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SOME NEW RESULTS ON THE TOPOLOGY OF NONSINGULAR REAL ALGEBRAIC SETS

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When is a closed smooth connected submanifold M^m of $\mathbf{R}^n$ isotopic to an algebraic subset? This is an old misunderstood problem whose answer is often thought to be known.

Evidently Seifert is the first one to make progress on this question. In his 1936 paper [S] he showed that if M has a trivial normal bundle, then it can be isotoped to a nonsingular component Z_0 of an algebraic subset Z. Furthermore, he showed that one can take $Z = Z_0$ if either $n - m = 1$, or $n - m = 2$ and M is orientable. His method in fact gives Z to be a complete intersection in $\mathbf{R}^n$. This conclusion makes his result the best possible; because it turns out that there are homotopy theoretical obstructions to isotoping submanifolds with trivial normal bundle to complete intersections [AK6].

In 1952 in his celebrated paper Nash [N] generalized Seifert's result with a weaker conclusion. He proved that in general M can be isotoped to a nonsingular sheet of an algebraic subset of $\mathbf{R}^n$ (the sheets might intersect each other). Then when $n \geq 2m + 1$, by a normalization process, he was able to separate the sheets in $\mathbf{R}^n$, thereby isotoping M to a nonsingular component of an algebraic subset of $\mathbf{R}^n$ provided $n \geq 2m + 1$. He conjectured the same holds without the dimension restriction.

Wallace's 1957 attempt [W] to prove this conjecture failed (see [K, page 823]). In 1973, Tognoli [T1] improved Nash's result by getting rid of the extra unwanted components in the conclusion of the Nash's theorem, i.e., showing that M is in fact isotopic to a nonsingular algebraic subset of $\mathbf{R}^n$ provided $n \geq 2m + 1$ (this result is sometimes incorrectly referred to as the solution of the Nash's conjecture). Later Tognoli [T1] and Ivanov [I] improved this result to $n > 3m/2$. Most recently another attempt to prove

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this conjecture in [T2] failed (see [AK4]). We prove this conjecture and generalize to immersions. We also give the necessary and sufficient homotopy theoretical conditions to removing the extra components, and give some further generalizations of these results.

Theorem 1. *If $M \subset \mathbf{R}^n$ is a smooth closed submanifold, then M is ε-isotopic to the nonsingular points of a real algebraic subset of $\mathbf{R}^n$. In particular, M is isotopic to a union of components of a real algebraic subset of $\mathbf{R}^n$.*

One can ask whether a similar result holds for immersed submanifolds of $\mathbf{R}^n$. To answer this we make the following definition for algebraic sets, which gives the correct notion of nonsingularity for algebraic sets that are images of immersions of smooth manifolds:

Definition. We say x is an *almost nonsingular point* of an algebraic set X of dimension d, if a neighborhood of x in X is a union of analytic manifolds of dimension d, and furthermore the complexification of these analytic manifolds form a neighborhood of x in the complexification $X_{\mathbf{C}}$ of X. An algebraic set consisting entirely of almost nonsingular points of dimension d is called *almost nonsingular algebraic set.*

Hence all nonsingular points are almost nonsingular, and the converse is true if X is normal. We can generalize Theorem 1 as follows:

Theorem 2. *If $f : M \to \mathbf{R}^n$ is a smooth immersion of a smooth closed manifold, then f is ε-regularly homotopic to a smooth immersion f' onto the almost nonsingular points of a real algebraic subset of $\mathbf{R}^n$. In particular $f'(M)$ is a union of components of a real algebraic subset of $\mathbf{R}^n$.*

We can further generalize these results when $\mathbf{R}^n$ is replaced by a nonsingular algebraic set V, except in this case we must cross with $\mathbf{R}$. Of course, in this case there is also the necessary bordism condition: Recall, we say that a bordism class of a map $f : M \to V$ is algebraic if it can be represented by a rational map $g : W \to V$ from a nonsingular algebraic set W. This bordism condition is satisfied if V has totally algebraic homology, i.e., all its $\mathbf{Z}_2$-homology is generated by algebraic sets [AK1, AK2]. The following gives a certain generalization of Proposition 2.3 in [AK3].

