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All compact manifolds are homeomorphic to totally algebraic real algebraic sets

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We say a real algebraic set V is *totally algebraic* if any $\mathbb{Z}/2\mathbb{Z}$ homology class of V is represented by a compact algebraic subset. In [BD] an example was given of a compact manifold which was not diffeomorphic to any totally algebraic nonsingular real algebraic set. In this paper we will prove the following:

THEOREM. *Any closed (i.e. compact unbounded) smooth manifold is homeomorphic to a totally algebraic real algebraic set.*

The real algebraic set we obtain will in general be singular. The homeomorphism will be piecewise differentiable. Total algebraicity is a very useful concept since it eiliminates many obstructions to making a topological situation algebraic, c.f. [AK1], [AK2], [AK5]. At first glance our result does not seem to be very useful since current algebraic approximation lemmas require nonsingularity. The value of this paper is really in a more complex situation. In particular, our singular algebraic set has a resolution of singularities which is totally algebraic. Thus in practice one would apply the existing approximation theorems to the nonsingular resolved manifold and then blow down to the manifold in which you are really interested. For a relatively simple example, suppose that you have a smooth map $f: N \to M$ between smooth manifolds that you wish to make algebraic. Perhaps M is such that it has no totally algebraic model, thus standard techniques do not hold. By Theorem 8 below there is an algebraic multiblowup $\pi : Z \to Y$ and a homeomorphism $h : M \to Y$ with Z totally algebraic. Suppose that $hf: N \to Y$ approximately lifts to a map $g : N \to Z$, so $\pi g \sim hf$. (Perhaps this was arranged after blowing up N, a common procedure in [AK2].) Then by standard algebraic approximation theorems, we may assume N is a nonsingular real algebraic set and g is an entire rational function. In particular, $f: N \to M$ is approximated by the algebraic situation $\pi g : N \to Y$. In doing so, M has

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become singular but in some contexts (e.g., [AK2]) one is always subdividing stratifications anyway so this does not really hurt.

For definitions of real algebraic variety, nonsingular, rational function, multi-blowup and other terms used the reader can refer to previous papers of ours. In particular, if $L \subset M$ is a proper submanifold then $\pi(M, L) : B(M, L) \to M$ denotes the blowup of M will center L. If M and L are algebraic, we put the usual algebraic structure on $B(M, L)$. Recall that in section 6 of [AK1] it was shown that V is totally algebraic if and only if any $\mathbb{Z}/2\mathbb{Z}$ homology class of V is $\varphi_*([X])$ for some entire rational function $\varphi : X \to V$ from a compact nonsingular algebraic set X. (Here $[X]$ is the fundamental class of X and φ_* is the map on homology.) (The requirement stated in [AK1] that V be nonsingular is totally unnecessary).

A *thickened blowup* of a manifold V is a map $\pi : V' \to V$ where $V' = B(V, L) \times \mathbb{R}^n$ for some center L and $n \geq 0$. The map π is the composition $B(V, L) \times \mathbb{R}^n \to B(V, L) \to V$ of projection to $B(V, L)$ and $\pi(V, L)$. A thickened blowup is *algebraic* if V and L are nonsingular real algebraic sets and V' is given the canonical algebraic structure. Let $M \subset V$ be a proper submanifold which contains L and let $h_t : V' \to V'$, $t \in [0, 1]$ be a smooth isotopy with $h_0 = \textit{identity}$ and let M' denote $h_1(B(M, L) \times 0)$. Then M' is called a *fuzzy transform* of M. We have a *fuzzy transform map* $M' \to M$ given by $\pi \circ h_1^{-1}$. In practice the isotopy h_t will be very small, so that M' is diffeomorphic to $B(M, L)$ via a diffeomorphism which carries points of $M' \cap \pi^{-1}(x)$ to $\pi(M, L)^{-1}(x)$ for all $x \in L$. In particular, $\pi(M')$ is isotopic to M via a small C^0 isotopy which is fixed on L, smooth on $M - L$ but perhaps does some bending of the manifold along L. This will all be generalized in Lemma 1 below.

A *thickened multiblowup* is a composition of thickened blowups. Likewise, if $V_n \to V_{n-1} \to \cdots \to V_1 \to V_0$ is a thickened multiblowup with centers $L_i \subset V_i$ and if $M_0 \supset L_0$ then a fuzzy transform of M_0 is an $M_n \subset V_n$ so that for some sequence $M_i \subset V_i$, M_i is a fuzzy transform of M_{i-1} for $i = 1, 2, \ldots, n$ and of course $L_i \subset M_i$ for $i = 0, 1, \ldots, n-1$. We say a fuzzy transform is *small* if all the isotopies are small. We have a fuzzy transform map $\theta : M_n \to M_0$ which is a composition of the fuzzy transform maps $M_i \to M_{i-1}$. Note that θ is also a multiblowup map, i.e. a composition of smooth blowup maps. An *algebraic thickened multiblowup* is a composition of algebraic thickened blowups. The *center image* $Z \subset V_0$ is the union of the images of all the centers of the thickened multlblowup, i.e. if $\pi_i : V_i \to V_0$ is the composition $V_i \to V_{i-1} \to \cdots \to V_1 \to V_0$ then $Z = L_0 \cup \pi_1(L_1) \cup \cdots \cup \pi_{n-1}(L_{n-1})$.

