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CONSTRUCTING A FAKE 4-MANIFOLD BY GLUCK CONSTRUCTION TO A STANDARD 4-MANIFOLD

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LET $S^3 \tilde{\times} S^1$ denote the twisted S^3 bundle over S^1. In this paper we will demonstrate an imbedding of a 2-sphere $f: S^2 \hookrightarrow S^3 \tilde{\times} S^1 \# S^2 \times S^2$ such that twisting $S^3 \tilde{\times} S^1 \# S^2 \times S^2$ along $f(S^2)$ (Gluck construction) produces a fake manifold M^4. In fact M^4 coincides with the fake $S^3 \tilde{\times} S^1 \# S^2 \times S^2$'s of [1] and [2]. More specifically, if $M_0^4 = M - \text{int}(B^3 \tilde{\times} S^1)$ (a fake $B^3 \tilde{\times} S^1 \# S^2 \times S^2$) and Q^4 is the Cappell–Shaneson's fake $\mathbb{R}P^4$ corresponding to the matrix,

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

then $Q^4 \# S^2 \times S^2 = (D^2 \tilde{\times} \mathbb{R}P^2) \bigcup_{\partial} M_0$. In fact we have discovered this construction by studying Q^4. So far this is the only example of a fake smoothing which is related to a standard one by Gluck construction to a *2-sphere* (even though there is a strong evidence that the more general type of Gluck twisting described in [3] gives many fake smoothings of 4-manifolds). We will construct M^4 from scratch and show that it is fake and prove the above claim almost simultaneously. But unfortunately the relation of M^4 to Q^4 is very long to prove. Since this is a side issue we will not prove it here. So basically, this paper is self contained. We should mention that as a result of the Gluck construction, M^4 has the property that if $P = \mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$ then,

$$M^4 \# P = S^3 \tilde{\times} S^1 \# S^2 \times S^2 \# P.$$

Throughout the paper we use handelbody presentation of 4-manifolds. The introduction section of [1] provides more than adequate explanation of our notations for the unfamiliar reader. Our result is previously announced in [2] (with a different picture of the imbedding f, in fact that imbedding is isotopic to f described in this paper).

THE CONSTRUCTION

Let Σ be the homology sphere which is the link of $z_0^2 + z_1^3 + z_2^7 = 0$, that is $\Sigma = \Sigma(2, 3, 7)$. Σ is obtained by doing -1 surgery to the figure eight knot. Hence Fig. 1 gives a 4-manifold N_1 whose boundary is $\Sigma \# -\Sigma$. By doing a handle slide and then surgering an imbedded 2-sphere gives Fig. 2. The dotted slice knot indicates the surgered 2-sphere (this means that first attach 2-handle with zero framing to this slice knot then surger the 2-sphere consisting

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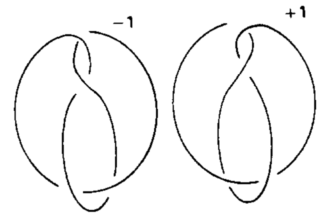
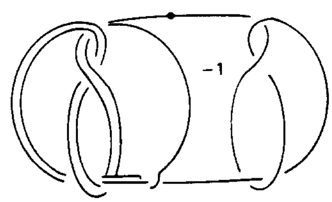


N_1

Fig. 1.

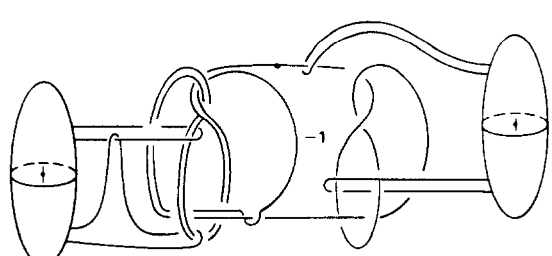


N_2

Fig. 2.

of the obvious slice disc in B^4 union the core of the 2-handle). So Fig. 2 is a homology ball N_2 with $\partial N_2 = \Sigma \# -\Sigma$. By attaching a 3-handle to N_2 along ∂N_2 we get N_3 which is a homology $S^3 \times I$ and $\partial N_3 = \Sigma \amalg -\Sigma$.

The 2-sphere, which the 3-handle is attached, is the 2-sphere which separates the two figure-eight knots in Fig. 1 (even though it is hard to see in Fig. 2). We don't draw the 3-handle even though it is there. So we continue to represent N_3 by Fig. 2. We then attach one 1-handle and two 2-handles to N_3 along ∂N_3, to get N_4. Figure 3 is the picture of N_4. The 1-handle is an orientation reversing 1-handle. Recall from [1] the convention is that the two balls in the picture are the attaching $S^0 \times B^3$ of the 1-handle and they are identified via the 1-handle with the diffeomorphism $(x, y, z) \to (x, -y, -z)$ with respect to the coordinate axis centered in the centers of the balls. In particular anything that enters into one of the balls emerges from the other with its orientation reversed. Notice that because of the presence of the 3-handle in Fig. 2 the attaching circles of the 2-handles of Fig. 3 are not allowed to puncture the attaching 2-sphere of the 3-handle. It can easily be checked that this is the case by going back to Fig. 1 and tracing back the attaching circles



N_4

Fig. 3.

