Introduction to Resolution Towers

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In this article we give an overview of the theory of resolution towers. This is a summary of our ongoing work on the topology of real algebraic sets. A real algebraic set is the solution of real polynomial equations $f_i(x_1, \cdots, x_m) = 0$, $i = 1, \cdots, k$ in $\mathbb{R}^m$. We are interested in classifying all possible topological types of such sets. First of all in the case of nonsingular algebraic sets this problem is solved (see [1], [2]):

$$\left\{\begin{matrix}\text{Nonsingular} \\ \text{Algebraic sets}\end{matrix}\right\} = \left\{\begin{matrix}\text{Interiors of smooth} \\ \text{compact manifolds}\end{matrix}\right\}$$

Also according to [1] a set is homeomorphic to an algebraic set if and only if the one point compactification is homeomorphic to an algebraic set. Hence it suffices to understand the compact singular algebraic sets. First we start with the case $m = 1$. For the sake of argument assume that the coefficients of the polynomials are in $\mathbb{Q}$ (it turns out that our methods give us such polynomials, so this is not a serious restriction). Recall that the elements of algebraic subsets of $\mathbb{R}$ are called algebraic numbers, and that an algebraic subset of $\mathbb{R}$ is a finite number of points in $\mathbb{R}$. We could consider this case to be trivial and go on to the case $m > 1$. But already in this case we see a glimpse of a certain internal structure, reflecting the fact that these points lie in an extension field of the rational numbers. For example, if x is an algebraic number such as $a + \sqrt{2}\, b$, $a,b \in \mathbb{Q}$, we can associate to it a set $\mathcal{J} = \{a,b,x^2 = 2\}$ consisting of two rational numbers and a monomial equation. We can say that the realization $|\mathcal{J}|$ of this set is the algebraic number x. Naively, we can consider the number x as being obtained by glueing two rational numbers with a monomial. For the high dimensional algebraic sets we might expect a similar structure. First of all we can consider the set of compact smooth manifolds as a high dimensional analogue of the rationals (recall that there are countable number of them). If we can define 'topological monomial maps' between smooth manifolds, we can conjecture that the objects such as $\mathcal{T} = \{V_i, f_{ji}\}$, where V_i, $i = 0, \cdots, n$ are closed smooth manifolds and $f_{ji} : V_i \to V_j$ are topological monomial maps, should classify real algebraic sets. Then, we can expect that the realization

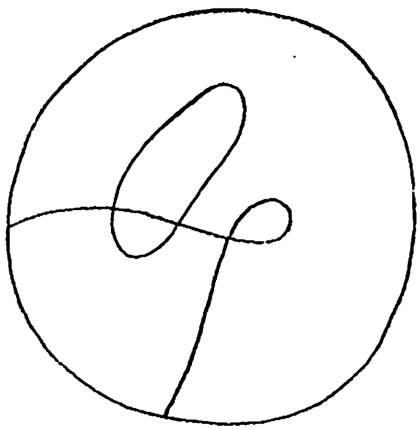
$$|\mathcal{T}| = \bigcup_{i=o}^{n} V_i / x \sim f_{ji}(x)$$

of $\mathcal{T}$ (i.e., the set obtained by glueing the smooth manifolds V_i, $i = 0, \cdots, n$ together by the maps f_{ji}) should describe the underlying stratified space of the algebraic set. Surprisingly, it turns out that the algebraic sets do possess such a structure. We call these structures $\mathcal{T}$ resolution towers. Historically, mathematicians attempted to understand algebraic sets by stratified spaces which are high dimensional analogues of real numbers. They are too general to characterize algebraic sets. It was clear that the complexification property of the real algebraic sets did impose some restriction on their topological types [12]. It was also suspected that the resolution of singularities theorem [11] should further restrict the possible topological types of stratified spaces that can occur as algebraic sets. We hope that the resolution towers give a definite answer to these expectations. A resolution tower $\mathcal{T}$ has a rich internal structure which imposes many strong restrictions on the topology of its realization $|\mathcal{T}|$. Roughly, the manifolds and the maps of $\mathcal{T}$ reflect the resolution and the complexification properties of the algebraic set $|\mathcal{T}|$, respectively. To be able to define these objects, we need to define topological monomial maps, which brings us to the notion of ticos (acronym for transversally intersection codimension one submanifolds).