In the following we must be precise about what we mean by a tubular neighborhood. Let M be a smooth submanifold of a smooth manifold N. Let $\rho : E \to M$ be the normal bundle of M which we think of as the quotient bundle $(TN|_M)/TM$. A tubular neighborhood of M in N is an imbedding $\tau : E \to N$ so that τ is the identity on the zero section and so that for each $v \in E$, if $\gamma_v : \mathbb{R} \to N$ is the curve $\gamma_v(t) = \tau(tv)$ then v is the derivative $[(\gamma_v)'(0)]$ where [ ] denotes the equivalence class in the quotient bundle. Tubular neighborhoods always exist. For example, the standard way of getting a tubular neighborhood is to put some Riemmanian method on N, identify E with $TM^{\perp}$ and let τ be the restriction of the exponential map (with suitable scaling).

When looking at blowup maps it will be convenient to have the following notation. If $y \in \mathbb{R}^n$ we let $y^{(i)}$ denote $(y_1 y_i, \ldots, y_{i-1} y_i, y_i, y_{i+1} y_i, \ldots, y_n y_i)$ and we let $y_{(i)}$ denote $(y_1/y_i, \ldots, y_{i-1}/y_i, y_i, y_{i+1}/y_i, \ldots, y_n/y_i)$. Notice that $(y^{(i)})_{(i)} = y$ and $(y_{(i)})^{(i)} = y$. See [AK4] or [AK2] for example for the elementary properties of blowing up which we use here.

LEMMA 1. *Let $\pi : V' \to V$ be a thickened blowup with center L and let $M \subset V$ be a proper submanifold containing L. Let $B = B(M, L) \times 0 \subset V'$, let $\rho' : E' \to B$ be the normal bundle of B in V' and let $\rho : E \to M$ be the normal bundle of M in V. Let $\pi_* : E' \to E$ be induced from the derivative map from the tangent space TV' to TV. Let $\tau : E \to V$ by a tubular neighborhood of M.*

a) There is a tubular neighborhood $\tau' : E' \to V'$ of B so that $\pi\tau' = \tau\pi_*$, i.e. the following diagram commutes.

$$\begin{array}{ccc} E' & \xrightarrow{\tau'} & V' \\ \downarrow{\scriptstyle \pi_*} & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow{\tau} & V \end{array}$$

b) If $\sigma' : B \to E'$ is a continuous section then there is a unique continuous section $\sigma : M \to E$ so that $\pi_* \circ \sigma' = \sigma \circ \pi|_B$. This section σ is 0 on L. If σ' is close to the zero section then σ will also be close.

Proof. We will first do the case where π is actually a blowup, so $V' = B(V, L)$. In this case we can easily define τ' restricted to $(\rho')^{-1}\pi^{-1}(M - L)$, it must be $\tau'(x) = \pi^{-1}\tau\pi_*(x)$. The claim is that this τ' extends uniquely to all of E'. To prove this it suffices to look locally, i.e. assume $V = \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c$, $M = \mathbb{R}^a \times \mathbb{R}^b \times 0$, $L = \mathbb{R}^a \times 0 \times 0$ and the map τ takes the equivalence class of a vector (u, v, z) at $(x, y, 0)$ to (x, y, z). The blowup $B(V, L)$ is covered by $b + c$ charts, but we will ignore c of them since those charts do not contain $B(M, L)$. The b charts we will use are charts $\varphi_i : \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c \to B(V, L)$ $i = 1, \ldots, b$ so that $\pi\varphi_i(x, y, z) = (x, y^{(i)}, y_i z)$. Thus

$$\varphi_i^{-1}(B) = \{(x, y, z) \mid z = 0\}$$

and

$$\varphi_i^{-1}\pi^{-1}(M-L) = \{(x, y, z) \mid z = 0 \quad \text{and } y_i \neq 0\}.$$

