

ON FAKE $S^3 \tilde{\times} S^1 \# S^2 \times S^2$

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In [A] we have constructed a fake $S^3 \tilde{\times} S^1 \# S^2 \times S^2$, that is a closed smooth manifold M^4 which is simple homotopy equivalent to $S^3 \tilde{\times} S^1 \# S^2 \times S^2$ but not diffeomorphic to it. In fact, they are not even normally cobordant to each other. Here $S^3 \tilde{\times} S^1$ denotes the twisted S^3 bundle over S^1 . It is interesting to note that by [FQ] M^4 is topologically standard. Here we announce a curious property of M^4 ; i.e. M^4 is obtained by twisting $S^3 \tilde{\times} S^1 \# S^2 \times S^2$ along an imbedded 2-sphere (Gluck construction). This implies that $M \# CP^2$ is standard. The proof of this fact is long; it will appear elsewhere. Instead here we specifically identify this imbedding of 2-sphere by drawing a picture. We end the paper by giving a quick alternative construction of a fake $S^3 \tilde{\times} S^1 \# S^2 \times S^2$. We would like to thank Larry Taylor for useful conversations on surgery theory.

§1. Notations

Recall that we can visualize the handlebody structures of connected smooth 4-manifolds by positioning ourselves at the boundary of the zero handle ($= S^3$). We see the attaching $S^0 \times B^3$ of a 1-handle $B^1 \times B^3$ as a pair of balls, and the attaching $S^1 \times B^2$ of a 2-handle $B^2 \times B^2$ as a framed knot. The framed knots are allowed to go over the 1-handles, hence, we only see the part of them which lie in S^3 , i.e., arcs entering one of the balls and leaving from the others as in Figure 1. The framings are determined by the vector field on the plane of the paper (plus the number of twists added when indicated as in Figure 1.)

There are many ways to attach 1-handles, here we use only one of them; namely, if we visualize coordinate axis at the centers of the balls, they are identified via $B^1 \times B^3$ by the map $(x, y, z) \rightarrow (x, -y, -z)$. Since 3 and 4- handles are attached the standard way, we don't need to visualize them. For more systematic description of 4-dimensional handlebodies, we refer the reader to [A].

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§2. Gluck construction

We start with the handlebody picture of $S^3 \tilde{\times} S^1 \# S^2 \times S^2$ which is $B_+^3 \tilde{\times} S^1 \# S^2 \times S^2 \cup_{\partial} B_-^3 \tilde{\times} S^1$. We draw $B_+^3 \tilde{\times} S^1 \# S^2 \times S^2$ in Figure 2. Now, we describe an imbedding of S^2 into $S^3 \tilde{\times} S^1 \# S^2 \times S^2$ as follows. Figure 3 is a picture of an imbedding $D_+^2 \hookrightarrow B_+^3 \tilde{\times} S^1 \# S^2 \times S^2$ (the shaded disc) such that $\gamma = \partial D_+^2$ is an unknot on the boundary ($= S^2 \tilde{\times} S^1$). One can verify the last claim by simply

cancelling $\circlearrowleft \circlearrowright$ ($= S^2 \times S^2$) by surgering one of the 2-spheres and tracing γ in the picture. Since γ is an unknot, it bounds a trivial 2-disc D_-^2 in $B_-^3 \tilde{\times} S^1$. Then let $K^2 = D_+^2 \cup_{\partial} D_-^2$, K^2 is an imbedded 2-sphere in $S^3 \tilde{\times} S^1 \# S^2 \times S^2$.

THEOREM 1. M^4 is obtained by doing the Gluck construction to $S^3 \tilde{\times} S^1 \# S^2 \times S^2$ along K . Namely,

$$M^4 = (S^3 \tilde{\times} S^1 \# S^2 \times S^2 - \text{int}(K \times D^2)) \cup_{\varphi} K \times D^2$$

where $K \times D^2$ is the closed tubular neighborhood of K , and φ is the self diffeomorphism of $\partial(K \times D^2) = S^2 \times S^1$ given by $(x, y) \rightarrow (\alpha(y)x, y)$ with $\alpha \in \pi_1 SO_3$ is the nontrivial element.