1. TICOS

Let $f : M \to N$ be a smooth map between closed smooth manifolds. We would like to find a topological condition which will make f 'look like' a monomial map. One futile attempt would be to use the coordinate charts to define a monomial map. If M,N are not already algebraic manifolds, this definition will not make sense. Since we need coordinates to define monomials and the coordinate charts are no help, we put fixed coordinates directly on M and N as follows:

DEFINITION: A tico $\mathcal{A}$ in M is a finite collection of properly immersed closed smooth codimension one submanifolds of M.



M

One should consider a tico to be the generalization of the coordinate hyperplanes $\{\mathbb{R}^n_i\}^n_{i=1}$ in $\mathbb{R}^n$, where $\mathbb{R}^n_i = \{x \in \mathbb{R}^n \mid x_i = 0\}$. We call the elements of $\mathcal{A}$ the sheets. We say $\mathcal{A}$ is a regular tico if each element is a properly imbedded submanifold. We define the realization of a tico $\mathcal{A}$ to be $|\mathcal{A}| = \cup_{S\in\mathcal{A}} S$. Of course $\{|\mathcal{A}|\}$ is also a tico consisting of a single sheet. For simplicity often we call the pair $(M,\mathcal{A})$ a tico. A tico $\mathcal{A}$ induces a natural stratification of M, where a codimension d stratum is a connected component of the d-fold self intersections of $|\mathcal{A}|$. We say a tico $(M,\mathcal{A})$ is an algebraic tico if M is a nonsingular algebraic set and each sheet of $\mathcal{A}$ is a nonsingular algebraic subset of M. In particular algebraic ticos are regular. Now by using ticos as coordinates we can define topological monomials.

DEFINITION. Let $(M,\mathcal{A})$, $(N,\mathcal{B})$ be a smooth manifold with ticos. A tico map $f : (M,\mathcal{A}) \to (N,\mathcal{B})$ is a smooth map from M to N with the following local property. Pick any $p \in M$, and pick any charts $\psi : (\mathbb{R}^m,0) \to (M,p)$ and $\theta : (\mathbb{R}^n,0) \to (N,f(p))$ such that $\psi^{-1}(|\mathcal{A}|) = \cup^a_{j=1} \mathbb{R}^m_j$, $\theta^{-1}(|\mathcal{B}|) = \cup^b_{i=1} \mathbb{R}^n_i$. Let $f_i(x)$ be the i-th coordinate of $\theta^{-1} \circ f \circ \psi(x)$. Then there are nonnegative integers α_{ij}, $1 \leq i \leq b$, $1 \leq j \leq a$ and smooth functions $\phi_i : \mathbb{R}^m \to \mathbb{R}$ such that:

$$f_i(x) = \prod_{j=1}^{a} x_j^{\alpha_{ij}} \phi_i(x)$$

for all x near 0 and $\phi_i(0) \neq 0$ for $i = 1, \cdots, b$. The following properties can easily be checked [5].

a) Up to permuation of i and j the exponents α_{ij} above depend only on f and on $p \in M$, not on the local charts ψ and θ.

b) $|\mathcal{A}| \supset f^{-1}(|\mathcal{B}|)$. Furthermore if $\mathcal{A}$ is regular then $f^{-1}(|\mathcal{B}|)$ is a union of components of sheets of $\mathcal{A}$.

c) We may pick $\phi_i(x)$ so that $\phi_i^{-1}(0) = \emptyset$.

The property (a) says that the exponents of a tico map is well defined, and (b) says that tico maps have an 'analytic continuation' like property. Because of (a) for every $S \in \mathcal{A}$ and $T \in \mathcal{B}$ we can define a function $\alpha_{ST} : S \to \mathbb{Z}$ as follows. For $p \in S$ if $f(p) \notin T$ let $\alpha_{ST}(p) = 0$, otherwise choose coordinates as above, say $\psi^{-1}(S) = \mathbb{R}^m_j$ and $\theta^{-1}(T) = \mathbb{R}^n_i$ and let $\alpha_{ST}(p) = \alpha_{ij}$. It follows from definitions that α_{ST} is continuous, hence it is locally constant on each component of S; furthermore

$$f^{-1}(T) = \bigcup_{S \in \mathcal{A}} \alpha_{ST}^{-1}(\mathbb{Z}-o)$$

One can impose some niceness conditions on tico maps to make them more useful, the two most important ones are:

DEFINITION. A tico map $f : (M,\mathcal{A}) \to (N,\mathcal{B})$ is called <u>type N</u> if for every $p \in M$ there are charts $\psi : (U,0) \to (M,p)$, $\theta : (\mathbb{R}^m,0) \to (N,f(p))$, where $0 \in U \subset \mathbb{R}^m$ is an open subset, such that

$$\psi^{-1}(|\mathcal{A}|) = (\bigcup_{j=1}^{a} \mathbb{R}^m_j) \cap U, \quad \theta^{-1}(|\mathcal{B}|) = \bigcup_{i=1}^{b} \mathbb{R}^n_i$$

and for some

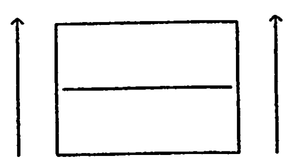
$$c \geq a \quad f_i(x) = \prod_{j=1}^{c} x_j^{\alpha_{ij}}$$

for $x \in U$, where $f_i(x)$ is the i-th coordinate of $\theta^{-1} \circ f \circ \psi(x)$; and the $b \times c$ matrix (α_{ij}) is onto and $m - c \geq n - b$. We say f is <u>submersive</u> if in addition, we have $f_i(x) = x_{i-n+m}$ for all $i > b$ and all $x \in U$.

So basically a type N tico map is a tico map such that under some choice of local coordinates it becomes a pure monomial ($\phi_i(x) \equiv 1$), and its exponent matrix is onto. The later condition means that it submerses the top stratum of $(M,\mathcal{A})$. A submersive tico map is a type N tico map which submerses each stratum of $(M,\mathcal{A})$ to a stratum of $(N,\mathcal{B})$.

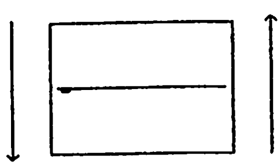
The local topological behavior of tico maps is quite restrictive: they are composition of folds and crushes. For example if $M = \mathbb{R}^2$ and $\mathcal{A} = \{\mathbb{R}^2_1, \mathbb{R}^2_2\}$, type N tico maps $f : (M,\mathcal{A}) \to (M,\mathcal{A})$ are of the form $f(x,y) = (x^a y^b, x^c y^d)$ where $ad - bc \neq 0$. The 'topological behavior' of f depends only on the parity of a,b,c,d. The case $b = 0$ arises naturally in classification of 3-dimensional algebraic sets [8]. They are up to sign compositions of four basic tico maps $I = (x,y)$, $g = (x,y^2)$, $h = (x^2,y)$, $k = (x,xy)$. They can be classified into eight different topological types. We can distinguish them according to how they map the square $[-1,1] \times [-1,1]$ into itself. In particular it is enough to know how the two vertical sides of the square is mapped into itself. In [8] these are symbolically denoted as follows

$(x,y) \to (x,y)$



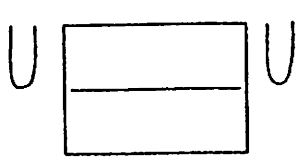
I

$(x,y) \to (x,xy)$



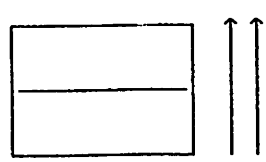
k

$(x,y) \to (x,y^2)$

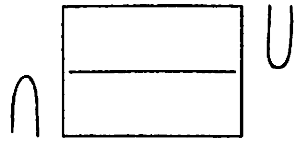


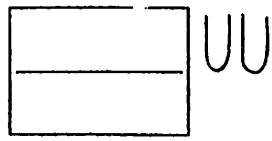
g

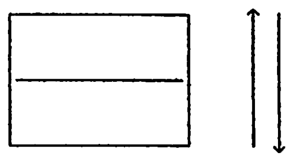
$(x,y) \to (x^2,y)$

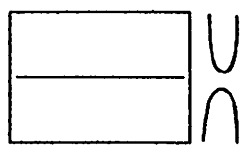


h

$(x,y) \to (x,xy^2)$  $k \cdot g$

$(x,y) \to (x^2,y^2)$  $h \cdot g$

$(x,y) \to (x^2,xy)$  $h \cdot k$

$(x,y) \to (x^2,xy^2)$  $h \cdot k \cdot g$

Finally we make the following definition to be able to talk about the germs of tico maps