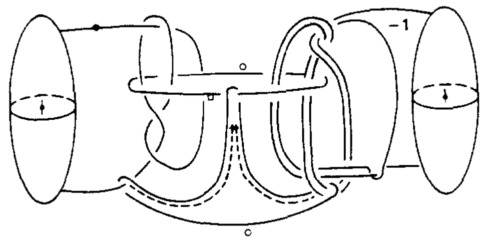


Fig. 4.

of the 2-handles. Figure 4 is the same as Fig. 3 moved by an isotopy. We claim that $\partial N_4 = S^3$. This can be seen by changing the interior of N_4. Namely first replace the dot on the slice knot with zero; this corresponds undoing the original surgery done going from N_1 to N_2. It other words, just surger the linking circle of the originally surgered 2-sphere from N_4. Then do the indicated handle slides (indicated by dotted arrows) in Fig. 4. We get Fig. 5, which is $S^1 \tilde{\times} B^3 \# S^2 \times S^2 \# S^2 \tilde{\times} S^2$ plus a 3-handle attached to the boundary $S^1 \tilde{\times} S^2$. Hence Fig. 5 is the punctured $S^1 \tilde{\times} S^3 \# S^2 \times S^2 \# S^2 \tilde{\times} S^2$, so the boundary is S^3. This shows that $\partial N_4 = S^3$, now define $M = N_4 \bigcup_{\partial} B^4$. We will continue to denote M by Fig. 3 (and Fig. 4). Now comes the important point! We claim that Fig. 6 with a three- and a four-handle is the same as M. To see this we can do the same modification (i.e. replace the dotted knot with zero then do the indicated handle slides) to the interior of the manifold in Fig. 6 and get the *same* Fig. 5. This shows that the attaching circle of the (-1)-framed handle of Fig. 4 and Fig. 6 are isotopic, so Fig. 4 is the same as Fig. 6. Hence Fig. 6 (with a three- and a four-handle) is M. Figure 7 is obtained from Fig. 6 by an isotopy so it is also M^4. Figure 7 clearly shows that M is obtained from $S^3 \tilde{\times} S^1 \# S^2 \times S^2$ by Gluck construction. Figure 7 without the dotted knot and (-1)-framed knot gives $S^3 \tilde{\times} S^1 \# S^2 \times S^2$. The dotted slice knot bounds the obvious slice disc D^2_- in the 0-handle, and since this knot is an unknot in the boundary of the 4-handle it also bounds an

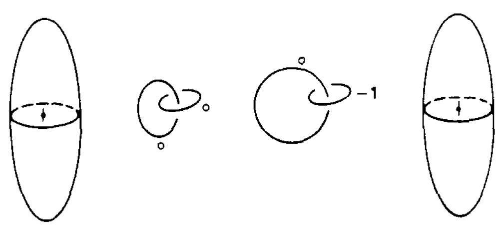


Fig. 5.

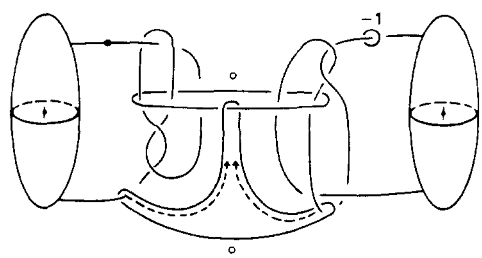
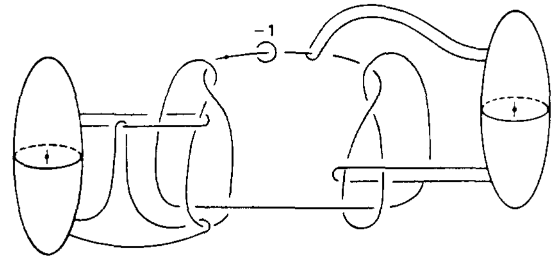


Fig. 6.



M

Fig. 7.

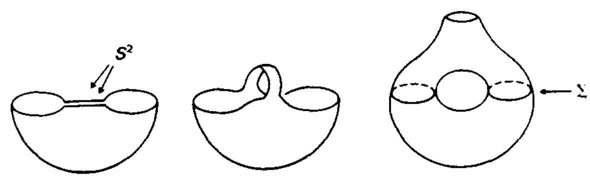


Fig. 8.

obvious disc D^2_+ in the 4-handle. $D^2_- \bigcup_{\partial} D^2_+$ gives an imbedding of 2-sphere $f: S^2 \hookrightarrow S^3 \tilde{\times} S^1 \# S^2 \times S^2$. Removing this 2-sphere (putting a dot on the slice knot) and attaching a 2-handle with (-1)-framing to the linking circle of this 2-sphere corresponds to the Gluck construction. Namely $M = (S^3 \tilde{\times} S^1 \# S^2 \times S^2 - \mathrm{int}(N)) \bigcup_{\varphi} N$ where $N \approx S^2 \times B^2$ is a closed tubular neighborhood of $f(S^2)$ and φ is the diffeomorphism $\partial N \to \partial N$ given by $(x, y) \to (\alpha(y)x, y)$ where $\alpha \in \pi_1 SO_3 = \mathbb{Z}_2$ is the nontrivial element.

