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A resolution theorem for homology cycles of real algebraic varieties

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Let V be a compact nonsingular real algebraic set then the main result of this paper (Theorem 13) implies the following resolution theorem for homology cycles.

Theorem. *There exists an algebraic resolution* $\pi: \tilde{V} \to V$ (i.e. *a composition of algebraic blowups along nonsingular centers and rational diffeomorphisms*) *such that* $H_k(\tilde{V}; \mathbb{Z}_2)$ *is generated by components of k-dimensional compact nonsingular algebraic subsets of* $\tilde{V}$ *for all k.*

The theorem is the best possible in the sense that the components of nonsingular algebraic subsets in the conclusion cannot be replaced by nonsingular algebraic subsets (see Remark 14). It is well known that the topology of the singularities of $\mathbb{Z}_2$-homology cycles is closely related to real algebraic structures on V [AK_2, AK_3]. Therefore it is natural to study to what extent general homology cycles differ from the cycles induced by submanifolds. The main theorem of this paper implies that all the singularities of homology cycles can be smoothed by a blowing-up process. This result should be viewed as a homology version of the resolution theorem [H]. The reason this result is not an easy consequence of [H] is that not all homology cycles of V may be represented by algebraic subsets (there are real obstructions, e.g. [AK_2]; even if they were, the map resolving them may introduce new bad homology cycles upstairs. The result turns out to be a useful device in topologically classifying real algebraic sets [AK_6], it also gives interesting Corollaries in algebraic topology ([AK_2]). Another result of this paper (Theorem 4) says that any smooth map $f: W \to V$ between nonsingular algebraic sets can be approximated by a rational map after changing W by a rational diffeomorphism, if the homotopy class of f contains a rational representative; this strengthens Proposition 2.3 of [AK_5].

For given algebraic sets $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$ a function $f: V \to W$ is called an *entire rational function* if $f(x) = P(x)/Q(x)$ where $P: \mathbb{R}^n \to \mathbb{R}^m$ and $Q: \mathbb{R}^n \to \mathbb{R}$

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are polynomials and $Q \neq 0$ on V. An entire rational function $f: V \to W$ is a *birational isomorphism* if there is an entire rational function $g: W \to V$ so that $f \circ g$ and $g \circ f$ are the identities on W and V respectively. Generally speaking we consider real algebraic sets to be the same if they are birationally isomorphic.

If V and W are nonsingular and $f: V \to W$ is an entire rational function which is also a diffeomorphism, we call f a *rational diffeomorphism.* Note that the inverse of rational diffeomorphism need not be a rational function, for example if $V = \{(x, y) \in \mathbb{R}^2 \mid y^3 + y = x\}$ and $f: V \to \mathbb{R}$ is given by $f(x, y) = x$.

If $M \subset V$ is an m-dimensional closed submanifold or an m-dimensional compact algebraic subset then [M] denotes the homology class induced by inclusion in $H_m(V; \mathbb{Z}_2)$ (see [AK_2]). If $X \subset V$ is a subset of an algebraic set V then the smallest algebraic set containing X is called the Zariski closure of X.

Definition. A nonsingular algebraic subset L of a nonsingular algebraic set V is called a *stable algebraic subset* if there are compact nonsingular algebraic subsets $\{V_i\}_{i=0}^r$ such that

1) $L = V_0 \subset V_1 \subset \dots V_{r-1} \subset V_r = V$
2) $\dim(V_{i+1}) = \dim(V_i) + 1$, for all i.

If a nonsingular algebraic set L is a component of the intersection of condimension one compact nonsingular algebraic subsets which intersect transversally, then V is a stable algebraic set.

Definition. Let V be a nonsingular algebraic set, then define the following subgroups of $H_k(V; \mathbb{Z}_2)$:

$H_k^A(V; \mathbb{Z}_2)$ = Subgroup generated by *algebraic subsets* of V

$AH_k(V; \mathbb{Z}_2)$ = Subgroup generated by *stable algebraic subsets* of V

$A_0H_k(V; \mathbb{Z}_2)$ = Subgroup generated by *connected components of stable algebraic subsets* of V

$H_k^{imb}(V; \mathbb{Z}_2)$ = Subgroup generated by *closed smooth submanifolds* of V.

We have the inclusions: $AH_k(V; \mathbb{Z}_2) \hookrightarrow A_0H_k(V; \mathbb{Z}_2) \hookrightarrow H_k^{imb}(V; \mathbb{Z}_2)$, and also $AH_k(V; \mathbb{Z}_2) \hookrightarrow H_k^A(V; \mathbb{Z}_2)$. Notice if $v = \dim(V)$, $AH_{v-1}(V; \mathbb{Z}_2)$ is just the subgroup generated by codimension one nonsingular algebraic subsets of V. It turns out that $H_k^A(V; \mathbb{Z}_2)$ is also the subgroup generated by $f_*[M]$ where M is a k-dimensional compact nonsingular algebraic set and $f: M \to V$ is an entire rational function. Also any rational diffeomorphism $\phi: V \to W$ between nonsingular algebraic sets induces an isomorphism $\phi_*: H_*^A(V; \mathbb{Z}_2) \xrightarrow{\cong} H_*^A(W; \mathbb{Z}_2)$; see [$AK_2$] for more discussion of these groups.

Proposition 1. *Let V^v be an algebraic set and let $M \subset V$ be a codimension one closed smooth submanifold which is contained in a nonsingular component of V. Then M is isotopic to a nonsingular algebraic subset of V by an arbitrarily small isotopy if and only if $[M] \in AH_{v-1}(V; \mathbb{Z}_2)$.*

Proof. Theorem 4.1 of [AK_2].

Proposition 2. *Let V, W be nonsingular algebraic sets and let $L \subset V$ be a nonsingular algebraic subset of V. Let $f: V \to W$ be a smooth function such that $f|_L = \phi$ where $\phi: L \to W$ is an entire rational function. Then there exists an* *algebraic set* $Z \subset V \times \mathbb{R}^n$ *for some* n, *a nonsingular component* Z_0 *of* Z, *and an entire rational function* $F: Z \to W$ such that

1) $f \circ \pi|_{Z_0}$ *is arbitrarily close to* $F|_{Z_0}$ *where* π *is the projection* $V \times \mathbb{R}^n \to V$,
2) $\pi|_{Z_0}: Z_0 \to V$ *is a diffeomorphism.*
3) $L \times 0 \subset Z_0$.
4) $F|_{L \times 0} = \phi$.