DEFINITION. Let $(M,\mathcal{A})$, $(N,\mathcal{B})$ be smooth manifolds with ticos and let $\mathcal{C} \subset \mathcal{A}$. We call a map $f : |\mathcal{C}| \to N$ a <u>mico</u> if there is a neighborhood U of $|\mathcal{C}|$ in M and a tico map $g : (U, U \cap \mathcal{A}) \to (N,\mathcal{B})$ such that $f = g|_{|\mathcal{C}|}$.

2. RESOLUTION TOWERS

DEFINITION. A resolution tower $\mathcal{T} = \{V_i, \mathcal{A}_i, p_i\}_{i=0}^{n}$ is a collection of compact smooth manifolds with ticos $(V_i, \mathcal{A}_i)$ $i = 0, \cdots, n$ and a collection of maps $p_i = \{p_{ji}\}_{j=o}^{i-1}$ with $p_{ji} : V_{ji} \to V_j$ such that each $V_{ji} = |\mathcal{A}_{ji}|$ for some $\mathcal{A}_{ji} \subset \mathcal{A}_i$ and

(I) $p_{ji}(V_{ji} \cap V_{ki}) \subset V_{kj}$ for $0 \le k < j < i \le n$

(II) $p_{kj} \circ p_{ji}| = p_{ki}|_{V_{ji} \cap V_{ki}}$ for $0 \le k < j < i \le n$

(III) $p_{ji}^{-1}(\cup_{k<m} V_{kj}) = \cup_{k<m} V_{ki}) \cap V_{ji}$

(IV) $\mathcal{A}_i = \cup_{j<i} \mathcal{A}_{ji}$ and $\mathcal{A}_{ji} \cap \mathcal{A}_{ki} = \emptyset$ if $j \ne k$

(Notice that this does not rule out $|\mathcal{A}_{ji}| \cap |\mathcal{A}_{ki}| \neq \emptyset$)

There are some extra properties which resolution towers can satisfy, the basic ones are:

R - Each tico $\mathcal{A}_i$ is regular

M - Each p_{ji} is a mico

N - Each p_{ji} is a type N mico

S - Each p_{ji} is a submersive mico

U - $p_{ji}|_S$ is a submersion for every stratum S of $(V_i,\mathcal{A}_i)$ with $S \subset V_{ji} - \cup_{k<j} V_{ki}$

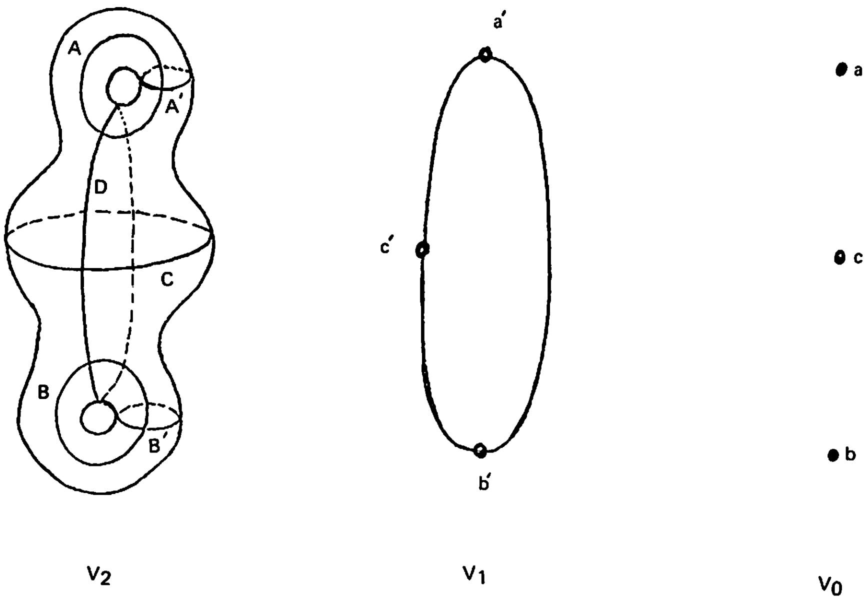
F - Each $(V_i,\mathcal{A}_i)$ is full. This means if S is V_i or any intersection of sheets of $\mathcal{A}_i$, $H_*(S;\mathbb{Z}/2\mathbb{Z})$ is generated by imbedded smooth submanifolds of S.

