A SOLUTION TO A CONJECTURE OF ZEEMAN

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ONE OF the useful properties of the 4-ball is that any loop γ in S^3 bounds a PL-disc in B^4 (not necessarily locally flat), namely the cone on γ. In [5] Zeeman conjectured that this property fails for contractible 4-manifolds, that is he conjectured that there exists a contractible 4-manifold W and a loop $\gamma \subset \partial W$ such that γ can not bound an imbedded PL disc in W. He proposed Mazur manifolds as possible source of counterexamples. Here we prove this conjecture. Amazingly our counterexample is almost the same Mazur manifold which he drew in his paper (to be specific our Mazur manifold differs by an overcrossing from his, and that particular one fails to be a counterexample). Along the way we construct a fake smooth structure on a ribbon disc complement Q in B^4 with fundamental group $\mathbb{Z}$. In particular Q is a fake smoothing of a 4-manifold homotopy equivalent to S^1.

Let $\gamma \subset \partial W$ be the loop on the boundary of the contractible 4-manifold W given by the following handle decomposition. Here we use the convention of [3].
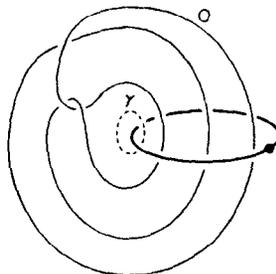

Fig. 1.

THEOREM 1. *The loop γ can not bound an imbedded PL disc in W.*

Let $f : \partial W \to \partial W$ be the involution obtained by first surgering $S^1 \times B^3$ to $B^2 \times S^2$ in the interior of W, then surgering the other imbedded $B^2 \times S^2$ back to $S^1 \times B^3$ (i.e. replacing the dots in Fig. 2). This diffeomorphism f takes the loop γ to the loop δ, i.e. $f(\gamma) = \delta$. Also notice that δ is smoothly slice in W.

Now recall the important property of W:

THEOREM 2 ([1]). *γ is not smoothly slice in W, in particular f does not extend to a self diffeomorphism of W. Moreover if we glue W to itself via f we get S^2, i.e. $S^4 = W \cup_f (-W)$.*

Proof. (of Theorem 1) Suppose γ bounds a PL imbedded 2-disc D in W. Then D is locally flat except a finite number of singular points, where D is a cone on a knot in B^4 near these
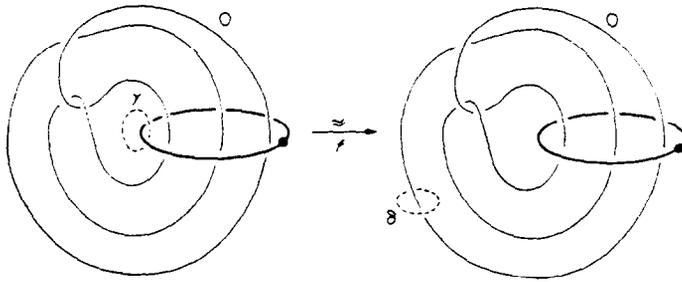
Fig. 2.

points. By isotopy we can push these points together, that is we can assume that there is only one singular point x_0. Then there is a 4-ball neighborhood V of x_0 in $int(W)$ and a knot K in S^3 such that there is a diffeomorphism $(V, D \cap V) \approx (B^4, cone\ (K))$. Now since $W \cup_f (-W) = S^4$ and δ is slice in $-W$ and $f(\gamma) = \delta$, K is slice in $W \cup_f (-W) - int(V) \approx B^4$. So K bounds a smooth disc D_0 in V. This implies that γ bounds a smooth disc in W, namely $D^* = (D - V \cap D) \cup D_0$ as indicated in Fig. 3. This contradicts Theorem 2.   ∎
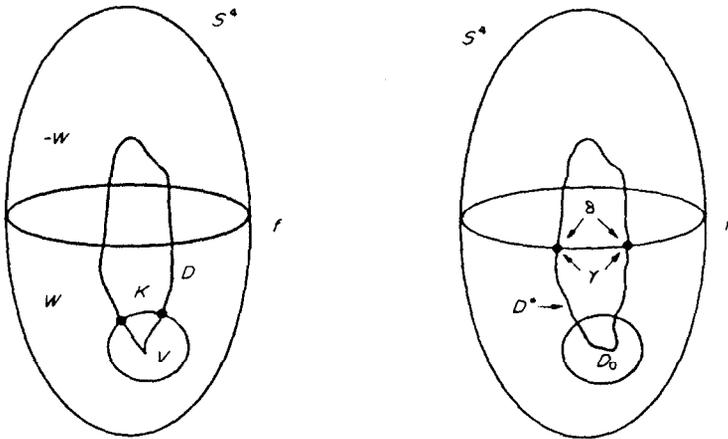

Fig. 3.