Proof. This is a special case of the normalization theorem of [AK_4] (Proposition 2.8 of [AK_1] or Lemma 2.9 of [AK_4]). □

Proposition 3. *If* V *is a nonsingular algebraic set and* $M \subset V$ *is a stable algebraic subset and* $K \subset V$ *is a smooth subcompiex, then there exists a small isotopy* $f_t: M \hookrightarrow V$ *with* $f_0(M) = M$ *so that* $f_1(M)$ *is a stable algebraic subset transverse to* K.

Proof. Proposition 4.3 of [AK_2]. □

Theorem 4. *Let* V, W *be nonsingular algebraic sets and let* $f: V \to W$ *be a smooth map which is homotopic to an entire rational function. Then there exists a nonsingular algebraic set* V', *an entire rational function* $\phi: V' \to W$, *and a rational diffeomorphism* $\alpha: V' \xrightarrow{\cong} V$ *such that* $f \circ \alpha$ *is arbitrarily close to* ϕ.

Proof. Let $G: V \times I \to W$ be a homotopy with $G(x,0) = f(x)$ and $G(x,1) = \beta(x)$ where $\beta: V \to W$ is an entire rational function. By doubling G we get a smooth function $H: V \times S^1 \to W$ with $H(x,a) = f(x)$ and $H(x,b) = \beta(x)$ where $a, b \in S^1$. By Proposition 2, there is a nonsingular algebraic set Z and entire rational functions $F: Z \to W$ and $\pi: Z \to V \times S^1$ and a nonsingular component Z_0 of Z such that $\pi|_{Z_0}$ is a diffeomorphism and $H \circ \pi|_{Z_0}$ is close to $F|_{Z_0}$ and $V = V \times b \subset Z_0$.

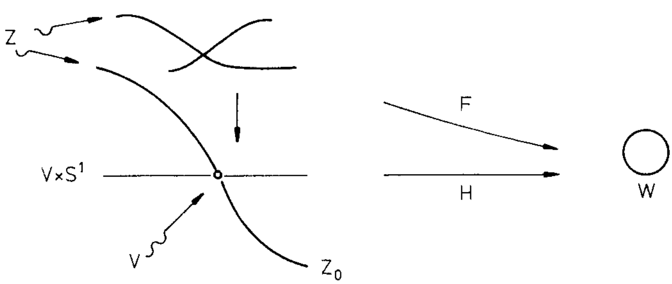


Fig. 1

Let $M = (\pi|_{Z_0})^{-1}(V \times a) \subset Z_0$, M is a diffeomorphic copy of V in Z_0. Since $[M] = [V] \in AH_v(Z; \mathbb{Z}_2)$, by Proposition 1 M is isotopic to a nonsingular algebraic subset V' of Z by a small isotopy. Call $F|_{V'} = \phi$, $\alpha = p \circ \pi|_{V'}: V' \to V$ where $p: V \times S^1 \to V$ is the projection. Then α is a rational function which is a diffeomorphism since $\pi(V')$ is a nearby copy of $V \times a \hookrightarrow V \times S^1$. Since $f \circ p|_{V \times a} = H|_{V \times a}$, $f \circ \alpha = f \circ p \circ \pi|_{V'} \sim H \circ \pi|_{V'} \sim F|_{V'} = \phi$, where $\sim$ means close to. We are done. □

Lemma 5. *Let V, L and L' be nonsingular algebric sets with $L\subset V$ and L compact and let $f: L'\to L$ be a rational diffeomorphism. Then there exist non-singular algebraic sets V' and L'' and a rational diffeomorphism $F: V'\to V$ and a birational isomorphism $G: L''\to L'$ so that $L''\subset V'$ and $F|_{L''}=f\circ G$.*

Proof. Suppose $L'\subset \mathbb{R}^n$. Pick a smooth function $\alpha: V\to\mathbb{R}^n$ with compact support so that $\alpha|_L=f^{-1}$. Then $\bar{V}=\{(\alpha(x),x)|x\in V\}\subset\mathbb{R}^n\times V$ is a diffeomorphic copy of V. $\bar{V}$ contains the algebraic subset $\bar{V}\cap(L'\times V)$ birationally isomorphic to L', and we let $L''=\bar{V}\cap(L'\times V)$. $\bar{V}$ has a trivial normal bundle in $\mathbb{R}^n\times V$.

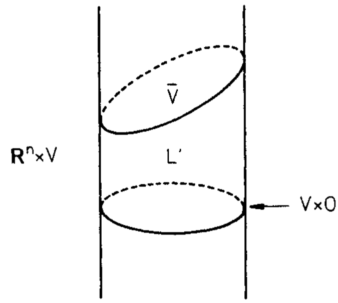


Fig. 2

Let $\rho: \mathbb{R}^n\times V\to\mathbb{R}$ be a proper polynomial with $\rho^{-1}(0)=L''$ (c.f. Lemma 2.4 of [AK$_4$]). Pick a smooth function $\beta: \mathbb{R}^n\times V\to[0,1]$ with compact support and $\beta(x,y)=1$ when $|x|<|\alpha(y)|$. Define $\gamma(x,y)=\beta(x,y)(x-\alpha(y))+(1-\beta(x,y))\rho^2(x,y)x$, then $\gamma: \mathbb{R}^n\times V\to\mathbb{R}^n$ has the property that γ is transverse to 0, $\gamma^{-1}(0)=\bar{V}$ and γ equals a polynomial $\rho^2(x,y)x$ outside a compact set. By Lemma 2.1 of [AK$_1$], $\gamma(x,y)-\rho^2(x,y)x$ can be approximated by an entire rational function $u: (\mathbb{R}^n\times V, L'')\to(\mathbb{R}^n,0)$ on all of $\mathbb{R}^n\times V$. Then $w(x,y)=u(x,y)+\rho^2(x,y)x$ is an entire rational function approximating $\gamma(x,y)$ on $\mathbb{R}^n\times V$. Let $V'=w^{-1}(0)$ then $L''\subset V'$. The properness of ρ insures that $V'=w^{-1}(0)$ is isotopic to $\bar{V}=\gamma^{-1}(0)$ fixing L''. Let $F: V'\to V$ be induced by projection $\mathbb{R}^n\times V\to V$ and we are done. □

We will need some facts about blowing up. Let V be a real algebraic set and let ϑ be an ideal in either the ring of polynomial functions on V or the ring of entire rational functions on V. Then we may define $B(V,\vartheta)$ as follows. Pick generators $p_1,\dots,p_m$ of ϑ. Then $B(V,\vartheta)$ is the Zariski closure of $\{(x,[u_1:\dots:u_m])\in V\times\mathbb{R}P^{m-1}|u_i=p_i(x),\ i=1,\dots,m$ and $p_j(x)\neq 0$ for some $j=1,\dots,m\}$. We may realize this as a real algebraic set and up to birational isomorphism it is independent of the generators chosen. We have a canonical projection $\pi(V,\vartheta): B(V,\vartheta)\to V$ induced by the projection $V\times\mathbb{R}P^{m-1}\to V$.