We say a resolution tower is type R if it satisfies R, is type RM if it satisfies both properties R and M, etc. We define $\partial\mathcal{T} = \{\partial V_i, \partial\mathcal{A}_i, p_i|\}$ where $\partial\mathcal{A}_i = \{\partial S \mid S \in \mathcal{A}_i\}$ and $p_i|$ is the restriction. An <u>algebraic resolution tower</u> is a resolution tower $\mathcal{T}$ such that each $(V_i,\mathcal{A}_i)$ is an algebraic tico and each p_{ji} is an entire rational function. To emphasize the fact that the resolution towers are purely topologically defined objects we sometimes call them <u>topological resolution towers</u>.

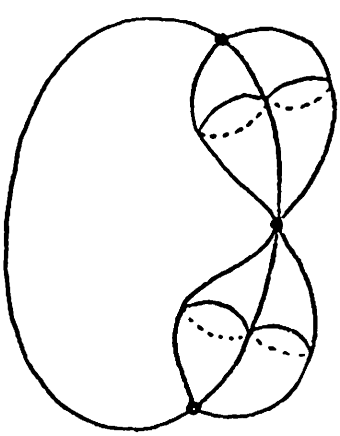
Let $\mathscr{T}$ be the set of topological resolution towers and $\mathscr{A}$ be the set of algebraic resolution towers. When we put subscripts R,M,N,S,U,F to $\mathscr{T}$ or $\mathscr{A}$ we mean the set of resolution towers of that type. For example $\mathscr{T}_{RM}$ is the set of resolution towers of type RM. Clearly we have the forgetful inclusion $\mathscr{A} \subset \mathscr{T}$, also $\mathscr{A}_R = \mathscr{A}$ and $\mathscr{T}_S \subset \mathscr{T}_{NU} \subset \mathscr{T}_{MU}$, $\mathscr{A}_S \subset \mathscr{A}_{NU} \subset \mathscr{A}_{MU}$. For $\mathcal{T} \in \mathscr{T}$ we define the <u>realization</u> of $\mathcal{T}$

$$|\mathcal{T}| = \bigcup_{i=o}^{n} V_i/x \sim p_{ji}(x)$$

That is, $|\mathcal{T}|$ is obtained by identifying the points x and $p_{ji}(x)$ in the disjoint union $\cup V_i$. $|\mathcal{T}|$ is a stratified space with strata $\{V_j - \cup_{r<j} V_{rj}\}$. Define $\dim(\mathcal{T})$ = dimension of $|\mathcal{T}|$. For example, the following is a resolution tower of type S : $\mathcal{T} = \{V_i,\mathcal{A}_i,p_i\}_{i=0}^{2}$ where V_2 is the surface of genus 2, $V_1 = S^1$, $V_0 = \{a,b,c\}$, $\mathcal{A}_{12} = \{D\}$, $\mathcal{A}_{02} = \{A,A',B,B',C\}$, and $\mathcal{A}_{01} = \{a',b',c'\}$, where A,A',B,B',C,D are circles on V_2 and a',b',c' are points on V_1 as indicated by the picture



p_{02} collapses A ∪ A' to a, B ∪ B' to b, C to c and p_{01} identifies a',b',c' to a,b,c respectively, and p_{12} folds D onto the arc $\overline{a'c'b'}$ on V_1. Then the realization $|\mathcal{T}|$ of this tower is



We can usually modify resolution towers to nicer ones without changing their realizations, for example:

PROPOSITION 1. ([5]). $\mathcal{T} \in \mathscr{T}$ then there is $\mathcal{T}' \in \mathscr{T}_R$ with $|\mathcal{T}'| = |\mathcal{T}|$. Furthermore $\mathcal{T}'$ has the same type as $\mathcal{T}$.