Figure 4 is the handlebody picture of M^4 .

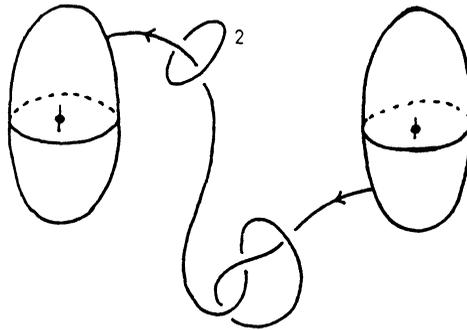


FIGURE 1



FIGURE 2

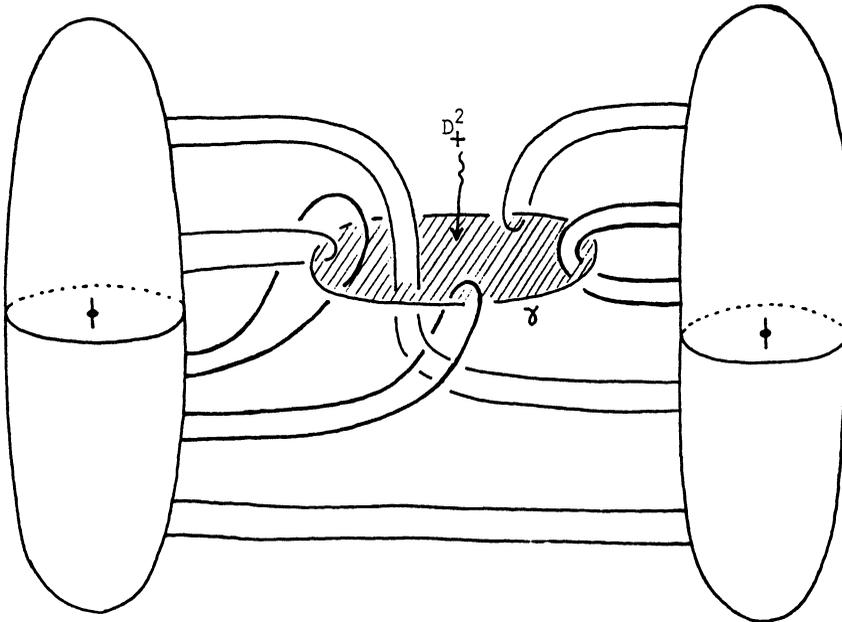
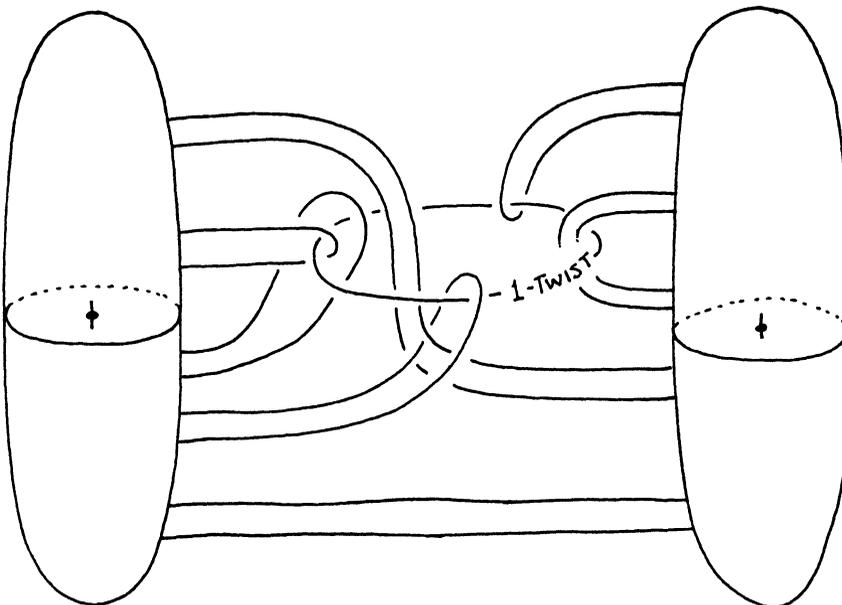


FIGURE 3



Here the above notation means that all the arcs going through γ are twisted once with a full right handed twist.