If $L\subset V$ is a real algebraic subset we define $\vartheta_V(L)$ to be the ideal of entire rational functions $p: V\to\mathbb{R}$ so that $p|_L=0$. Then we define $B(V,L)$ and $\pi(V,L)$ to be $B(V,\vartheta_V(L))$ and $\pi(V,\vartheta_V(L))$ respectively. It is not hard to show that $B(V,\vartheta_V(L))$ and $B(V,\vartheta)$ are birationally isomorphic if ϑ is the ideal of polynomials vanishing on L. We call L the *center* of the blowup.

If V and L are nonsingular then $B(V, L)$ is nonsingular also and $\pi(V, L)^{-1}(L)=P$ is a codimension one nonsingular algebraic subset of $B(V, L)$ which is diffeomorphic to the projectivized normal bundle of L in V. The map $\pi(V, L)=\pi$ is a diffeomorphism in the complement of P and crushes P fibre-wise to L. If $M \subset V$ is a closed subset then $\tilde{M}=\text{closure }(\pi^{-1}(M-L))$ is called the *strict preimage* of M under the blowing up projection $\pi: B(V, L) \to V$. In case M is a nonsingular algebraic subset transverse to L, then $\tilde{M}$ is a nonsingular algebraic set, in fact $\tilde{M}=B(M, M \cap L)$.

If $f: V' \to V$ is a rational diffeomorphism and $M \subset V$ we define the strict preimage of M to be $f^{-1}(M)$.

Definition. Let V be a nonsingular real algebraic set. Then $\pi: \tilde{V} \to V$ is an *uzunblowup* if π is the composition

$$\tilde{V}=V_n \xrightarrow{\pi_n} V_{n-1} \xrightarrow{\pi_{n-1}} \dots \longrightarrow V_1 \xrightarrow{\pi_1} V_0=V$$

where each π_{i+1} is either a rational diffeomorphism or a blowup of V_i along some nonsingular center $L_i \subset V_i$. We call $\{L_i\}$ the centers of $\pi: \tilde{V} \to V$.

A *multiblowup* is an uzunblowup where each π_i is a blowup. If $X \subset V$ is an algebraic subset then we define $\tilde{X} \subset \tilde{V}$ to be the *strict preimage* of X under π if $\tilde{X}$ is the strict preimage of $X_{n-1} \subset V_{n-1}$ under π_n, and X_{n-1} is the strict preimage of X under $\pi_1 \circ \pi_2 \circ \dots \circ \pi_{n-1}$.

If $V \subset W$ are nonsingular algebraic sets then any multiblowup $\pi: \tilde{V} \to V$ induces a multiblowup $\pi': \tilde{W} \to W$. $\tilde{V}$ is in fact the strict preimage of V under π'. Because of Lemma 5, the same statement is true for uzunblowups. The induced multiblowup is unique up to birational diffeomorphism, but an induced uzunblowup is not. It depends on some choice every time we apply Lemma 5.

Proposition 6. *Let V be a nonsingular algebraic set and let $X \subset V$ be a compact algebraic subset. Then there exists a multiblowup $\pi: \tilde{V} \to V$ such that the strict preimage $\tilde{X} \subset \tilde{V}$ of X is a stable algebraic subset.*

Proof. By [H] there exists a multiblowup $\pi_1: \bar{V} \to V$ such that the strict preimage $\bar{X}$ of X nonsingular. By Theorem 3.5 of [AK_2] there exists a multiblowup π_2: $\tilde{V} \to \bar{V}$ such that the strict preimage $\tilde{X}$ of $\bar{X}$ is a stable algebraic subset. Then $\pi=\pi_1 \circ \pi_2$: $\tilde{V} \to V$ has the required property. □

Lemma 7. *Let $\pi: \tilde{V} \to V$ be an uzunblowup of a nonsingular algebraic set V, then*

(a) *The induced map π_* on $\mathbb{Z}_2$ homology is onto*

(b) *$A_0H_k(V; \mathbb{Z}_2) \subset \pi_* A_0H_k(\tilde{V}; \mathbb{Z}_2)$, for all k*

(c) *$AH_k(V; \mathbb{Z}_2) \subset \pi_* AH_k(\tilde{V}; \mathbb{Z}_2)$, for all k.*

Proof. (a) holds since π is a degree one map with $\mathbb{Z}_2$-coefficients. Since rational diffeomorphisms pull back nonsingular algebraic sets to nonsingular algebraic sets it suffices to prove (b) when $\tilde{V}=B(V, L) \xrightarrow{\pi} V$ is just a single blowup along a nonsingular center. Pick $\theta \in A_0H_k(V; \mathbb{Z}_2)$. Then θ is represented by $\bigcup_\alpha Z_\alpha^0$ where each Z_α^0 is a component of a stable algebraic set Z_α. By Proposition 3, Z_α can be made to be transverse to L. Let $\tilde{Z}_\alpha$ be the strict preimage of Z_α. Then $\pi_*(\tilde{\theta}) = \theta$ where $\tilde{\theta}$ is the cycle represented by $\bigcup \tilde{Z}_\alpha^0$ where each $\tilde{Z}_\alpha^0$ is the component of $\tilde{Z}_\alpha$ lying over Z_α^0. The proof of (b) gives (c). □

Lemma 8. *Let $\tilde{V} \xrightarrow{\pi} V$ be an uzunblowup of a nonsingular algebraic set V with centers $\{L_i\}$. Then the following statements hold:*

(a) *If $H_*(L_i; \mathbb{Z}_2) = A_0 H_*(L_i; \mathbb{Z}_2)$ for all i then* $\ker \pi_* \subset A_0 H_*(\tilde{V}; \mathbb{Z}_2)$.