We will now calculate τ' restricted to $(\rho')^{-1}\pi^{-1}(M-L)$ which we have already defined as $\pi^{-1}\tau\pi_*$. Take a normal vector $\zeta = (0, 0, z)$ to $\varphi_i^{-1}(B)$ at a point $(x, y, 0)$ with $y_i \neq 0$. Then $\pi_*\varphi_{i*}([\zeta])$ is the equivalence class of the vector $(0, 0, y_i z)$ at $(x, y^{(i)}, 0)$. Then $\tau\pi_*\varphi_{i*}(\zeta) = (x, y^{(i)}, y_i z)$ so $\tau'\varphi_{i*}([\zeta]) = \pi^{-1}\tau\pi_*\varphi_{i*}(\zeta) = \varphi_i(x, y, z)$. Now the extension of τ' to all of E' is obvious and is clearly an imbedding. The extension is also unique by continuity. Condition a) follows from continuity. We now verify b). We must pick $\sigma = \pi_*\sigma'\pi^{-1}$ on $M - L$. On L we pick σ to be 0. We must now show that this is continuous. Again this is a local question so we only need prove continuity of σ at $(0, 0, 0)$ for $V = \mathbb{R}^a \times \mathbb{R}^b \times \mathbb{R}^c$ and M, L, τ and φ_i as above. Let $\theta_i : \mathbb{R}^a \times \mathbb{R}^b \to \mathbb{R}^c$ be such that $\varphi_{i*}^{-1}\sigma'\varphi_i(x, y, 0) = [(0, 0, \theta_i(x, y))]$. Let $K \subset \mathbb{R}^a \times \mathbb{R}^b$ be the compact set

$$K = \{(x, y) \mid |x| \le 1 \text{ and all } |y_j| \le 1 \quad j = 1, \ldots, b\}.$$

By compactness we may pick a number N so that $|\theta_i(x, y)| \le N$ for each i and all $(x, y) \in K$. Now pick any $(x, y) \in K$ with $y \neq 0$. Let i be such that $|y_i| \ge |y_j|$ for all $j = 1, \ldots, b$. Then

$$\begin{aligned}\sigma(x, y, 0) &= \pi_*\sigma'\pi^{-1}(x, y, 0) = (\pi\varphi_i)_*\varphi_{i*}^{-1}\sigma'\varphi_i\varphi_i^{-1}\pi^{-1}(x, y, 0)\\ &= (\pi\varphi_i)_*\varphi_{i*}^{-1}\sigma'\varphi_i(x, y_{(i)}, 0)\\ &= (\pi\varphi_i)_*([0, 0, \theta_i(x, y_{(i)})])\\ &= [(0, 0, y_i\theta_i(x, y_{(i)}))].\end{aligned}$$

In particular $|\sigma(x, y, 0)| \le N|y_i|$. Hence $\sigma(x, y, 0)$ approaches 0 as y approaches 0, so σ is continuous.

Now we must prove this lemma in the general case where $V' = B(M, L) \times \mathbb{R}^n$. Let $\rho'' : E'' \to B(M, L)$ be the normal bundle of $B(M, L)$ in $B(V, L)$ and let $\pi'' : B(V, L) \to V$ denote $\pi(V, L)$. Let $\pi' : B(V, L) \times \mathbb{R}^n \to B(V, L)$ be projection, so $\pi = \pi'' \circ \pi'$. Then by the above there is a unique tubular neighborhood $\tau'' : E'' \to B(V, L)$ so that $\tau \circ \pi''_* = \pi'' \circ \tau''$. But $E' = E'' \times \mathbb{R}^n$ with $\rho'(x, y) = (\rho''(x), 0)$ and there is an obvious tubular neighborhood $\tau'(x, y) = (\tau''(x), y)$ for $(x, y) \in E'' \times \mathbb{R}^n$. Then

$$\begin{aligned}\pi \circ \tau'(x, y) &= \pi'' \circ \pi'(\tau''(x), y) = \pi'' \circ \tau''(x) = \tau \circ \pi''_*(x)\\ &= \tau \circ \pi''_* \circ \pi'_*(x, y) = \tau \circ \pi_*(x, y),\end{aligned}$$

