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On approximating submanifolds by algebraic sets and a solution to the Nash conjecture*

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In [N] Nash showed that any compact smooth submanifold M of $\mathbb{R}^n$ is ε-isotopic to a union of connected components of a real algebraic variety if $n > 2\dim(M)$. (ε-isotopic just means that there are arbitrarily small smooth isotopies to components of real algebraic varieties.) For smaller n however, he was not able to get such a strong result but conjectured it was still true. In [W] a proof of this conjecture was given which unfortunately had a gap. Later, [I] and [T2] gave correct proofs in the case where n is bigger than roughly $3\dim(M)/2$. Among other results in this paper, we give a proof of this conjecture of Nash. In particular, we will show:

Theorem A *If $M \subset \mathbb{R}^n$ is a compact smooth submanifold, then M is ε-isotopic to the nonsingular points of a real algebraic subset of $\mathbb{R}^n$. In particular, M is isotopic to a union of components of a real algebraic subset of $\mathbb{R}^n$.*

As a corollary we obtain:

Theorem B *If $M \subset \mathbb{R}^n$ is a compact smooth submanifold, then M is ε-isotopic to a nonsingular real algebraic subset of $\mathbb{R}^{n+1}$.*

Theorem A is a special case of:

Theorem C *If $f: M \to \mathbb{R}^n$ is a smooth immersion of a smooth compact manifold M, then f is ε-regularly homotopic to a smooth immersion f' onto the almost nonsingular points of a real algebraic subset of $\mathbb{R}^n$. In particular, $f'(M)$ is a union of components of a real algebraic subset of $\mathbb{R}^n$.*

The ε-regularly homotopic condition means that there is a small homotopy of f to f' through immersions. The notion of an almost nonsingular point x of a real algebraic set X is new. It is as close as the image of an immersion can be to nonsingular. We say x is an *almost nonsingular* point of X if a neighborhood of x in X is a finite union of analytic manifolds with dimension equal to the dimension of X and the analytic complexifications of these analytic manifolds form a neighborhood of x in the algebraic complexification $X_{\mathbb{C}}$ of X. Thus all nonsingular points

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are almost nonsingular. Furthermore, if X is normal and almost nonsingular at x then x is a nonsingular point of X. This is because normality implies there is only one analytic branch at x. Almost nonsingularity then says that a neighborhood of x in $X_{\mathbb{C}}$ is a complex analytic manifold. But then x is a nonsingular point of X, (see [M, p. 13]). Likewise, if the f in Theorem C above is an imbedding then f' will be an imbedding so all the almost nonsingular points of $f'(M)$ will in fact be nonsingular. So Theorem A is a corollary of Theorem C.

If $\mathbb{R}^n$ is replaced by a general nonsingular real algebraic set, then we get analogous theorems except that we must always cross with $\mathbb{R}$. Thus we have the following analogue of Theorem B:

Theorem D *If $M \subset V$ is a compact smooth submanifold of a nonsingular real algebraic set V and the bordism class of the inclusion $f: M \hookrightarrow V$ is algebraic, then M is ε-isotopic to a nonsingular real algebraic subset of $V \times \mathbb{R}^2$.*

The bordism condition in Theorem D is a certain necessary condition, a bordism class is algebraic if it is represented by a regular map $\rho: W \to V$ from a nonsingular algebraic set W. This bordism condition is satisfied if V has totally algebraic homology, i.e. all its $\mathbb{Z}/2\mathbb{Z}$ homology is generated by algebraic subsets, see [AK1, Lemma 2.5].

We also get the following analogues of Theorems C and A:

Theorem E *If $f: M \to V$ is a smooth immersion of a smooth compact manifold M to a nonsingular real algebraic set V and the bordism class of f is algebraic, then f is ε-regularly homotopic to a smooth immersion f' onto the almost nonsingular points of a real algebraic subset of $V \times \mathbb{R}$. In particular, $f'(M)$ is a union of components of a real algebraic subset of $V \times \mathbb{R}$.*

