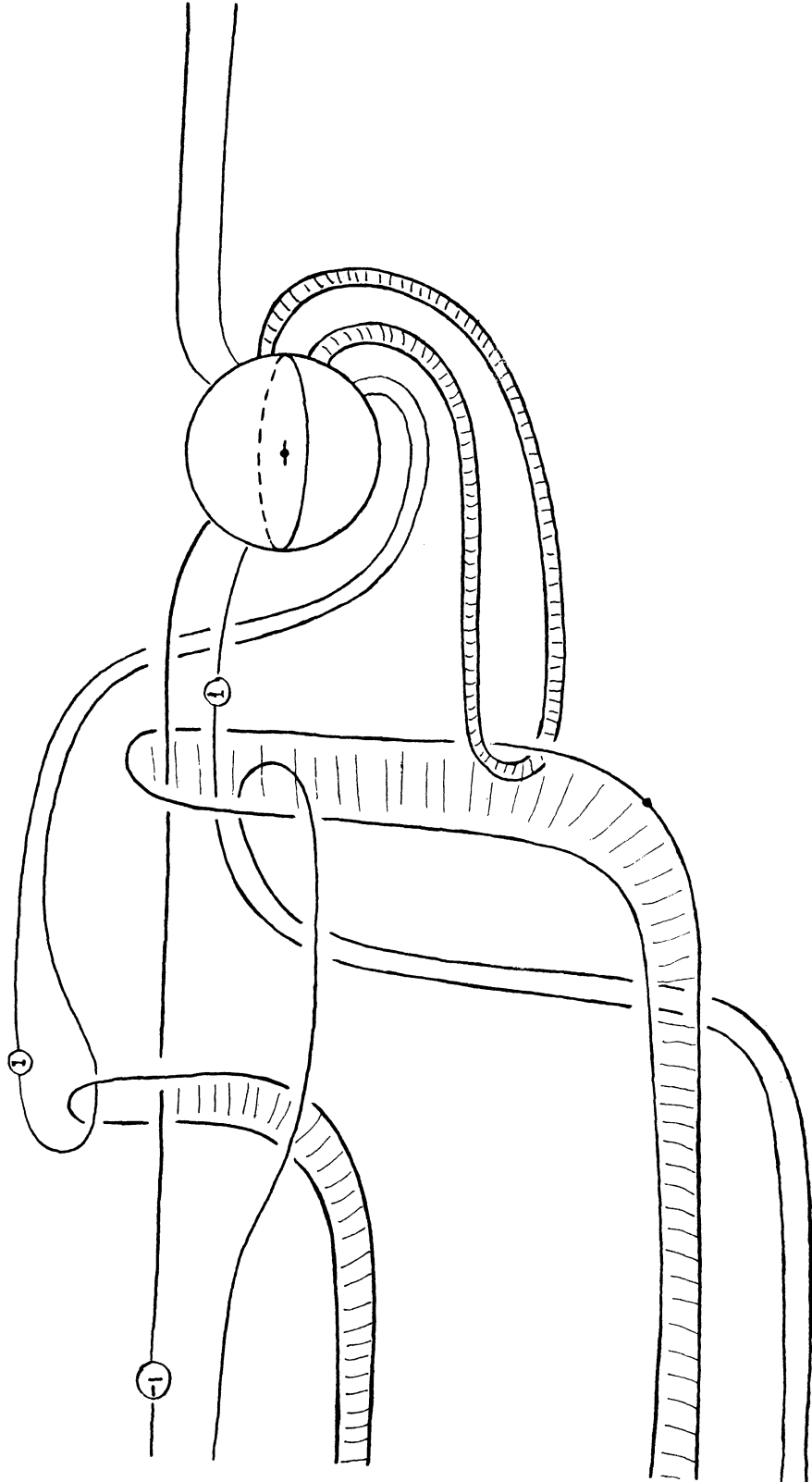


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