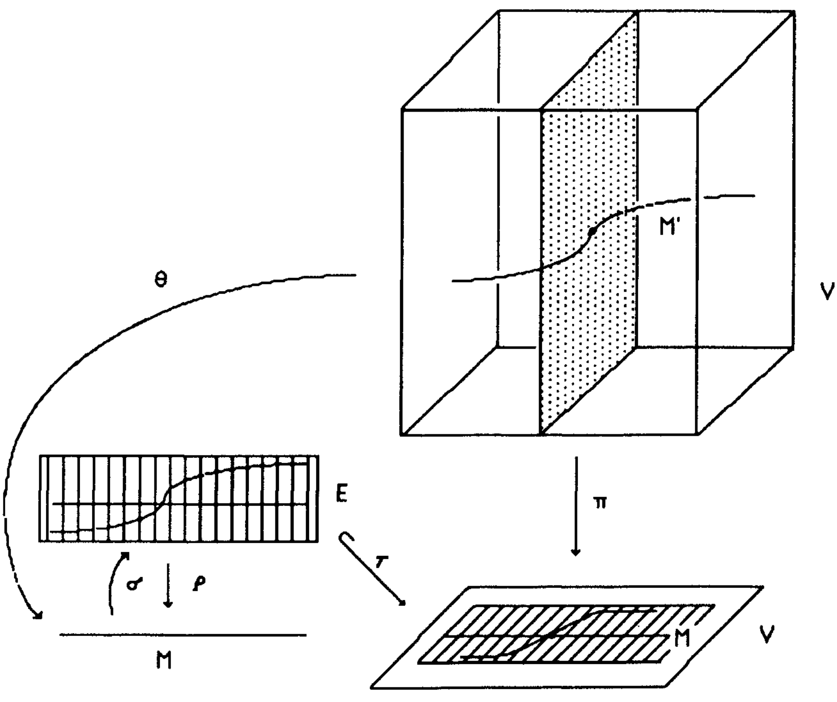
so a) is proven. Let $\eta : B(M, L) \to B$ be the diffeomorphism $\eta(x) = (x, 0)$. To prove b), note that if $\sigma' : B \to E'$ is a section, then $\sigma'' = \pi'_* \circ \sigma' \circ \eta$ is a section of $\rho'' : E'' \to B(M, L)$ so by the above there is a continuous section $\sigma : M \to E$ vanishing on L so that $\pi''_* \circ \sigma'' = \sigma \circ \pi''|_{B(M,L)}$ and hence $\pi_* \circ \sigma' = \pi''_* \circ \pi'_* \circ \sigma' = \pi''_* \circ \sigma'' \circ \eta^{-1} = \sigma \circ \pi'' \circ \eta^{-1} = \sigma \circ \pi'' \circ \pi'|_B = \sigma \circ \pi|_B$. □

PROPOSITION 2. *Let $\pi : V' \to V$ be a thickened multiblowup and let $M' \subset V'$ be a small fuzzy transform of a manifold $M \subset V$. Let $\rho : E \to M$ be the normal bundle of M in V and let $\tau : E \to V$ be a tubular neighborhood of M. Then the following are true.*

a) $\pi(M')$ contains the center image.

b) There is a small continuous section $\sigma : M \to E$ such that $\tau\sigma(M) = \pi(M')$.

c) The map $\tau\sigma\theta : M' \to \pi(M')$ is homotopic to $\pi|_{M'}$ where θ is the fuzzy transform map.



Proof. We prove this by induction on the number of thickened blowups in the thickened multiblowup. If π is the identity, i.e. there are no thickened blowups, it is trivial. So suppose π is not the identity. Let $\pi = \pi'' \circ \pi'$ where $\pi'' : V'' \to V$ is a thickened blowup with center L and $\pi' : V' \to V''$ is a thickened multiblowup. Let $B = B(M, L) \times 0$. Let $M'' \subset V''$ be a small fuzzy transform of M so that M' is a small fuzzy transform of M''. Let $h_t : V'' \to V''$ be the small isotopy taking B to M''. Let $\theta' : M' \to M''$ and $\theta'' : M'' \to M$ be the fuzzy transform maps, so $\theta = \theta'' \circ \theta'$ and $\theta'' = \pi'' \circ h_1^{-1}$.

Let $\rho'' : E'' \to B$ be the normal bundle of B in V''. By Lemma 1a) there is a tubular neighborhood $\tau'' : E'' \to V''$ so that $\pi'' \circ \tau'' = \tau \circ \pi''_*$. Since h_t is small, there is a small smooth section $\sigma'' : B \to E''$ so that $M'' = \tau''(\sigma''(B))$ and furthermore the diffeomorphism $h_1 : B \to M''$ is isotopic to $\tau'' \circ \sigma''$. We may identify the normal bundle of M'' with the normal bundle of B by letting it be the composition $\tau'' \circ \sigma'' \circ \rho'' : E'' \to M''$. This gives us a tubular neighborhood of M'', $\tau' : E'' \to V''$ where $\tau'(x) = \tau''(x + \sigma''\rho''(x))$.

By induction there is a small continuous section σ' of $E'' \to M''$ so that $\pi'(M') = \tau'(\sigma'(M''))$ and $\tau' \circ \sigma' \circ \theta' : M' \to \pi'(M')$ is homotopic to $\pi'|_{M'}$. Let $\sigma''' : B \to E''$ be the small section of $\rho'' : E'' \to B$ given by $\sigma'''(x) = \sigma'\tau''\sigma''(x) + \sigma''(x)$. Now $\tau'' \circ \sigma'' \circ \rho'' \circ \sigma'$ is the identity, so $\tau'' \circ \sigma'' \circ \rho'' \circ \sigma' \circ \tau'' \circ \sigma''(x) = \tau'' \circ \sigma''(x)$ so $\sigma'' \circ \rho'' \circ \sigma' \circ \tau'' \circ \sigma''(x) = \sigma''(x)$ by injectivity of τ''. Hence

$$\tau'' \circ \sigma'''(x) = \tau''(\sigma' \circ \tau'' \circ \sigma''(x) + \sigma'' \circ \rho'' \circ \sigma' \circ \tau'' \circ \sigma''(x)) = \tau' \circ \sigma' \circ \tau'' \circ \sigma''(x)$$

for all $x \in B$.