It remains to show that M^4 is fake, in other words, it is simple homotopy equivalent but not diffeomorphic (or P.L. homeomorphic) to $S^3 \tilde{\times} S^1 \# S^2 \times S^2$. This is shown in [2], but to make this paper self contained we will sketch the proof here. The complement of N_3 in M is homotopy equivalent to $S^3 \times I \# S^2 \times S^2$. We can extend this homotopy equivalence to a (simple) homotopy equivalence $f: M \to S^3 \tilde{\times} S^1 \# S^2 \times S^2$ with that Σ is the transverse inverse image of a fibre $S^3 \subset S^3 \tilde{\times} S^1 \# S^2 \times S^2$. By composing f with the collapsing map we get a degree 1 normal map $\bar{f}: M \to S^3 \times S^1$, so $(M, \bar{f})$ is an element of the set $[S^3 \tilde{\times} S^1; G/0]$ of degree 1 normal cobordism classes of degree 1 normal maps to $S^3 \times S^1$. In [4] an invariant $\sigma: [S^3 \tilde{\times} S^1; G/0] \to \mathbb{Z}$ is defined to detect nontrivial elements. It is defined as follows, pick $(N, g) \in [S^3 \tilde{\times} S^1; G/0]$ make $g: N \to S^3 \times S^1$ transverse to a fibre S^3 and let Σ be the framed 3-manifold $g^{-1}(S^3)$, and let $W = N - \Sigma$ then $\sigma(N, g) = 2\mu(\Sigma) - \mathrm{Sign}(W) (\mathrm{mod}\ 32)$ where $\mu(\Sigma)$ is the Rohlin invariant of the framed Σ, and $\mathrm{Sign}(W)$ is the signature of W. It is a simple exercise to see that σ is well defined, i.e., it is invariant under a degree 1 normal cobordism. In particular

$$\begin{aligned}\sigma(M, \bar{f}) &= 2\mu(\Sigma) - \mathrm{Sign}(S^3 \times I \# S^2 \times S^2) \qquad (\mathrm{mod}\ 32)\\ &= 2\mu(\Sigma(2, 3, 7)) \qquad (\mathrm{mod}\ 32)\\ &= 16 \qquad (\mathrm{mod}\ 32).\end{aligned}$$

We claim that M can not even be s-cobordant to $S^3 \tilde{\times} S^1 \# S^2 \times S^2$. To see this, suppose that there is an s-cobordism H with $\partial_- H = M$, $\partial_+ H = S^3 \tilde{\times} S^1 \# S^2 \times S^2$. By attaching to a 3-handle to H along $\partial_+ H$ we get $\bar{H}$ with $\partial_- H = M, \partial_+ \bar{H} = S^3 \tilde{\times} S^1$. Since $\pi_2(S^3 \tilde{\times} S^1) = 0$ we can extend $\bar{f}$ to a degree 1 normal map $F: \bar{H} \to S^3 \tilde{\times} S^1$ which restricts to a homotopy equivalence $g = F|_{\partial^+ \bar{H}}: S^3 \tilde{\times} S^1 \to S^3 \tilde{\times} S^1$. Hence

$$\begin{aligned}\sigma(M, \bar{f}) &= \sigma(S^3 \tilde{\times} S^1, g) \\ &= 2\mu(\Sigma') \qquad (\text{mod } 32),\end{aligned}$$

where Σ' is the transverse inverse image $g^{-1}(S^3)$. Hence $\mu(\Sigma') = 8$ (mod 16). Σ' imbeds into the universal cover $S^3 \times \mathbb{R}$, as a homologous copy of S^3. So in particular Σ' imbeds into $S^3 \times [-a, a]$ for some large $a \in \mathbb{R}$ as a framed submanifold this implies that $\mu(\Sigma') = \mu(S^3) = 0$ (mod 16), contradiction. □

Remark. By changing the 1-handle in the above construction to an orientation preserving 1-handle we get a simple homotopy equivalence $f: M' \to S^3 \times S^1 \# S^2 \times S^2$ which is not homotopic to a diffeomorphisms and M' is obtained from $S^3 \times S^1 \# S^2 \times S^2$ by a Gluck construction to a 2-sphere. However, unlike M we won't be able to tell if M' itself is fake. Also by attaching a 2-handle to N_4 then surgering a 2-sphere we get the fake $\mathbb{R}P^4$ of [4] (this is pointed out in the introduction).

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