(b) *If $H_*(L_i; \mathbb{Z}_2) = AH_*(L_i; \mathbb{Z}_2)$ for all i then* $\ker \pi_* \subset AH_*(\tilde{V}; \mathbb{Z}_2)$.

Proof. We do induction on the 'length' n of the uzunblowup:

$$\tilde{V} = V_n \xrightarrow{\pi_n} V_{n-1} \longrightarrow \dots \longrightarrow V_1 \xrightarrow{\pi_1} V_0 = V.$$

Pick $\theta \in \ker \pi_*$ and let r be the largest number with $\pi(r+1)_*(\theta) = 0$ where $\pi(r+1) = \pi_{r+1} \circ \pi_{r+2} \circ \dots \circ \pi_n$. Then $0 \neq \pi(r+2)_*(\theta) \in \ker(\pi_{r+1})_*$. Hence $\pi_{r+1}: V_{r+1} \to V_r$ is a blowup along a nonsingular center L_r (i.e. it is not a rational diffeomorphism). Let $P_{r+1} = \pi_{r+1}^{-1}(L_r)$, then the homology exact sequences of pairs induces the following commutative diagram where the top and the bottom rows are exact

$$\begin{array}{ccccccccc}
\dots \longrightarrow & H_k(P_{r+1}) & \xrightarrow{i_*} & H_k(V_{r+1}) & \longrightarrow & H_{k+1}(V_{r+1}, P_{r+1}) & \longrightarrow \dots \\
& \downarrow & & \downarrow (\pi_{r+1})_* & & \downarrow \approx \text{excision} & \\
\dots \longrightarrow & H_k(L_r) & \longrightarrow & H_k(V_r) & \longrightarrow & H_{k+1}(V_r, L_r) & \longrightarrow \dots
\end{array}$$

where $i: P_{r+1} \hookrightarrow V_{r+1}$ is the inclusion. Hence kernel $(\pi_{r+1})_* \subset \operatorname{im}(i_*)$. Since $P_{r+1} \xrightarrow{\pi_{r+1}} L_r$ is the projectivized normal bundle, i.e. an $\mathbb{R}P^s$-bundle over L_r where $s+1 = \operatorname{codim}(L_r)$, $H^*(P_{r+1}; \mathbb{Z}_2)$ is generated by classes of the form $(\pi_{r+1})^*(\alpha) \cup \xi^i$ $i = 1, 2, \dots, s$, where $\alpha \in H^*(L_r; \mathbb{Z}_2)$ and $\xi \in H^1(P_{r+1}; \mathbb{Z}_2)$ is the first Stiefel-Whitney class of the normal bundle of P_{r+1} in V_{r+1}. This follows from Theorem 5.7.9 of [S] since the map $\eta: H^*(\mathbb{R}P^s, \mathbb{Z}_2) \to H^*(P_{r+1}, \mathbb{Z}_2)$ is a cohomology extension of the fiber where $\eta(\alpha^k) = \xi^k$, $\alpha \in H^1(\mathbb{R}P^s, \mathbb{Z}_2)$ is the generator. (To see that η is a cohomology extension of the fiber, take any fiber F in P_{r+1}. Then the restriction to F of the normal bundle of P_{r+1} is the canonical line bundle over F, so $i^*(\xi)$ is the first Stiefel-Whitney class of the canonical line bundle over F which is the generator of $H^1(F, \mathbb{Z}_2)$. Hence $i^* \circ \eta: H^*(\mathbb{R}P^s, \mathbb{Z}_2) \to H^*(F, \mathbb{Z}_2)$ is an isomorphism, where i is the inclusion map $F \subset P_{r+1}$.)

We can find a representative for the Poincare dual of ξ in the following way. First isotop P_{r+1} via a small isotopy to a submanifold P_{r+1}^1 in V_{r+1} which is transverse to P_{r+1}. Then $[P_{r+1} \cap P_{r+1}^1]$ is the Poincare dual of ξ in $H_*(P_{r+1}; \mathbb{Z}_2)$.

Take $\alpha \in H^*(L_r; \mathbb{Z}_2)$. Then since $H_*(L_r; \mathbb{Z}_2) = A_0 H_*(L_r; \mathbb{Z}_2)$ the Poincare dual of α is represented by some component Z^0 of a stable algebraic subset Z of L_r. Then the Poincare dual of $\pi_{r+1}^*(\alpha)$ is represented by $\pi_{r+1}^{-1}(Z^0)$ and the Poincare dual of $\pi_{r+1}^*(\alpha) \cup \xi^i$ is represented by $\pi_{r+1}^{-1}(Z^0) \cap P^1 \cap \dots \cap P^i$ where P^j

$j=1, \ldots, i$ are isotopic copies of P_{r+1} and the manifolds $\pi_{r+1}^{-1}(Z)$, $P^1, \ldots, P^i$ are in general position in V_{r+1}. By Proposition 1 we may assume that each P^j is a nonsingular real algebraic set. Hence $\pi_{r+1}^{-1}(Z^0) \cap P^1 \cap \ldots \cap P^i$ is a component of the stable algebraic subset $\pi_{r+1}^{-1}(Z) \cap P^1 \cap \ldots \cap P^i$. Therefore $\pi(r+2)_*(\theta) \in A_0 H_k(V_{r+1}; \mathbb{Z}_2)$. By Lemma 7(b) we can find $\theta_0 \in A_0 H_k(\tilde{V}; \mathbb{Z}_2)$ with $\pi(r+2)_*(\theta_0) = \pi(r+2)_*(\theta)$, hence $\theta - \theta_0 \in \ker \pi(r+2)_*$. Then by induction $\theta - \theta_0 \in A_0 H_k(\tilde{V}; \mathbb{Z}_2)$, therefore $\theta \in A_0 H_k(\tilde{V}; \mathbb{Z}_2)$. The proof of (b) is similar; it follows from the proof of (a). □

Lemma 9. *If V is a nonsingular algebraic set, then there exists a multiblowup π: $\tilde{V} \to V$ with*

$$\text{(1)} \quad \pi_* A_0 H_k(\tilde{V}; \mathbb{Z}_2) = H_k(V; \mathbb{Z}_2) \quad \textit{for all } k$$

$$\text{(2)} \quad \pi_* A H_k(\tilde{V}; \mathbb{Z}_2) = H_k^A(V; \mathbb{Z}_2) \quad \textit{for all } k.$$

Proof. Since for any multiblowup $\pi: \tilde{V} \to V$. π_* is onto and the elements of $A_0 H_k(V; \mathbb{Z}_2)$ lift to $A_0 H_k(\tilde{V}; \mathbb{Z}_2)$ (Lemma 7), the result (1) follows by repeated application of the following claim.