By Lemma 1b), there is a small continuous section σ of E so that $\pi''_* \circ \sigma''' = \sigma \circ \pi''|_B$ and σ is 0 on L. Hence

$$\begin{aligned}\pi(M') &= \pi''(\pi'(M')) = \pi''(\tau'(\sigma'(M''))) = \pi''(\tau'(\sigma'(\tau''(\sigma''(B))))) \\ &= \pi''(\tau''(\sigma'''(B))) = \tau(\pi''_*(\sigma'''(B))) = \tau(\sigma(\pi''(B))) = \tau(\sigma(M)),\end{aligned}$$

so b) is proven. Also

$$\begin{aligned}\tau \circ \sigma \circ \theta &= \tau \circ \sigma \circ \theta'' \circ \theta' = \tau \circ \sigma \circ \pi'' \circ h_1^{-1} \circ \theta' = \tau \circ \pi''_* \circ \sigma''' \circ h_1^{-1} \circ \theta' \\ &= \pi'' \circ \tau'' \circ \sigma''' \circ h_1^{-1} \circ \theta'\end{aligned}$$

which is homotopic to

$$\pi'' \circ \tau'' \circ \sigma''' \circ (\tau'' \circ \sigma'')^{-1} \circ \theta' = \pi'' \circ \tau' \circ \sigma' \circ \theta'$$

which is homotopic to

$$\pi'' \circ \pi'|_{M'} = \pi|_{M'},$$

so c) is proven. To prove a), let $Z \subset V$ and $Z'' \subset V''$ be the center images of

$\pi : V' \to V$ and $\pi' : V' \to V''$. By definition, $Z = \pi''(Z'') \cup L$ and by induction we know that $Z'' \subset \pi'(M')$. So $Z = \pi''(Z'') \cup L \subset \pi''(\pi'(M')) \cup L = \pi(M') \cup L$. But $L \subset \pi(M')$ since by Lemma 1b) the section σ is 0 on L, so a) is proven.

COROLLARY 3. *Let $\pi : V' \to V$ be a thickened multiblowup and let $M' \subset V'$ be a small fuzzy transform of a manifold $M \subset V$. Then there is a small C^0 isotopy of V carrying $\pi(M')$ to M. In particular, M is homeomorphic to $\pi(M')$.*

We now prove the following blowing down lemma which is a consequence of Lemma 5.3.2 of [AK2].

LEMMA 4. *Let X, V and L be real algebraic sets with X compact and with V and L nonsingular and with $L \subset V$. Let $\pi : V' \to V$ be the blowup of V with center L. Let $\varphi : X \to V'$ be an entire rational function which is injective when restricted to $\varphi^{-1}(V' - \pi^{-1}(L))$. Then there are an algebraic set Y and entire rational functions $\eta : Y \to V$ and $\mu : X \to Y$ so that:*

a) $\eta(Y) = \pi(\varphi(X)) \cup L$.

b) η is a topological imbedding.

c) $\eta \circ \mu = \pi \circ \varphi$.

Proof. By Proposition 2.6.1 of [AK2] there are a real algebraic set Y, entire rational functions $\mu : X \to Y$, $\theta : L \to Y$ and $\eta : Y \to V$ so that

1) $\eta\mu = \pi\varphi$ and $\eta\theta =$ inclusion.

2) $\theta : L \to \eta^{-1}(L)$ is a birational isomorphism.

3) $\mu| : X - \varphi^{-1}\pi^{-1}(L) \to Y - \eta^{-1}(L)$ is a birational isomorphism.

To prove a), note $Y = \mu(X) \cup \theta(L)$ so $\eta(Y) = \eta\mu(X) \cup \eta\theta(L) = \pi\varphi(X) \cup L$ by 1). Finally, to prove b), it suffices to prove η is injective since $\mu(X)$ is compact and $\eta|_{\theta(L)}$ is an imbedding. So suppose $\eta(y) = \eta(y')$ and $y \neq y'$. By 2) we know $\eta(y) \notin L$. Then by 3) we know $y = \mu(x)$ and $y' = \mu(x')$ for $x, x' \in X - \varphi^{-1}\pi^{-1}(L) = \varphi^{-1}(V' - \pi^{-1}(L))$. By hypothesis, $\varphi(x) \neq \varphi(x')$, but then $\pi\varphi(x) \neq \pi\varphi(x')$ because π is injective on $V' - \pi^{-1}(L)$. But $\pi \circ \varphi(x) = \eta \circ \mu(x) = \eta \circ \mu(x') = \pi \circ \varphi(x')$, a contradiction. □

PROPOSITION 5. *Let $\pi : V' \to V$ be an algebraic thickened multiblowup and let $M' \subset V'$ be an algebraic set which is also the small fuzzy transform of a smooth compact manifold $M \subset V$. Then there is an algebraic set Y and entire rational functions $\eta : Y \to V$ and $\mu : M' \to Y$ so that:*

a) $\eta(Y) = \pi(M')$.

b) $\eta : Y \to \pi(M')$ is a homeomorphism.

c) $\pi|_{M'} = \eta \circ \mu$.