Theorem F *If $M \subset V$ is a compact smooth submanifold of a nonsingular real algebraic set V and the bordism class of the inclusion $f: M \hookrightarrow V$ is algebraic, then M is ε-isotopic to the nonsingular points of a real algebraic subset of $V \times \mathbb{R}$. In particular, M is isotopic to a union of components of a real algebraic subset of $V \times \mathbb{R}$.*

Recently, [T1] claimed a result which is even stronger then the conjecture of Nash – that every compact smooth submanifold M of $\mathbb{R}^n$ can be ε-isotoped to a nonsingular algebraic subset of $\mathbb{R}^n$. A close examination of the proof reveals several errors. We will discuss this proof and provide counterexamples to some of its crucial steps. In a forthcoming paper [AK4] we show that this stronger result holds if the immersed cobordism class of the submanifold contains an algebraic representative.

To establish some notation, if A is a subset of a complex algebraic set we will let $\mathrm{CL}_{\mathbb{C}}(A)$ denote the Zariski closure of A, the smallest complex algebraic set containing A. Likewise if A is a subset of a real algebraic set we will let $\mathrm{Cl}_{\mathbb{R}}(A)$ denote the real Zariski closure of A, the smallest real algebraic set containing A. If $X \subset \mathbb{C}^n$ is a complex algebraic set, $X_{\mathbb{R}}$ will denote $X \cap \mathbb{R}^n$. If $X \subset \mathbb{R}^n$ is a real algebraic set, $X_{\mathbb{C}} \subset \mathbb{C}^n$ will denote the complexification $\mathrm{Cl}_{\mathbb{C}}(X)$. If X is an algebraic set, $\mathrm{Sing}(X)$ will denote the singular points of X and $\mathrm{Nonsing}(X)$ will denote $X - \mathrm{Sing}(X)$. A function $f: X \to Y$ between algebraic sets will be called *regular* if it is a rational function (which is defined everywhere on X). In earlier papers we called this an entire rational function which is more descriptive, but less standard. We denote a point in projective space $\mathbb{RP}^n$ or $\mathbb{CP}^n$ by $[x_0: \ldots : x_n]$. The symbol $\sim$ means approximately equal to. We will sometimes refer to finite regular maps. These have a technical definition (see [S]) but the main properties we will use are finite-to-oneness and properness (which implies that the image of an algebraic set is algebraic).

We say that a complex algebraic set $Z \subset \mathbb{C}^n$ is *defined over* $\mathbb{R}$ if it is the set of zeroes of polynomials with real coefficients. Equivalently, it is invariant under complex conjugation, $\bar{Z} = Z$. Likewise, a rational function $f: Z \to \mathbb{C}^k$ is *defined over* $\mathbb{R}$ if it is locally given as the quotient of polynomials with real coefficients. Equivalently it is equivariant, $\overline{f(z)} = f(\bar{z})$.

An analysis of Tognoli's proof

Let us first review the main idea of the proof in [T1]. We will follow the terminology of [T1]. The first step is to isotop M to a smooth algebraic subset of $\mathbb{R}^{n+1}$ (i.e. our Theorem B). In the second step, the techniques used in the first step are refined to make M algebraic in $\mathbb{R}^n$. We will give counterexamples to assertions claimed in the first step. They will automatically be counterexamples to corresponding assertions in the second step which is more delicate. The proof of the first step in [T1] is as follows. Using well-known results, one first finds a smooth real algebraic set W and a rational function $\pi: W \to \mathbb{R}^n$ which imbeds W onto a manifold $\pi(W)$ which is ε-isotopic to M. Let $W' = \mathrm{Cl}_{\mathbb{R}}(\pi(W))$. It is well-known that there is a real algebraic set S with $\dim(S) < \dim(M)$ so that $W' = \pi(W) \cup S$. Let $N(c(W'))$ be the normalization of the complexification $c(W')$ of W' and $\theta: N(c(W')) \to c(W')$ the normalization map. One component of the real points of $N(c(W'))$ is a real algebraic set $|C(\theta)^{-1}(\pi(W))|$ so that θ restricts to a diffeomorphism from $|C(\theta)^{-1}(\pi(W))|$ to $\pi(W)$.