FIGURE 4

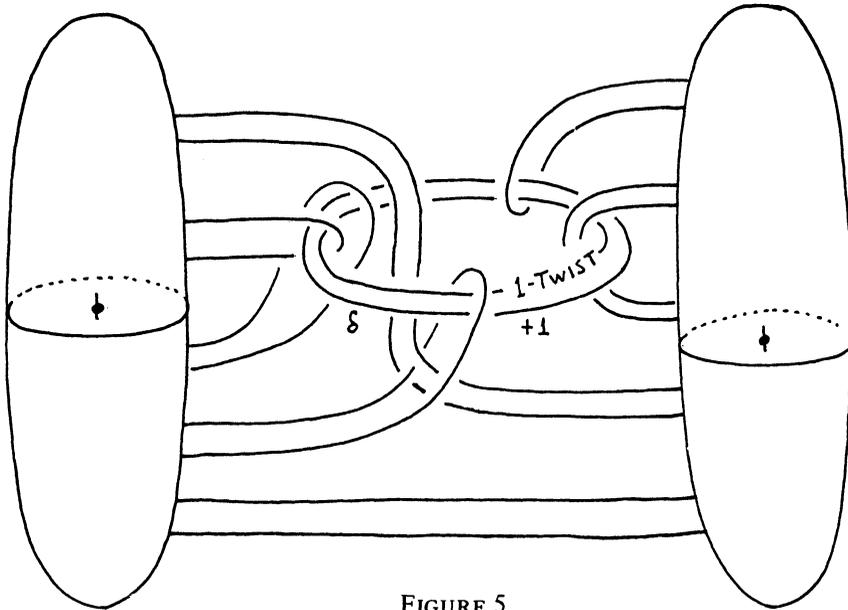


FIGURE 5

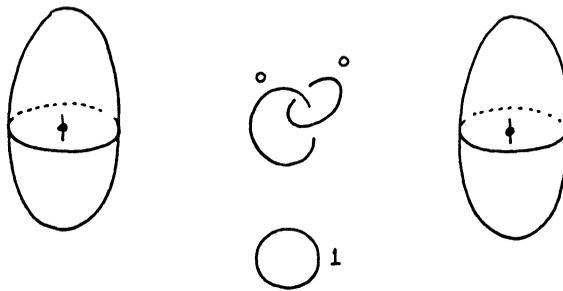


FIGURE 6

COROLLARY 2. $M \# CP^2 = S^3 \tilde{\times} S^1 \# S^2 \times S^2 \# CP^2$.

PROOF. Since ∂D_+ is an unknot in $S^2 \tilde{\times} S^1$ the circle δ in Figure 5 is also an unknot in $S^2 \tilde{\times} S^1$. $M \# CP^2$ is obtained by attaching a 2-handle to an unknot of Figure 4 with +1 framing. Hence, if we attach a 2-handle with +1 framing along δ as in Figure 5, we get $M \# CP^2$; but then by sliding all 2-handles over this new 2-handle gives Figure 6 which is (along with $B^3 \tilde{\times} S^1$) is $S^3 \tilde{\times} S^1 \# S^2 \times S^2 \# CP^2$. \square