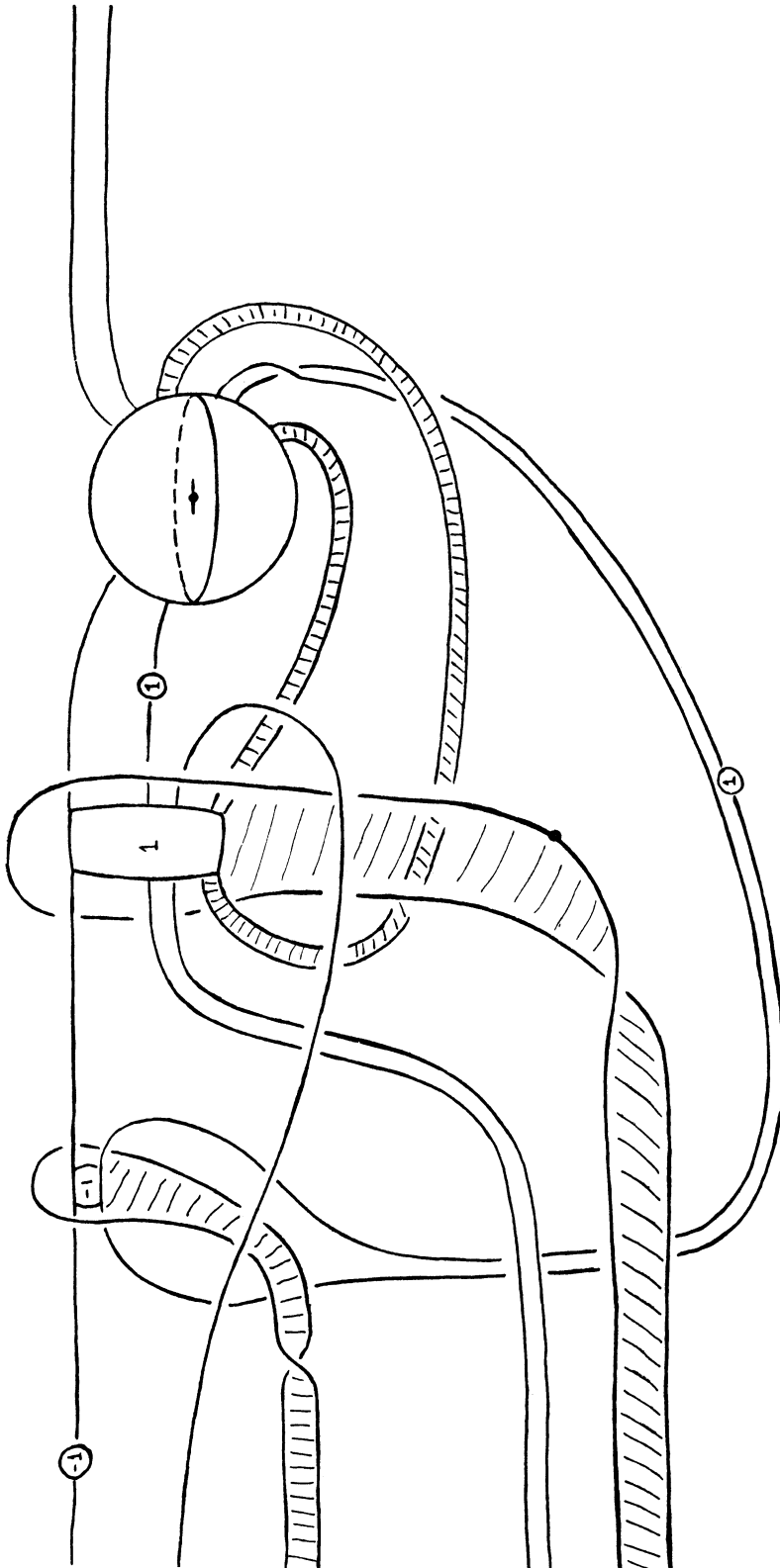


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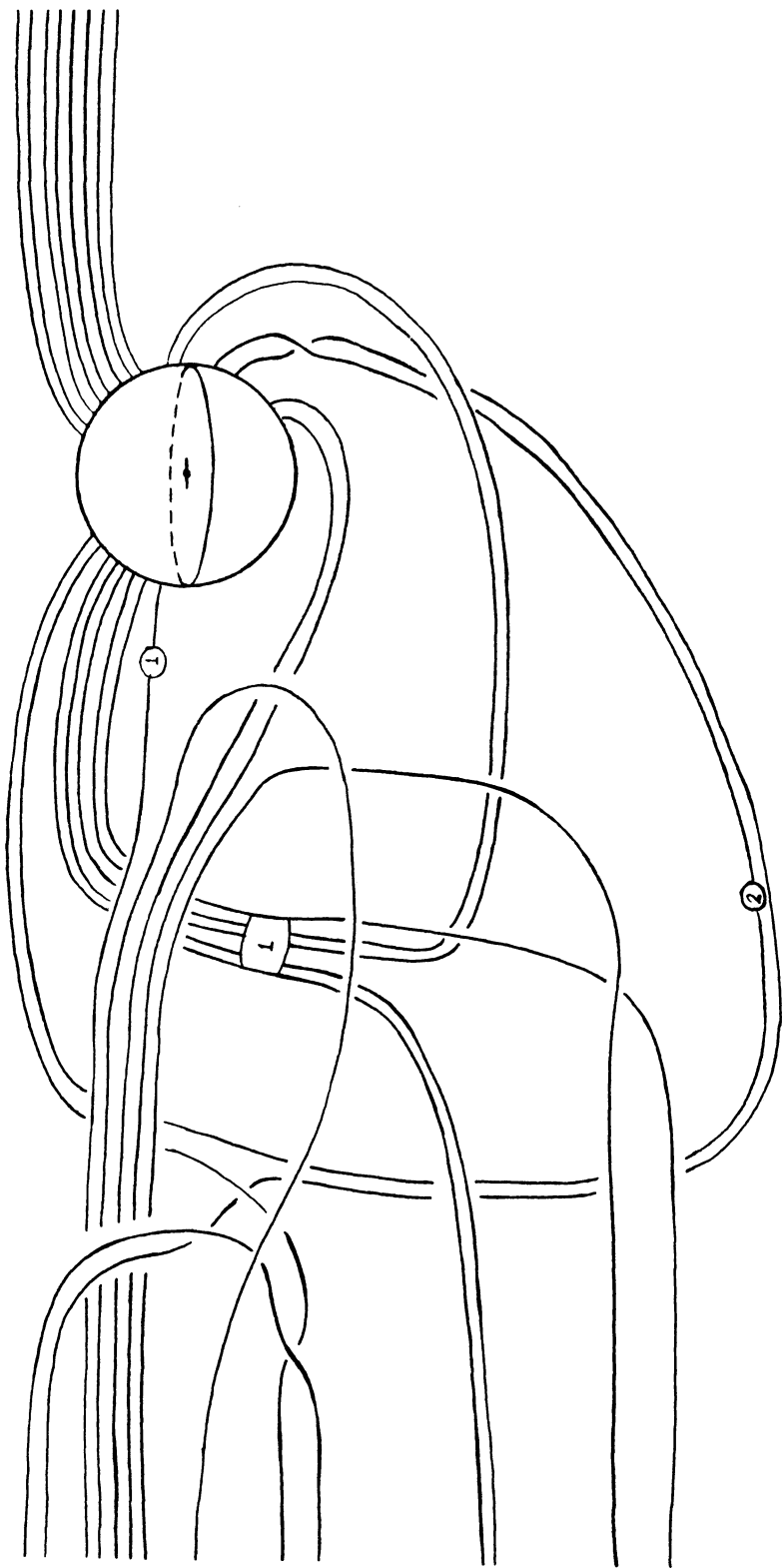