Proof. We prove this by induction on the number of thickened blowups in the thickened multiblowup. If π is the identity, i.e. there are no thickened blowups, we may take $Y = M'$, $\eta =$ inclusion. So suppose π is not the identity. Let $\pi = \pi'' \circ \pi'$ where $\pi'' : V'' \to V$ is a thickened blowup with center L and $\pi' : V' \to V''$ is a thickened multiblowup. By induction we have an algebraic set Y' and entire rational functions $\eta' : Y' \to V''$ and $\mu' : M' \to Y'$ so that $\eta'(Y') = \pi'(M')$, $\eta' : Y' \to \pi'(M')$ is a homeomorphism and $\pi'|_{M'} = \eta' \circ \mu'$. Let $V'' = B(V, L) \times \mathbb{R}^n$ and let $\kappa : V'' \to B(V, L)$ be projection.

We wish to apply Lemma 4 to the map $\varphi = \kappa\eta' : Y' \to B(V, L)$. Let us see whether the hypotheses of Lemma 4 are satisfied. First, Y' is compact since it is the continuous image of the compactum M'. So we must only show that φ restricted to $\eta'^{-1}\pi''^{-1}(V - L)$ is injective. Since η' is an imbedding, it suffices to show that κ restricted to $\pi'(M')$ is injective. Let $M'' \subset V''$ be a small fuzzy transform of M such that M' is a small fuzzy transform of M''. Since M'' is isotopic to $B(M, L) \times 0$ by a very small isotopy, we know that $\kappa|_{M''}$ is an imbedding of M'' onto a smooth submanifold K of $B(V, L)$. Let $\rho' : E \to K$ be the normal bundle of K in $B(V, L)$ and let $\tau' : E \to B(V, L)$ be a tubular neighborhood. Let $\theta : K \to \mathbb{R}^n$ be the smooth function such that $(x, \theta(x)) \in M''$ for all $x \in K$. Then $\rho : E \times \mathbb{R}^n \to M''$ is the normal bundle of M'' where $\rho(x, y) = (\rho'(x), \theta\rho'(x) + y)$ and we have a tubular neighborhood $\tau : E \times \mathbb{R}^n \to V''$ of M'' given by $\tau(x, y) = (\tau'(z), \theta\rho'(z) + y)$. By Proposition 2, there is a small section σ of ρ so that $\tau\sigma(M'') = \pi'(M')$. So we must show that $\kappa\tau\sigma$ is injective. But $\kappa\tau\sigma = \tau'\sigma'\kappa|_{M''}$ for some section σ' of ρ' so in particular, $\kappa\tau\sigma$ is injective.

Now by Lemma 4, there is an algebraic set Y and there are entire rational functions $\eta : Y \to V$ and $\mu'' : M' \to Y$ so that:

a′) $\eta(Y) = \pi(V, L)(\varphi(Y')) \cup L = \pi(M') \cup L$.

b′) η is imbedding.

c) $\eta \circ \mu'' = \pi(V, L) \circ \varphi = \pi'' \circ \eta'$.

But a′) and Proposition 2a) imply that $\eta(Y) = \pi(M')$, so a) is proven. Then b′) implies b). Let $\mu = \mu'' \circ \mu'$. To prove c), note that $\pi|_{M'} = \pi'' \circ \pi'|_{M'} = \pi'' \circ \eta' \circ \mu' = \eta \circ \mu'' \circ \mu' = \eta \circ \mu$. □

We say that a smooth manifold is *full* if its $\mathbb{Z}/2\mathbb{Z}$ homology is generated by closed smooth submanifolds.

LEMMA 6. *Suppose V is a nonsingular totally algebraic real algebraic set and $M \subset V$ is a compact smooth manifold which is full. Then for some n there are arbitrarily small isotopies of $V \times \mathbb{R}^n$ carrying $M \times 0$ to a nonsingular totally algebraic real algebraic subset of $V \times \mathbb{R}^n$.*

Proof. Since M is full, we may pick submanifolds $M_i \subset M$ which generate the $\mathbb{Z}/2\mathbb{Z}$ homology of M. We may isotop these M_i until they are in general position. By Theorem 2.10 of [AK3] there are arbitrarily small isotopies of $V \times \mathbb{R}^n$ which take M and each M_i to algebraic sets. But if h_1 is such an isotopy, then $h_1(M \times 0)$ is totally algebraic since its homology is generated by the algebraic sets $h_1(M_i \times 0)$. □