§3. A quick construction of a fake $S^3 \tilde{\times} S^1 \# S^2 \times S^2$

By attaching two 2-handles to B^4 we can obtain a smooth W^4 , with $W \simeq S^2 \times S^2 - B^4$ and $\partial W = \Sigma^3 \# \Sigma^3$, where Σ^3 is a Rochlin invariant 8 homology sphere. To see this, start with the 4-manifold N^4 of Figure 7. $\partial N = \Sigma^3 \# \Sigma^3$, where $\Sigma^3 = \Sigma(2, 3, 7)$ is the Rochlin invariant 8 homology sphere which is the

link of the singularity $z_1^2 + z_2^3 + z_3^7 = 0$ in \mathbb{C}^3 . Attach 2-handles to N^4 as in Figure 8 to obtain $Q^4 = N^4 \cup h_1^2 \cup h_2^2$ with $\partial Q = S^3$. The fact that $\partial Q = S^3$ can be seen by the handle slides indicated in Figure 9. To obtain W^4 , we start with B^4 and attach the dual 2-handles of h_1^2 and h_2^2 to ∂B^4 , then clearly $\partial W = \partial N = \Sigma \# \Sigma$ and $W \simeq S^2 \times S^2 - B^4$ (check framings).

By attaching a 3-handle H^3 to W along the obvious S^2 of ∂W , we get $W_1 = W \cup h^3$ with $\partial W_1 = \Sigma^3 \cup \Sigma^3$. Let Q^4 be the closed manifold obtained by identifying the two boundary components of W_1 .

THEOREM 3. Q^4 is a fake $S^3 \tilde{\times} S^1 \# S^2 \times S^2$.

PROOF. From the obvious homotopy equivalence $W_1 \rightarrow S^3 \times I \# S^2 \times S^2$, we get a simple homotopy equivalence $f : Q^4 \rightarrow S^3 \tilde{\times} S^1 \# S^2 \times S^2$ with the property that Σ^3 is the transverse inverse image of a fibre $S^3 \subset S^3 \tilde{\times} S^1 \# S^2 \times S^2$. Consider the map $\bar{f} : Q \rightarrow S^3 \tilde{\times} S^1$ obtained by pinching $S^2 \times S^2$ to a point; then $(Q, \bar{f}) \in [S^3 \tilde{\times} S^1; G/0]$ which is the normal cobordism classes of degree 1 normal maps to $S^3 \tilde{\times} S^1$. According to [CS] $\alpha(Q, \bar{f}) = 2\mu(\Sigma) -$