Claim. *Let $0 \neq \theta \in H_k(V; \mathbb{Z}_2)$. Then there exists a multiblowup π: $\tilde{V} \to V$, a k-dimensional stable algebraic subset Z^k of $\tilde{V}$ and a component Z_0^k of Z^k such that $\pi_*[Z_0] = \theta$.*

Proof of Claim. This is Theorem 6.1 of [AK$_2$], but for completeness we include the proof here. By Steenrod representability [T] we can choose a map f: $M^k \to V$ where M is a closed smooth manifold and $f_*[M] = \theta$. By transversality we may also assume that f is one to one almost everywhere. We can assume that M^k is a nonsingular algebraic set. By Proposition 2 we can find an algebraic set Q, and an entire rational function ϕ: $Q \to V$, and a component Q_0 of Q which is diffeomorphic to M such that $\phi_*[Q] = \theta$ and ϕ is one to one almost everywhere on Q_0. Now $\phi(Q_0)$ is a semi-algebraic set of dimension k. Let Y be the Zariski closure of $\phi(Q_0)$, then $\dim(Y) = k$. By Proposition 6, there exists a multiblowup π: $\tilde{V} \to V$ such that strict preimage Z of Y is a k-dimensional stable algebraic subset. Let $\tilde{Q}$ be the strict preimage of the nonsingular algebraic set $\Gamma_\phi = \{x, \phi(x)) \mid x \in Q\} \subset Q \times V$ under the multiblowup $id \times \pi$: $Q \times \tilde{V} \to Q \times V$. We have natural maps p and $\tilde{\phi}$ induced by projections $Q \times \tilde{V} \to Q \times V \to Q$ and $Q \times \tilde{V} \to \tilde{V}$ respectively making the following diagram commute

$$\begin{array}{ccc} \tilde{Q} & \xrightarrow{\tilde{\phi}} & \tilde{V} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ Q & \xrightarrow{\phi} & V. \end{array}$$

Notice $\tilde{\phi}(\tilde{Q}) \subset Z$ since $\phi(Q) \subset Y$. For simplicity, assume Q_0 is connected. Hence $p^{-1}(Q_0) = \tilde{Q}_0$ is connected. Let $Z_0 = \tilde{\phi}(\tilde{Q}_0)$, then Z_0 is connected. Notice $\tilde{\phi}$ is one to one almost everywhere on $\tilde{Q}_0$ since ϕ is one to one almost everywhere on Q_0. Hence $[Z_0] = \tilde{\phi}_*[\tilde{Q}_0]$. So

$$\pi_*[Z_0] = \pi_* \tilde{\phi}_*[\tilde{Q}_0] = \phi_* p_*[\tilde{Q}_0] = \phi_*[Q_0] = \theta.$$

Since $\dim(Z)=k$ and Z_0 is a connected subset representing a non-zero k-dimensional homology class, Z_0 must be a whole component of Z.

The proof of (2) is simpler. As in the proof of (1) it suffices to prove if $0 \neq \theta \in H_k^A(V; \mathbb{Z}_2)$ then there exists a multiblowup $\pi: \tilde{V} \to V$ and a stable algebraic subset Z of $\tilde{V}$ such that $\pi_*[Z]=\theta$. This follows from Proposition 6. □

Lemma 10. *Let Y be a real algebraic subset of an algebraic set Z. Let x be a nonsingular point of both Y and Z and let p_i: $(Z, Y) \to (\mathbb{R}, 0)$ $i=1, 2, \ldots, k$, be entire rational functions so that $k=\dim Z - \dim Y$ and x is a regular point of the map $(p_1, \ldots, p_k)$: $Z \to \mathbb{R}^k$, (i.e. the differential of $(p_1, \ldots, p_k)$ has rank k at x). Then if q is any entire rational function vanishing on Y, there are rational functions v_i: $U \to \mathbb{R}$ $i=1, \ldots, k$ so that $q|_U = \sum_{i=1}^{k} v_i p_i|_U$ where U is a Zariski open neighborhood of x in Z.*

Proof. By clearing denominators, we may as well assume that p_i $i=1, \ldots, k$ and q are all polynomials.

If $Z \subset \mathbb{R}^n$, pick $p_{k+1}, \ldots, p_{k+m}$ vanishing on Z so that $m=n-\dim Z$ and the gradients ∇p_i at x are linearly independent, $i=1, \ldots, k+m$. Take the complexifications $Y_{\mathbb{C}}$, $Z_{\mathbb{C}}$, $p_{i\mathbb{C}}$ and $q_{\mathbb{C}}$ of Y, Z, p_i and q. By Corollary 1.20 of [M] we known that $r q_{\mathbb{C}} = \sum_{i=1}^{k+m} h_i p_{i\mathbb{C}}$ for some complex polynomials r and h_i with $r(x) \neq 0$. But if $r=r' + \sqrt{-1}\, r''$ and $h_i = h_i' + \sqrt{-1}\, h_i''$ for polynomials r', r'', h_i' and h_i'' we must also have $r' q_{\mathbb{C}} = \sum_{i=1}^{k+m} h_i' p_{i\mathbb{C}}$ and $r'' = \sum_{i=1}^{k+m} h_i'' p_{i\mathbb{C}}$. Hence since either $r'(x) \neq 0$ or $r''(x) \neq 0$ we may assume that r and h_i $i=1, \ldots, k+m$, are complexifications of real polynomials r^* and h_i^*. Let $U = Z - r^{*-1}(0)$ and $v_i = h_i^*/r^*$. Since $p_i(z)=0$ for $z \in Z$ and $i>k$ we are done. □