We can draw Fig. 1 as in Fig. 4. Then clearly the involution f is induced by 180˚ rotation along z axis and f fixes the isotopy class of the loop α. Notice α bounds an obvious disc D^2 in W. Hence we can extend f over D^2, so f extends over a tubular neighborhood $N(D^2)$ of D^2. Let $Q = W - N(D^2)$. f induces a map $f^*: \partial Q \to \partial Q$. Clearly f^* can not extend smoothly over Q otherwise this would give an extension of f over W contradicting Theorem 2. The affect of removing a tubular neighborhood of D^2 from W is the introducing a 1-handle as in Fig. 5, i.e. putting a dot on α (c.f. [2]). So Fig. 5 is the handlebody of Q. By sliding the 2-handle over the 1-handle as indicated in Fig. 5 we get Fig. 6 and by another handle slide we obtain Fig. 7 which also describes Q. The shading in the figures indicates the ribbon discs which are pushed inside of B^4 and removed from B^4. In Fig. 7 we also draw the pictures of the loops γ and δ which the induced map f^* switches. Easily from Fig. 7 we see that $\pi_1(Q) = \mathbb{Z}$. f^* extends to a self homotopy equivalence of Q, and by [4] f^* extends to a homeomorphism $F: Q \to Q$. Let Q^* be the smooth structure on Q obtained by pulling back
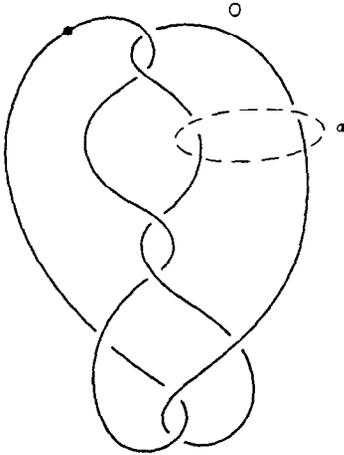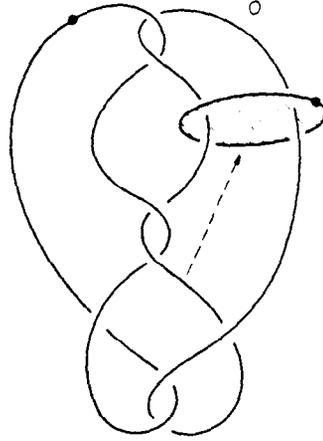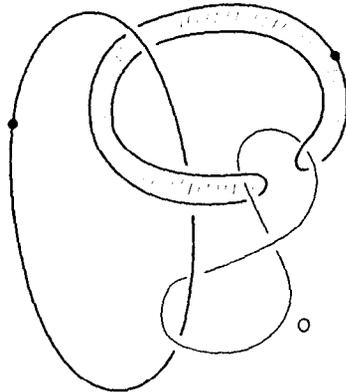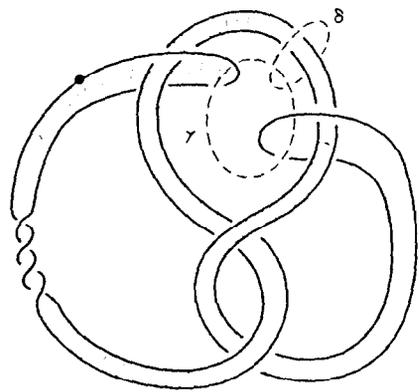
Fig. 4.

Fig. 5.

Fig. 6.

Fig. 7.

the smooth structure of Q by F. Then clearly Q^* can not be diffeomorphic rel boundary to Q, i.e. the indentity can not extend to a diffeomorphism $Q \to Q^*$.

REFERENCES

1. S. AKBULUT: A fake compact contractible manifold, *J. of Diff. Geom.* (to appear).
2. S. AKBULUT: On two dimensional homology classes of 4-manifolds, *Math. Proc. Camb. Phil. Soc.* **82** (1977), 99–106.
3. S. AKBULUT and R. KIRBY: Mazur manifolds, *Michigan Math. J.* **26** (1979), 259–284.
4. M. FREEDMAN: The topology of 4-dimensional manifolds, *J. of Diff. Geom.* **17** (1982), 357–453.
5. E. C. ZEEMAN: On the dunce hat, *Topology* **2** (1963), 341–358.

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