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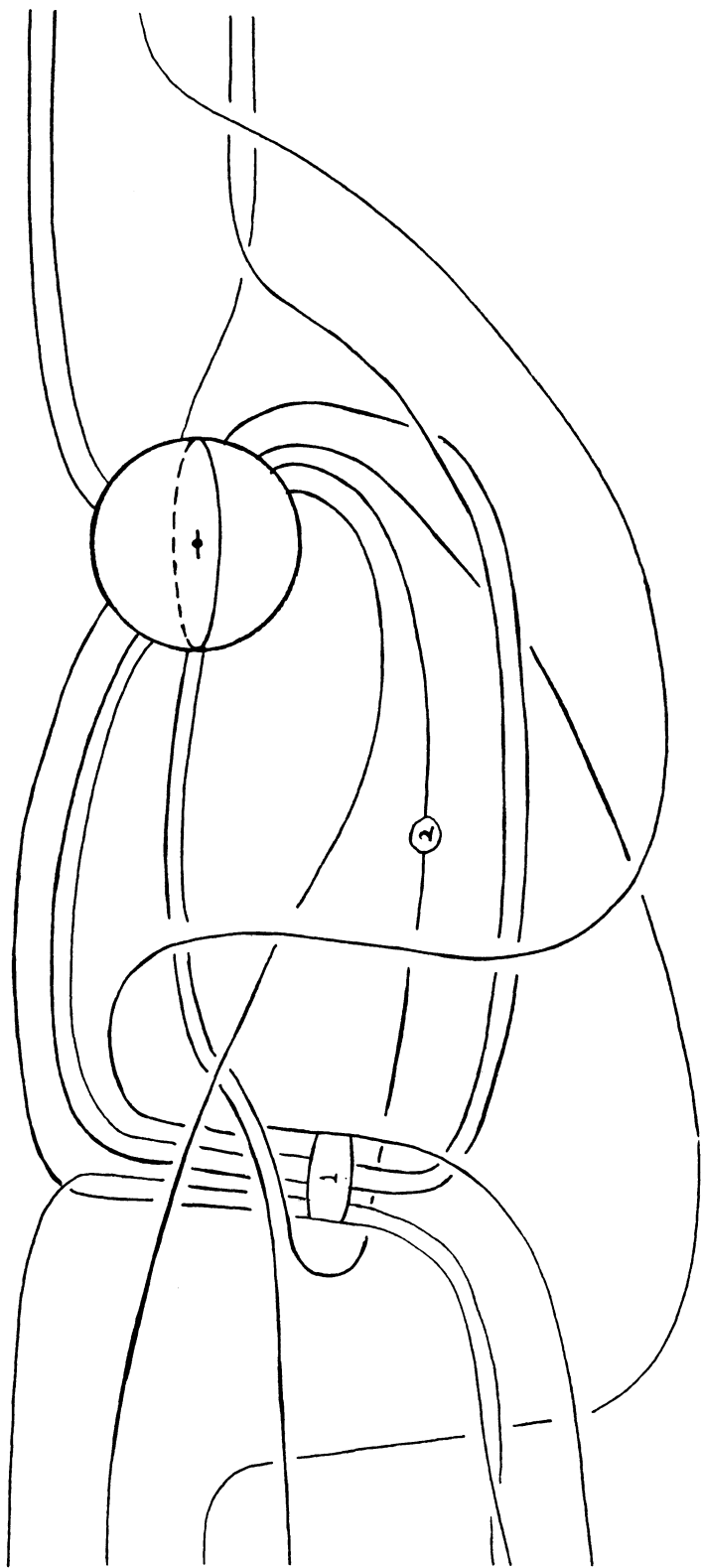