LEMMA 7. *Let M be a closed smooth manifold. then there is a full multiblowup of M, i.e. a sequence of smooth blowups $M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 = M$ with centers L_i so that M_n and each L_i are full.*

Proof. We may assume M is a nonsingular real algebraic set. By Theorem 13 of [AK5] there is an A_0H_*-uzunblowup $\pi : M' \to M$ so that $H_k(M'; \mathbb{Z}/2\mathbb{Z}) = A_0H_k(M'; \mathbb{Z}/2\mathbb{Z})$ for all k. This means that $\pi : M' \to M$ is a sequence of blowups of M (perhaps interspersed with entire rational functions which are diffeomorphisms) and all the centers L satisfy $H_k(L; \mathbb{Z}/2\mathbb{Z}) = A_0H_k(L; \mathbb{Z}/2\mathbb{Z})$ for all k. The subgroup $A_0H_k(L; \mathbb{Z}/2\mathbb{Z}) \subset H_k(L; \mathbb{Z}/2\mathbb{Z})$ is defined to be the subgroup generated by connected components of k dimensional nonsingular algebraic subsets of L with certain properties. In particular, M' and all centers L are full. So by ignoring the algebraic structure, $\pi : M' \to M$ is a full multiblowup of M. □

LEMMA 8. *Suppose L and V are totally algebraic nonsingular real algebraic sets with L compact and $L \subset V$. Then $B(V, L)$ is totally algebraic.*

Proof. The proof is similar to results in [AK1] and [AK5] but was apparently never done explicitly. It is convenient to use the terminology of [AK1] and [AK5] that if Y is a real algebraic set then $H_*^A(Y)$ is the subgroup of $H_*(Y, \mathbb{Z}/2\mathbb{Z})$ generated by compact algebraic subsets. Also if Y is compact and nonsingular then $H_A^*(Y) \subset H^*(Y, \mathbb{Z}/2\mathbb{Z})$ is the subgroup of Poincare duals of elements of $H_*^A(Y)$. Thus we know $H_*^A(L) = H_*(L, \mathbb{Z}/2\mathbb{Z})$ and $H_*^A(V) = H_*(V, \mathbb{Z}/2\mathbb{Z})$.

Let W denote $B(V, L)$, let $\pi : W \to V$ be $\pi(V, L)$ and let $P = \pi^{-1}(L)$. We first show that $\pi_*(H_*^A(W)) = H_*(V, \mathbb{Z}/2\mathbb{Z})$ and consequently $H_*^A(W) + \ker \pi_* = H_*(W, \mathbb{Z}/2\mathbb{Z})$. Pick any homology class $\alpha \in H_i(V, \mathbb{Z}/2\mathbb{Z})$. We may pick an entire rational function $\varphi : X \to V$ from a compact nonsingular real algebraic set X so that $\varphi_*([X]) = \alpha$. We may approximate φ by a smooth function f which is transverse to L. But then by Theorem 4 of [AK5] we may assume that f is rational after replacing X by some other algebraic set diffeomorphic to X. What this all boils down to is that we could have assumed that φ is transverse to L. But then by Proposition 11 of [AK5] we have an entire rational function $\varphi' : B(X, \varphi^{-1}(L)) \to W$ so that $\pi \circ \varphi' = \varphi \circ \pi(X, \varphi^{-1}(L))$. Then if $\beta = \varphi'_*([B(X, \varphi^{-1}(L))]) \in H_*^A(W)$ we have $\alpha = \pi_*(\beta)$.

So we only need show that $\ker \pi_* \subset H_*^A(W)$. We now do an argument similar to that of Lemma 8 of [AK5]. By comparing the exact sequences of the pairs (W, P) and (V, L) we see that $\ker \pi_*$ is contained in the image of $\imath_*$ where $\imath : P \to W$ is inclusion. So it suffices to show that P is totally algebraic. It is most convenient to show it using cohomology, so we will show that $H_A^*(P) = H^*(P, \mathbb{Z}/2\mathbb{Z})$.

Now P is a bundle over L with projective space fiber. By Theorem 5.7.9 of [S] we know the $\mathbb{Z}/2\mathbb{Z}$ cohomology ring of P is generated by ζ and $(\pi|_P)^*(H^*(L, \mathbb{Z}/2\mathbb{Z}))$ where ζ is the first Stiefel–Whitney class of the normal bundle of P in W. By Theorem 6.6 of [AK1], $H_A^*(P)$ is closed under cup products, so it suffices to show that ζ and $(\pi|_P)^*(H^*(L, \mathbb{Z}/2\mathbb{Z}))$ are in $H_A^*(P)$. If $\alpha \in H^*(L, \mathbb{Z}/2\mathbb{Z})$, its Poincare dual is represented by an algebraic subset $X \subset L$, but then the algebraic subset $\pi^{-1}(X) \subset P$ is Poincare dual to $(\pi|_P)^*(\alpha)$. So we only need show $\zeta \in H_A^*(P)$. But ζ is Poincare dual to $P \cap P'$ where P' is an isotoped copy of P transverse to P (since the Poincare dual to ζ is the zeroes of a transverse section of the normal bundle of P in W). By Theorem 4 of [AK5] there is a nonsingular algebraic set Q diffeomorphic to P and an entire rational function $\mu : Q \to W$ so that μ is transverse to P and closely isotopic to an imbedding onto P. Then $\mu| : \mu^{-1}(P) \to P$ represents the Poincare dual of ζ. □

We can now prove an interesting theorem.