Theorem 3. *If $f : M \to V$ is an immersion of a smooth closed manifold into a nonsingular algebraic set V and the bordism class of f is algebraic, then f is ε-regularly homotopic to an immersion f' onto the almost nonsingular points of a real algebraic subset of $V \times \mathbf{R}$. In particular $f'(M)$ is a union of components of a real algebraic subset of $V \times \mathbf{R}$.*

We can also ask the strong version of the Nash conjecture, that is if M is isotopic to an algebraic set in $\mathbf{R}^n$ (not just a component of it). As a corollary to the above theorems we show:

Theorem 4. *If $M \subset \mathbf{R}^n$ is a smooth closed manifold, then M is ε-isotopic to a nonsingular algebraic subset in $\mathbf{R}^n \times \mathbf{R} = \mathbf{R}^{n+1}$.*

Theorem 5. *If $M \subset V$ is a smooth closed submanifold of a nonsingular algebraic set V and the bordism class of the inclusion $f : M \to V$ is algebraic, then M is ε-isotopic to a nonsingular algebraic subset in $V \times \mathbf{R}^2$.*

In general we reduce this problem of getting rid of the extra components to a cobordism problem as follows: Recall that the cobordism classes of codimension k imbedded smooth closed submanifolds of $\mathbf{R}^n$ are classified by the homotopy group $\pi_n(MO_k)$, where MO_k is the Thom space of the universal $\mathbf{R}^k$-bundle. Similarly, $\pi_n(\Omega^\infty\Sigma^\infty MO_k)$ classifies the cobordism classes of immersed codimension k submanifolds of $\mathbf{R}^n$, [We]. Let $\pi_n^{\mathrm{alg}}(\Omega^\infty\Sigma^\infty MO_k)$ be the subgroup of $\pi_n(\Omega^\infty\Sigma^\infty MO_k)$ generated by almost nonsingular algebraic subsets of $\mathbf{R}^n$. Then we can characterize the immersed submanifolds of $\mathbf{R}^n$ that can be made algebraic in $\mathbf{R}^n$ as follows:

Theorem 6. *If $f : M \to \mathbf{R}^n$ is a codimension k immersion of a smooth closed manifold, then f is ε-regularly homotopic to a smooth immersion f' onto an almost nonsingular real algebraic subset of $\mathbf{R}^n$ if and only if the cobordism class of f lies in $\pi_n^{\mathrm{alg}}(\Omega^\infty\Sigma^\infty MO_k)$.*

There is also a version of this theorem when $\mathbf{R}^n$ is replaced with a nonsingular algebraic set V, except it requires some extra assumptions since we don't have a group structure for the cobordisms of immersions into V. Discussion of this along with the proofs of the above-stated results appear in [AK4, AK5]. The methods of [AK4] imply that the image of the map

$$\pi_{n-1}(\Omega^\infty\Sigma^\infty MO_{k-1}) \to \pi_n(\Omega^\infty\Sigma^\infty MO_k)$$

is contained in $\pi_n^{\mathrm{alg}}(\Omega^\infty \Sigma^\infty MO_k)$. This means that if an immersed submanifold of $\mathbf{R}^n$ deforms through an immersed cobordism into $\mathbf{R}^{n-1}$, then it can be isotoped to an algebraic set in $\mathbf{R}^n$.

We now give an outline of the proofs. We first prove three propositions. The first one is a generalization of our normalization theorem from [AK1, (Proposition 2.8)]. The second surprisingly tells when we can approximate a smooth function with a complex polynomial; the third one gives a sufficient condition for an image of a nonsingular component of an algebraic set under an immersion to be a component of an algebraic set.

Proposition A. *Let $f : M \to V$ be a smooth immersion from a compact smooth manifold to a nonsingular algebraic set. Suppose the bordism class of f is algebraic. Then there is a real algebraic set Z, a diffeomorphism $h : M \to \mathrm{Nonsing}(Z)$, and a polynomial $\rho : Z \to V$ such that:*

(a) *$\rho \circ h : M \to V$ approximates f.*

(b) *If $\rho_{\mathbf{C}} : Z_{\mathbf{C}} \to V_{\mathbf{C}}$ is the complexification of ρ, then $\rho_{\mathbf{C}}$ is a finite regular map.*

(c) *$\mathrm{Nonsing}(Z)$ is a union of connected components of $\rho_{\mathbf{C}}^{-1}(V)$.*

Furthermore, in case $V = \mathbf{R}^n$ (and hence the bordism condition is always satisfied) we may pick $Z_{\mathbf{C}}$ to be nonsingular.