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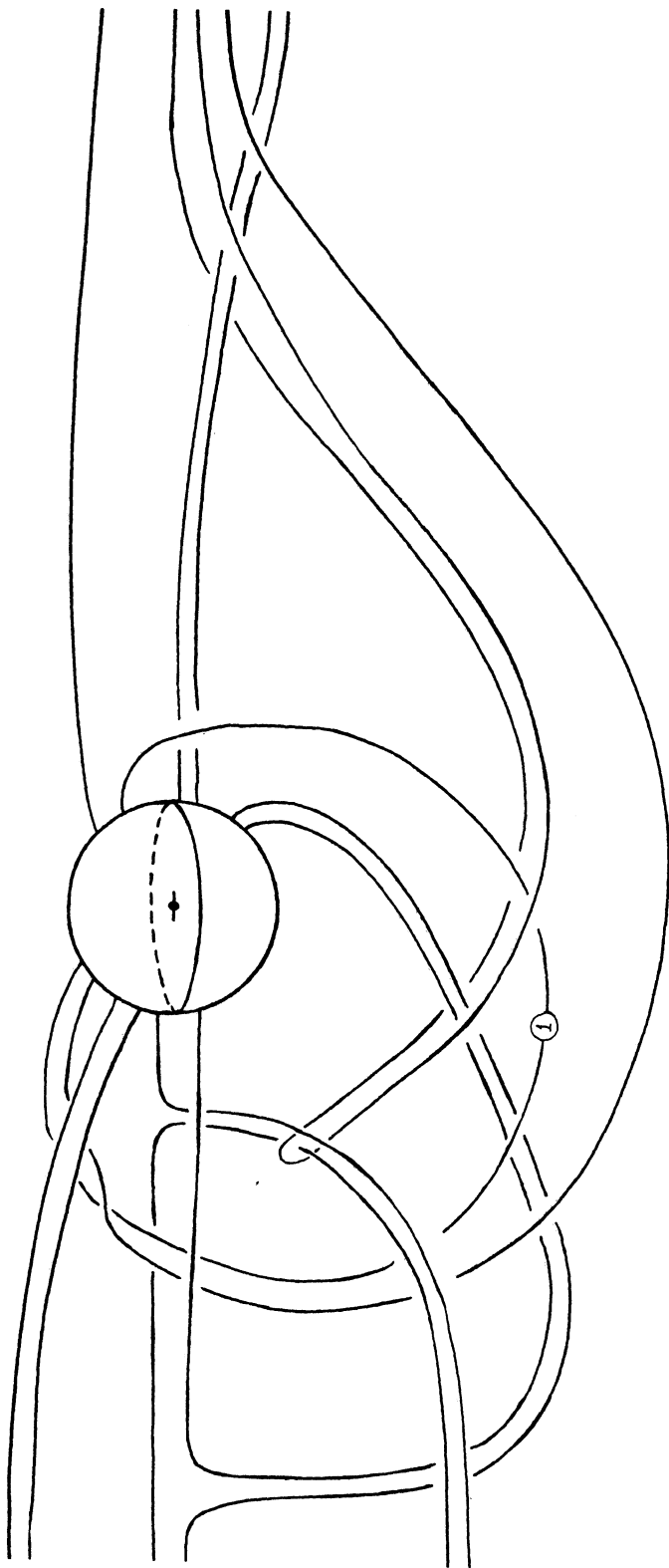


Figure 4.5

Figure 4.6