Proposition 11. *Let W, V be nonsingular algebraic sets, and let $L \subset V$ be a nonsingular algebraic subset, and let f: $W \to V$ be an entire rational function which is transverse to L. Then there exists an entire rational function f': $B(W, f^{-1}(L)) \to B(V, L)$ such that the following diagram commutes*

$$\begin{array}{ccc} B(W, f^{-1}(L)) & \xrightarrow{\ f'\ } & B(V, L) \\ \Big\downarrow{\scriptstyle \pi(W, f^{-1}(L))} & & \Big\downarrow{\scriptstyle \pi(V, L)} \\ W & \xrightarrow{\ f\ } & V. \end{array}$$

Proof. Let $\phi_0, \ldots, \phi_n$ be generators of $\vartheta_V(L)$. We claim that $\phi_0 \circ f, \ldots, \phi_n \circ f$ generate $\vartheta_W(f^{-1}(L))$. Then using the above generators we construct $B(W, f^{-1}(L)) = Cl\{(x, [u_1 : \ldots : u_n]) \in W \times \mathbb{R}P^{n-1} \mid x \notin f^{-1}(L)$ and $u_i = \phi_i \circ f(x)$ $i=1, \ldots, n\})$ and $B(V, L) = Cl(\{(y, [u_1 : \ldots : u_n]) \in V \times \mathbb{R}P^{n-1} \mid y \notin L$ and $u_i = \phi_i(y)$, $i=1, \ldots, n\})$, where Cl denotes Zariski closure. The map f' is induced by $f \times id$: $W \times \mathbb{R}P^{n-1} \to V \times \mathbb{R}P^{n-1}$. To prove the claim, pick any entire rational function $q: (W, f^{-1}(L)) \to (\mathbb{R}, 0)$. We will show that q is in the ideal ϑ generated by $\phi_1 \circ f, \ldots, \phi_n \circ f$. Pick any $x \in f^{-1}(L)$. After renumbering, we may assume that

$(\phi_1, \ldots, \phi_k)\colon V \to \mathbb{R}^k$ has a regular point at $f(x)$, where $k = \dim V - \dim L$. So by transversality, $(\phi_1 \circ f, \ldots, \phi_k \circ f)$ has a regular point at x. Hence by Lemma 10 we may find polynomials $h_i^x\colon V \to \mathbb{R}$ and $r^x\colon V \to \mathbb{R}$ so that $r^x(x) \neq 0$ and $r^x q = \sum_{i=1}^{k} h_i^x(\phi_i \circ f)$. Set $h_i^x = 0$ if $k < i \leq n$. Then $r^x q = \sum_{i=1}^{n} h_i^x(\phi_i \circ f)$. Do this for every x in $f^{-1}(L)$. Since $f^{-1}(L)$ is compact in the Zariski topology, we may pick $x_1, \ldots, x_m$ so that for every $y \in f^{-1}(L)$, $r^{x_j}(y) \neq 0$ for some $j = 1, \ldots, m$. Let $r_j = r^{x_j}$ and $h_{ij} = h_i^{x_j}$, $j = 1, \ldots, m$, $i = 1, \ldots, n$. Then $r_j q = \sum_{i=1}^{n} h_{ij}(\phi_i \circ f)$ $j = 1, \ldots, m$ and $f^{-1}(L) \cap \bigcap_{j=1}^{m} r_j^{-1}(0)$ is empty. Let $g = \sum_{i=1}^{n} (\phi_1 \circ f)^2 + \sum_{j=1}^{m} r_j^2$. Then

$$gq = \sum_{i=0}^{n} (\phi_i \circ f)^2 q + \sum_{j=0}^{m} r_j \sum_{i=0}^{n} h_{ij}(\phi_i \circ f)$$

so gq is in ϑ. Since $g > 0$ on W, $1/g$ is an entire rational function on W. Consequently, q is in ϑ. □

Proposition 12. *Let W be a nonsingular real algebraic set and let $\pi\colon X \to W$ be an uzunblowup. Suppose $W \subset Y$ where Y is a nonsingular real algebraic set and let $p\colon Z \to Y$ be an uzunblowup induced by π, so $X \subset Z$ and $p|_X = \pi$. Then there is a unique entire rational function $h\colon B(Z, X) \to B(Y, W)$ so that the following diagram commutes.*

$$\begin{array}{ccc} B(Z, X) & \xrightarrow{\;h\;} & B(Y, W) \\ \Big\downarrow{\scriptstyle \pi(Z,X)} & & \Big\downarrow{\scriptstyle \pi(Y,W)} \\ Z & \xrightarrow{\;p\;} & Y. \end{array}$$

Proof. By induction on the number of blowups and rational diffeomorphisms it suffices to prove this for a single blowup or rational diffeomorphism.

If p is a rational diffeomorphism then h exists by Proposition 11 since p is transverse to W and $X = p^{-1}(W)$.

So suppose p is a blowup with nonsingular center $L \subset W$. In fact, $p^*(\vartheta_Y(W)) = \vartheta_Z(X) \cdot \vartheta_Z(p^{-1}(L))$ and $\vartheta_Z(p^{-1}(L))$ is a locally principal ideal (i.e. Z can be covered with Zariski open subsets so that the restriction of $\vartheta_Z(p^{-1}(L))$ to each of these subsets is a principal ideal). It then follows that $B(Z, \vartheta_Z(X))$ and $B(Z, \vartheta_Z(X) \cdot \vartheta_Z(p^{-1}(L)))$ are isomorphic and the Proposition follows. Rather than fill in the details of the above argument we will give a more direct bare hands proof.

Let $q_1, \ldots, q_m$ be generators of $\vartheta_Y(W)$ and pick $q_{m+1}, \ldots, q_n$ so that $\vartheta_Y(L)$ is generated by $q_1, \ldots, q_n$. Then Z is the Zariski closure of

$$\{(y, [v_1 : \ldots : v_n]) \in Y \times \mathbb{R}P^{n-1} \mid y \in Y - L,\ v_i = q_i(y)\ i = 1, \ldots, n\}$$

and X is the set of points of Z so that $v_i = 0$, $i = 1, \ldots, m$. Also $B(Y, W)$ is the Zariski closure of the set

$$\{(y, [v_1 : \ldots : v_m] \in Y \times \mathbb{R}P^{m-1} \mid y \in Y - W,\ v_i = q_i(y)\ i = 1, \ldots, m\}.$$