PROPOSITION 2. ([5]). If $\mathcal{T} \in \mathscr{T}_M$ (or $\mathscr{A}_M$) then there is $\mathcal{T}' \in \mathscr{T}_{RN}$ (or $\mathscr{A}_{RN}$) with $|\mathcal{T}'| = |\mathcal{T}|$. Furthermore $\mathcal{T}'$ has the same type as $\mathcal{T}$.

Our main theorems relating algebraic sets to resolution towers are the following:

THEOREM 1. ([7]). If $\mathcal{T} \in \mathscr{A}_U$ then $|\mathcal{T}|$ is isomorphic to an algebraic set as a stratified space.

THEOREM 2. ([6]). Any real algebraic set is homeomorphic to $|\mathcal{T}|$ for some $\mathcal{T} \in \mathscr{A}_{FUN}$. This homeomorphism is an isomorphism of stratified sets for some algebraic stratification of the algebraic set.

This theorem turns out to be true for complex algebraic sets where the complex algebraic version $\mathscr{A}^{\mathbb{C}}$ of $\mathscr{A}$ is similarly defined, except in this case we can only take $\mathcal{T} \in \mathscr{A}^{\mathbb{C}}_{UN}$.

THEOREM 3. ([7]). If $\mathcal{T} \in \mathscr{T}_{FS}$ then there exists $\mathcal{T}' \in \mathscr{A}_{FS}$ such that $|\mathcal{T}| = |\mathcal{T}'|$. In particular $|\mathcal{T}|$ is isomorphic, as a stratified set, to an algebraic set (by Theorem 1).

Let us make the convention that if $\mathscr{B}$ is a subset of $\mathscr{T}$ or $\mathscr{A}$ then $|\mathscr{B}| = \{\, |\mathcal{T}| \mid \mathcal{T} \in \mathscr{B}\}$. Consider the set $|\mathscr{B}|$ up to P.L. isomorphism, that is two stratified sets in $|\mathscr{B}|$ are equivalent if they have the same isomorphic subdivisions. Also, let Alg denote the set of P.L. isomorphism classes of all compact real algebraic sets. Then we can summarize the above results by the following diagram.

$$
\begin{array}{ccccccc}
\text{Alg} = |\mathscr{A}_{FUN}| & \xleftarrow{\;i\;} & |\mathscr{A}_{FS}| & = & |\mathscr{T}_{FS}| & = & |\mathscr{T}_{RFS}| \\
\| & & & & \uparrow & & \\
|\mathscr{A}_U| & \nwarrow j & & & & & \\
\uparrow & & & & & & \\
|\mathscr{A}_S| & \xrightarrow{\quad k \quad} & & & |\mathscr{T}_S| & = & |\mathscr{T}_{RS}|
\end{array}
$$

where i,j,k are induced by inclusions. Hence proving i onto would topologically classify real algebraic sets; it would imply that Alg = $|\mathcal{T}_{FS}|$. Alternatively proving j and k onto would give the topological classification Alg = $|\mathcal{T}_S|$. The fact $|\mathcal{A}_U|$ = Alg = $|\mathcal{A}_{FUN}|$ suggests that getting condition F should be easy to establish, i.e., we should expect to have $|\mathcal{T}_S| = |\mathcal{T}_{FS}|$. This would imply k onto. So an important remaining problem is to refine the proof of Theorem 2 to show the surjectivity of j. We conjecture that this is the case. In low dimensional case i is an isomorphism, for example,

$$\{x \in \text{Alg} \mid \dim X \leq 3\} = \{|\mathcal{T}| \in \mathcal{A}_{FS} \mid \dim \mathcal{T} \leq 3\}$$

By using this one can even combinatorially classify all real algebraic sets of dimension $\leq$ 3 ([8]). One of the nice properties of resolution towers is that they enjoy many properties of manifolds. For example, we can define the cobordism groups of resolution towers, whereas it would be hard to make sense of cobordisms of stratified spaces. This allows us to talk about the cobordism groups of algebraic sets. In [8] the cobordism group of algebraic sets of dimension ≤ 2 is defined and computed to be $(\mathbb{Z}/2\mathbb{Z})^{15}$ and the pictures of 15 generators are given. Resolution towers naturally generalize the notion of A-spaces of [10]. Finally, to justify the need and the importance of type F resolution towers we refer the reader to [3], [4].

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