Proposition B. *Let $\theta : \mathbf{C}^m \to \mathbf{C}^n$ be a polynomial defined over $\mathbf{R}$. Let T be a compact subset of $\theta^{-1}(\mathbf{R}^n)$ which is invariant under complex conjugation, and $g : T \to \mathbf{C}$ be a continuous function with $g(\overline{z}) = \overline{g(z)}$ for all $z \in T$. Suppose $\theta|_T$ is finite-to-one. Then there exists a polynomial $\gamma : \mathbf{C}^m \to \mathbf{C}$ defined over $\mathbf{R}$ such that $\gamma|_T C^0$-approximates to g. If g can be locally C^k-approximated by polynomials defined over $\mathbf{R}$, then we may conclude that $\gamma|_T$ is a C^k-approximation to g.*

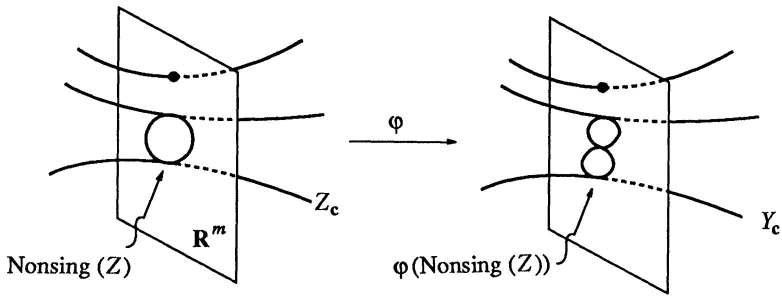
Proposition C. *Let $Z \subset \mathbf{R}^m$ be a real algebraic set and $Z_{\mathbf{C}} \subset \mathbf{C}^m$ be its complexification. Let $\varphi : Z_{\mathbf{C}} \to \mathbf{C}^n$ be a proper polynomial map defined over $\mathbf{R}$. Let Y be the Zariski closure of $\varphi(Z)$ in $\mathbf{R}^n$. Assume:*

(a) *$\mathrm{Nonsing}(Z)$ is closed.*

(b) *$\varphi|_{\mathrm{Nonsing}(Z)}$ is a smooth immersion.*

(c) *$\varphi^{-1}\varphi(z) \subset \mathrm{Nonsing}(Z)$ for all $z \in \mathrm{Nonsing}(Z)$.*

Then $\varphi(\mathrm{Nonsing}(Z))$ is the set of almost nonsingular points of Y.



We can now outline the proof of Theorem 3:
We apply Proposition A to the given immersion $f : M \to V$, and get the conclusion of this proposition.

Let $K \subset V$ be a compact set containing a neighborhood of $\rho(\text{Nonsing}(Z))$. Then since $\rho_{\mathbf{C}}$ is proper, the set $T = \rho_{\mathbf{C}}^{-1}(K)$ is compact. Define $g : T \to \mathbf{C}$ by $g(z) = 0$ for all $z \in \text{Nonsing}(Z)$, and $g(z) = 2$ for all $z \in T - \text{Nonsing}(Z)$. By Proposition B, there is a polynomial $\gamma : Z_{\mathbf{C}} \to \mathbf{C}$ defined over $\mathbf{R}$ such that $\gamma|_T$ approximates g. Let $\varphi = (\rho_{\mathbf{C}}, \gamma) : Z_{\mathbf{C}} \to Y_{\mathbf{C}} \times \mathbf{C}$ where $Y_{\mathbf{C}} = \rho_{\mathbf{C}}(Z_{\mathbf{C}})$. Then φ is proper, hence $\varphi(Z_{\mathbf{C}})$ is a complex algebraic subset of $Y_{\mathbf{C}} \times \mathbf{C}$. Since $\varphi \circ h$ is close to $(f, 0)$, M is ε-isotopic to $\varphi(\text{Nonsing}(Z))$. To conclude Theorem 3, we apply Proposition C; i.e., we only have to show (c) of Proposition C. Let $z \in \text{Nonsing}(Z)$; if $\varphi(z) = \varphi(w)$, then $\rho_{\mathbf{C}}(z) = \rho_{\mathbf{C}}(w)$, hence $w \in T$. But if $w \in T - \text{Nonsing}(Z)$, then $\gamma(w)$ is near 2, but $\gamma(z)$ is near 0, hence we can not have $\gamma(z) = \gamma(w)$. Therefore $w \in \text{Nonsing}(Z)$. Hence $\varphi^{-1}\varphi(z) \subset \text{Nonsing}(Z)$. We are done. Theorem 5 easily follows from Theorem 3. The other theorems are more sophisticated versions of these procedures.

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