On $Z-X$, blowing up along X does nothing so we must find $h|: Z-X \to B(Y, W)$ so that $\pi(Y, W) \circ h| = p|$. We just take $h(y, [v_1 : \ldots : v_n]) = (y, [v_1 : \ldots : v_m])$ which makes sense since we must have $v_i \neq 0$ for some $i = 1, \ldots, m$. So take a point of X. After reordering we may assume $v_n \neq 0$, so this point is contained in an open subset of Z birationally isomorphic to the Zariski closure U of

$$\{(y, z) \in Y \times \mathbb{R}^{n-1} \mid y \in Y - q_n^{-1}(0),\ z_i = q_i(y)/q_n(y),\ i = 1, \ldots, n-1\}.$$

But $U \cap X$ is the set of points of U so that $z_i = 0$, $i = 1, \ldots, m$ and $\vartheta_U(U \cap X) = (z_1, \ldots, z_m)$. (Notice $q_i(y) = z_i q_n(y)$.) So $B(U, U \cap X)$ is the Zariski closure of

$$\{(y, z, [x_1 : \ldots : x_m]) \in U \times \mathbb{R}P^{m-1} \mid (y, z) \in U - X \text{ and } x_i = z_i,\ i = 1, \ldots, m\}.$$

We then have a map $h| : B(U, U \cap X) \to B(Y, W)$ given by $h(y, z, [x_1 : \ldots : x_m]) = (y, [x_1 : \ldots : x_m])$. So we have defined h now on all of $B(Z, X)$. It is easy to see that our various definitions of h agree on overlaps. The reason is that h must be p on $\pi(Z, X)^{-1} p^{-1}(Y-W)$ and also $\pi(Z, X)^{-1} p^{-1}(Y-W)$ is dense in $B(Z, X)$. So if h exists, it is unique by continuity. □

For the sake of making the statements and the proofs of the following theorems simpler we make the following definition:

Definition. *We call an uzunblowup* π: $\tilde{V} \to V$ *an* $A_0 H_*$-uzunblowup *if all the centers* $\{L_i\}$ *have the property* $H_*(L_i; \mathbb{Z}_2) = A_0 H_*(L_i; \mathbb{Z}_2)$. *Similarly, we call it an* AH_*-uzunblowup *if* $H_*(L_i; \mathbb{Z}_2) = AH_*(L_i; \mathbb{Z}_2)$ *for all i.*

Theorem 13. *Let V be a compact nonsingular algebraic set. Then there exists an* $A_0 H_*$*-uzunblowup* π: $\tilde{V} \to V$ *such that*

(a) $H_k(\tilde{V}; \mathbb{Z}_2) = A_0 H_k(\tilde{V}; \mathbb{Z}_2)$ *for all k*

(b) $\pi_* A H_k(\tilde{V}; \mathbb{Z}_2) = H_k^A(V; \mathbb{Z}_2)$ *for all k.*

Proof. First we make the following claim.

Claim. *Let V be a nonsingular algebraic set and* p: $\bar{V} \to V$ *be a multiblowup. Then there exists a* $A_0 H_*$*-uzunblowup* π: $\tilde{V} \to V$ *and an entire rational function* g: $\tilde{V} \to \bar{V}$ *such that the following diagram commutes up to* $\mathbb{Z}_2$*-homology.*

$$\begin{array}{ccc} \bar{V} & \xleftarrow{\ g\ } & \tilde{V} \\ {\scriptstyle p}\downarrow & \swarrow {\scriptstyle \pi} & \\ V & & \end{array}$$

Let $A(v)$ and $C(v)$ be the statements of the theorem and the claim for all V with $\dim(V) \leq v$ respectively. It suffices to prove $C(v) \Rightarrow A(v)$ and $A(v-1) \Rightarrow C(v)$.

Proof of $C(v) \Rightarrow A(v)$. For a given V with $\dim(V) = v$, from Lemma 9 we get a multiblowup p: $\bar{V} \to V$ with the properties

$$p_* A_0 H_*(\bar{V}; \mathbb{Z}_2) = H_*(V; \mathbb{Z}_2)$$
$$p_* A H_*(\bar{V}; \mathbb{Z}_2) = H_*^A(V; \mathbb{Z}_2).$$

By $C(v)$ there exists an A_0H_*-uzunblowup $\pi'\colon \bar{\bar{V}} \to V$ and an entire rational function $g\colon \bar{\bar{V}} \to \bar{V}$ making the following commute up to homology

$$\begin{array}{ccc} \bar{V} & \xleftarrow{g} & \bar{\bar{V}} \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle \pi'} & \\ V & & \end{array}$$

Isotop g to a smooth function f which is transverse to all given submanifolds, in particular the generators of $AH_*(\bar{V}; \mathbb{Z}_2)$. By Theorem 4 there is a nonsingular algebraic set $\tilde{V}$ and an entire rational function $\phi\colon \tilde{V} \to \bar{V}$ and a rational diffeomorphism $\alpha\colon \tilde{V} \xrightarrow{\approx} \bar{\bar{V}}$ such that ϕ is close to $f \circ \alpha$. Therefore ϕ is transverse to all the generators $\{S_i\}$ of $AH_*(\bar{V}; \mathbb{Z}_2)$. Hence $\phi^{-1}(S_i)$ gives homology classes in $AH_*(\tilde{V}; \mathbb{Z}_2)$. We conclude that $AH_*(\bar{V}; \mathbb{Z}_2) \subset \phi_* AH_*(\tilde{V}; \mathbb{Z}_2)$. If $\pi = \pi' \circ \alpha$ then $\pi\colon \tilde{V} \to V$ is a A_0H_*-uzunblowup such that:

$$\begin{aligned} \pi_* A_0 H_*(\tilde{V}; \mathbb{Z}_2) &= \pi'_* \alpha_* A_0 H_*(\tilde{V}; \mathbb{Z}_2) \\ &= +p_* \phi_* A_0 H_k(\tilde{V}; \mathbb{Z}_2) \\ &\supset p_* A_0 H_*(\bar{V}; \mathbb{Z}_2) = H_*(V; \mathbb{Z}_2) \end{aligned}$$

hence $\pi_* A_0 H_*(\tilde{V}; \mathbb{Z}_2) = H_*(V; \mathbb{Z}_2)$. Surjectivity of π_* implies $H_*(\tilde{V}; \mathbb{Z}_2) = \ker \pi_* \oplus G$ where $\pi_*\colon G \xrightarrow{\cong} H_*(V; \mathbb{Z}_2)$ is an isomorphism and $G \subset A_0 H_*(\tilde{V}; \mathbb{Z}_2)$. By Lemma 8, $\ker \pi_* \subset A_0 H_*(\tilde{V}; \mathbb{Z}_2)$, hence $H_*(\tilde{V}; \mathbb{Z}_2) = A_0 H_*(\tilde{V}; \mathbb{Z}_2)$ which proves (a). Clearly $H_*^A(V; \mathbb{Z}_2) \supset \pi_* AH_*(\tilde{V}; \mathbb{Z}_2)$. Since $\pi_* = p_* \circ \phi_*$, $\pi_* AH_*(\tilde{V}; \mathbb{Z}_2) \supset p_* AH_*(\bar{V}; \mathbb{Z}_2) = H_*^A(V; \mathbb{Z}_2)$. Hence $\pi_* AH_*(\tilde{V}; \mathbb{Z}_2) = H_*^A(V; \mathbb{Z}_2)$ which proves (b).