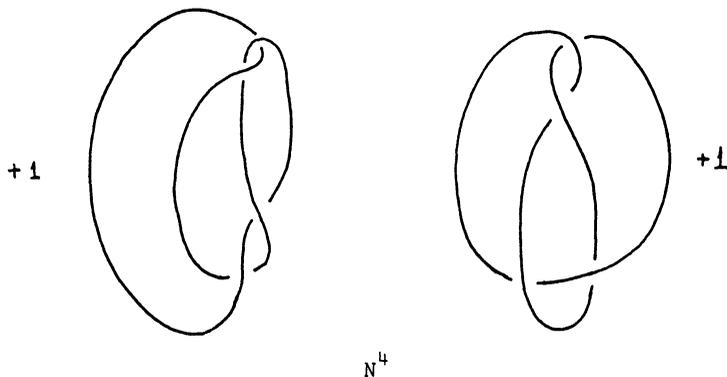


FIGURE 7

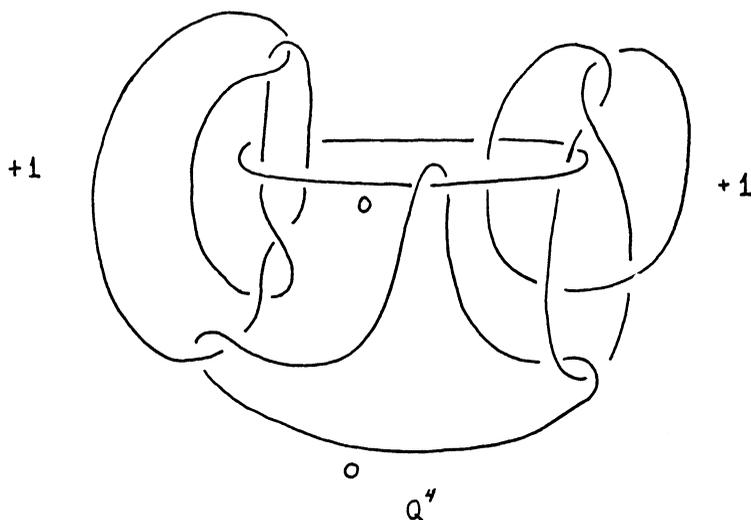


FIGURE 8

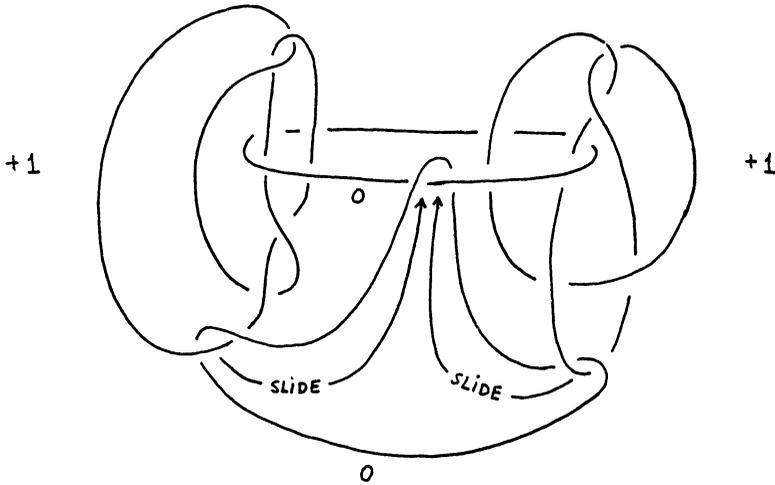


FIGURE 9

$\text{Sign}(W_1) \pmod{32} = 16 \pmod{32}$ is an invariant of $[S^3 \tilde{\times} S^1; G/0]$, where $\mu(\Sigma)$ denotes the Rochlin invariant of Σ , and $\text{Sign}(W)$ denotes the signature of W .

We claim that Q cannot be s -cobordant to $S^3 \tilde{\times} S^1 \# S^2 \times S^2$. If there was an s -cobordism H^5 with $\partial H = H_+ \cup H_-$, $H_- = Q$ and $H_+ = S^3 \tilde{\times} S^1 \# S^2 \times S^2$; by attaching a 3-handle to H_+ , we get a cobordism \bar{H} with $\partial \bar{H} = Q \cup S^3 \tilde{\times} S^1$. Since $\pi_2(S^3 \tilde{\times} S^1) = 0$ the map \bar{f} extends to a degree 1 normal map $F : \bar{H} \rightarrow S^3 \tilde{\times} S^1$. Let $F|_{S^3 \tilde{\times} S^1} = h$, then $\alpha(S^3 \tilde{\times} S^1, h) = \alpha(Q, \bar{f}) \neq 0$. Let L^3 be the framed manifold which is a transverse inverse image of S^3 under h , let C be a closed tubular neighborhood of L^3 and let $Z^4 = S^3 \tilde{\times} S^1 - \text{int}(C)$. By Mayer-Vietoris sequence of Z and C we get $H_2(\partial Z) \rightarrow H_2(Z)$ onto which implies that $\text{Sign}(Z) = 0$. By taking the universal cover $S^3 \times \mathbb{R}$ of $S^3 \tilde{\times} S^1$ we get an imbedding of L^3 into $S^3 \times I$, L^3 separates $S^3 \times I$ into two parts N_1 and N_2 . Then by applying Mayer-Vietoris to N_1, N_2 we get $H_2(\partial N_1) \rightarrow H_2(N_1)$ onto, hence $\text{Sign}(N_1) = 0$. Therefore $\mu(L^3) = \text{Sign}(N_1) \pmod{16} = 0$. By [CS] we compute

$$\begin{aligned} \alpha(S^3 \tilde{\times} S^1, h) &= 2\mu(L^3) - \text{Sign}(N) \pmod{32} \\ &= 0 \qquad \qquad \qquad \text{contradiction } \square \end{aligned}$$

Remark: The proof also shows that (Q, \bar{f}) cannot be normally cobordant to $(S^3 \tilde{\times} S^1 \# S^2 \times S^2, \bar{a})$.

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