THEOREM 9. *Let M be a closed smooth manifold, then M is homeomorphic to an algebraic set Y with totally algebraic homology.*

Proof. By Lemma 7 there is a full multiblowup of M, $M_n \to M_{n-1} \to \cdots \to M_1 \to M_0 = M$ with centers L_i so that M_n and each L_i are full. We claim there is an algebraic thickened multiblowup $V_n \to V_{n-1} \to \cdots \to V_1 \to V_0$ with centers K_i and imbeddings $\varphi_i : M_i \to V_i$ so that:

1) $K_i = \varphi_i(L_i)$ for $i = 0, 1, \ldots, n-1$.
2) $\varphi_n(M_n)$ and each V_i and K_i is nonsingular and totally algebraic.
3) Each $\varphi_i(M_i)$ $i \le n$ is a small fuzzy transform of $\varphi_0(M_0)$.

Let N_i denote $\varphi_i(M_i)$, let $\pi : V_n \to V_0$ be the thickened multiblowup map and let $X = \pi(N_n)$. By Proposition 2, there is a homeomorphism $\tau\sigma : N_0 \to X$. By Proposition 5, there is a real algebraic set Y and a homeomorphism $\eta : Y \to X$ such that $\eta^{-1} \circ \pi| : N_n \to Y$ is an entire rational function. We claim Y is totally algebraic. To see this, take a homology class $\alpha \in H_i(Y, \mathbb{Z}/2\mathbb{Z})$. Let $\theta : N_n \to N_0$ be the fuzzy transform map. Since θ is also a multiblowup map, we know that it is onto $\mathbb{Z}/2\mathbb{Z}$ homology by Lemma 6.5a of [AK1]. Hence there is a $\beta \in H_i(N_n, \mathbb{Z}/2\mathbb{Z})$ so that $\theta_*(\beta) = (\tau\sigma)^{-1}{}_*\eta_*(\alpha)$. Since N_n has totally algebraic homology, we may represent β by an i-dimensional algebraic set $W \subset N_n$. Now by Proposition 2c),

$\pi|_*(\beta) = (\tau\sigma)_*\theta_*(\beta)$, but $(\tau\sigma)_*\theta_*(\beta) = \eta_*(\alpha)$, hence $(\eta^{-1} \circ \pi|)_*(\beta) = \alpha$. But $\eta^{-1} \circ \pi$ is an entire rational function so α is an algebraic class.

So to prove this result we only need to prove our claims 1), 2) and 3) above. Suppose that for some $k \geq 0$ we have an algebraic thickened multiblowup $V_k \to V_{k-1} \to \cdots \to V_1 \to V_0$ with centers K_i and imbeddings $\varphi_i : M_i \to V_i$ so that:

1′) $K_i = \varphi_i(L_i)$ for $i = 0, 1, \ldots, k-1$.

2′) Each V_i $i \leq k$ and K_i $i \leq k-1$ is nonsingular and has totally algebraic homology.

3′) Each $\varphi_i(M_i)$ $i \leq k$ is a small fuzzy transform of $\varphi_0(M_0)$.

For example, we can do this for $k = 0$ by taking $V_0 = \mathbb{R}^m$. First let us do the case $k < n$. By Lemma 6, after replacing V_k by some $V_k \times \mathbb{R}^m$, we may assume that $\varphi_k(L_k)$ is a nonsingular totally algebraic subset of V_k. We may then set $K_k = \varphi_k(L_k)$, $V_{k+1} = B(V_k, K_k)$ and let $\varphi_{k+1} : B(M_k, L_k) \to B(V_k, K_k)$ be the canonical map. Note that V_{k+1} is totally algebraic by Lemma 8. Thus by induction we may as well assume that $k = n$. But then Lemma 6 again implies that, after replacing V_n by some $V_n \times \mathbb{R}^m$, we may assume that $\varphi_n(M_n)$ is a nonsingular totally algebraic subset of V_n. So we are done. □

COROLLARY 10. *Let M be a closed smooth manifold. Then there are algebraic sets Y and Z and an entire rational function $\pi : Z \to Y$ so that Z is nonsingular, π is one to one on π^{-1} (Nonsing Y), Y and Z are totally algebraic and Y is homeomorphic to M.*

Proof. This follows from the proof of Theorem 9 by setting $Z = N_n$. □

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