Proof of $A(v-1) \Rightarrow C(v)$. Pick V with $\dim(V) = v$ and $p\colon \bar{V} \to V$ a composition

$$\bar{V} = V_n \xrightarrow{p_n} V_{n-1} \longrightarrow \dots V_1 \xrightarrow{p_1} V_0 = V$$

of maps where each $V_{i+1} \xrightarrow{p_{i+1}} V_i$ is a blowup along a nonsingular center $L_i \subset V_i$.

Assume that we have constructed an A_0H_**-uzunblowup* $\pi(i)\colon Z_i \to V$ and an entire rational function $g_i\colon Z_i \to V_i$ such that the following commute up to $\mathbb{Z}_2$-homology

$$\begin{array}{ccc} V_i & \xleftarrow{g_i} & Z_i \\ {\scriptstyle p(i)}\downarrow & \swarrow{\scriptstyle \pi(i)} & \\ V & & \end{array}$$

where $p(i) = p_1 \circ p_2 \circ \dots \circ p_i$, $Z_0 = V$, and $g_0 = p(0) = \pi(0) = id$. If we call this statement $B(i)$, by induction, it suffices to show $B(i) \Rightarrow B(i+1)$. By Theorem 4, we can find a nonsingular algebraic set Z'_i and a rational diffeomorphism $\alpha\colon Z'_i \xrightarrow{\approx} Z_i$ and an entire rational function $\phi\colon Z'_i \to V_i$ which is transverse to L_i and $\phi_* = (g_i)_* \circ \alpha_*$. Then $N'_i = \phi^{-1}(L_i)$ is a nonsingular algebraic set of dimension $< v$

in Z_i'. By $A(v-1)$ there is A_0H_*-uzunblowup $N_i'' \to N_i'$ with $H_*(N_i''; \mathbb{Z}_2) = A_0H_*(N_i''; \mathbb{Z}_2)$. Let $Z_i'' \xrightarrow{\psi} Z_i'$ be the induced A_0H_*-uzunblowup. Consider the blowups $Q = B(Z_i', N_i') \xrightarrow{\beta} Z_i'$ and $Z_{i+1} = B(Z_i'', N_i'') \xrightarrow{\gamma} Z_i''$, then by Proposition 11 and 12 we have entire rational functions δ_1, δ_2 making the following diagram commute as shown up to $\mathbb{Z}_2$ homology:

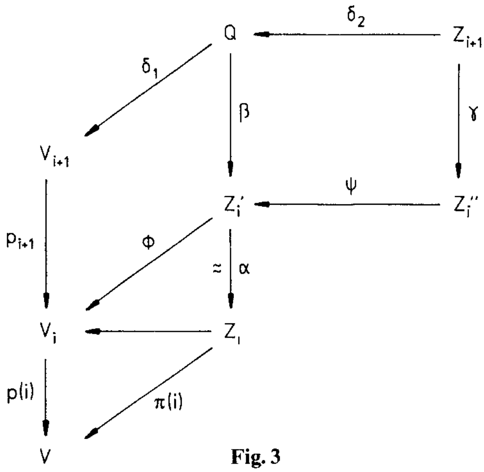
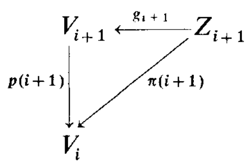


Fig. 3

Then $\pi(i+1)$: $Z_{i+1} \to V$ is an A_0H_*-uzunblowup, where $\pi(i+1) = \pi(i) \circ \alpha \circ \psi \circ \gamma$. If we let $g_{i+1} = \delta_1 \circ \delta_2$ we get the homology commutative diagram



as desired. □

Remark 14. In the conclusion of Theorem 12 we cannot hope to make $H_*(\tilde{V}; \mathbb{Z}_2) = AH_*(\tilde{V}; \mathbb{Z}_2)$ since there are algebraic sets V with $H_*(V; \mathbb{Z}_2) \neq H_*^A(V; \mathbb{Z}_2)$ (see [AK$_2$]).

Theorem 15. *Let V be a compact nonsingular algebraic set with $H_k(V; \mathbb{Z}_2) = H_k^A(V; \mathbb{Z}_2)$ for all k. Then there exists a AH_*-uzunblowup $\pi: \tilde{V} \to V$ such that $H_k(\tilde{V}; \mathbb{Z}_2) = AH_k(\tilde{V}; \mathbb{Z}_2)$ for all k.*

Proof. The proof is similar and simpler than the proof of Theorem 13, except in the statement of the claim we take $\pi: \tilde{V} \to V$ to be an AH_*-uzunblowup. □

Remark 16. The reason that we used the homology groups AH_* and A_0H_* instead of homology generated by nonsingular algebraic sets and the components of nonsingular algebraic sets in this paper is that the stable algebraic sets obey transversality (Proposition 3) which makes the techniques of this paper work. Alternatively we could have used RH_* and R_0H_* instead and all the results would have gone through, where

$RH_*(V;\mathbb{Z}_2)$ = Subgroup generated by $\phi_*[M^k]$ where $\phi: M \hookrightarrow V$ is an entire rational map from a compact nonsingular algebraic set M, and also ϕ is a smooth imbedding.

$R_0H_*(V;\mathbb{Z}_2)$ = The same as above, except M is a component of a nonsingular algebraic set. (This is actually the same as $H_*^{imb}(V;\mathbb{Z}_2)$, c.f. the weak version of Prop 2.8 of [AK_1].)

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