At this point, the proof is unclear. A certain set $|C(\theta)^{-1}(C(S))|$ is defined. However, as the proof reads, it will always be empty since for dimension reasons no irreducible analytic component of the germ $c(W'_x)$ could be contained in $c(S_x)$. From later stages in the proof we see the properties it must have:

(1) $|C(\theta)^{-1}(C(S))|$ must be a complex algebraic set defined over $\mathbb{R}$.
(2) $|C(\theta)^{-1}(C(S))| \cap |C(\theta)^{-1}(\pi(W))|$ must be empty.
(3) We must have either $|C(\theta)^{-1}(C(S))| \supset \theta^{-1}(W') - |C(\theta)^{-1}(\pi(W))|$ or perhaps only $|C(\theta)^{-1}(C(S))| \supset \theta^{-1}(S - \pi(W))$.

The last condition is not certain since it is not clear how strong a result the paper is claiming. However, we shall bypass this point by presenting a counterexample to the weaker condition $|C(\theta)^{-1}(C(S))| \supset \theta^{-1}(S - \pi(W))$.

Now [T1] takes a projective closure $N(C(W'))$ of $N(c(W'))$ so that θ extends to a regular function to the projective closure $C(W')$ of $c(W')$. Now you take a hyperplane H and an entire rational function $h: N(C(W')) - H \to \mathbb{C}$ defined over $\mathbb{R}$ so that H misses $|C(\theta)^{-1}(\pi(W))|$, so that H contains no irreducible component of $|C(\theta)^{-1}(C(S))|$, so that $|C(\theta)^{-1}(C(S))| \subset h^{-1}(0)$ and so that $h|_{|C(\theta)^{-1}(\pi(W))|}$ is approximately 1. Then we have a rational function $(\theta, 1/h - 1)$: $N(C(W')) - (h^{-1}(0) \cup H) \to c(W') \times \mathbb{C}$. It is claimed in [T1] that the image of this rational function is a complex algebraic set. In fact this is not true, the Zariski closure of its image will contain the set $\theta(H \cap \mathrm{Cl}_{\mathbb{C}}(h^{-1}(0))) \times \mathbb{C}$ since that is the image of the indeterminate points of the rational function from $N(C(W'))$ to $C(W') \times \mathbb{CP}^1$ which extends $(\theta, 1/h - 1)$. We give an example of this below in

Example 3. However, it is not hard to fix this part of the proof by always taking H to be the standard hyperplane at ∞. (The requirement that H not contain any irreducible component of $|C(\theta)^{-1}(C(S))|$ is unnecessary.) Do not bother to take the projective closure. Pick a polynomial $h: N(c(W')) \to \mathbb{C}$ defined over $\mathbb{R}$ so that $h^{-1}(0) \supset |C(\theta)^{-1}(C(S))|$ and $h|_{|C(\theta)^{-1}(\pi(W))|}$ is approximately 1. Now $(\theta, 1/h - 1)$: $N(c(W')) - h^{-1}(0) \to c(W') \times \mathbb{C}$ is a proper rational function, hence its image is a complex algebraic set. The proof now finishes by noting that the real points of $(\theta, 1/h - 1)(N(c(W') - (h^{-1}(0) \cup H))$ form an analytic manifold isotopic to $\pi(W) \times 0$. This real algebraic set would be nonsingular if the stronger condition $|C(\theta)^{-1}(C(S))| \supset \theta^{-1}(W') - |C(\theta)^{-1}(\pi(W))|$ were satisfied.

One point we would like to make is not terribly crucial to the proof but indicates a phenomenon apparently not considered by the author. Suppose that in the above, $\pi(W)$ is a real algebraic set. Then we could take S to be empty and the proof in [T1] would then allow us to take $|C(\theta)^{-1}(C(S))|$ empty and h identically 1. Then [T1] claims that $\pi(W)$ is a nonsingular algebraic set. However this is not necessarily so, Example 1 below shows that there are compact singular irreducible real algebraic sets which are analytic manifolds. This gap is easily fixed by defining S to be $\mathrm{Sing}(W')$. However, it does indicate that we can give two interpretations to algebraically approximating a manifold. The weak interpretation is that it is ε-isotopic to an analytic manifold which is a real algebraic set. The stronger and more useful interpretation is that it is ε-isotopic to a nonsingular real algebraic set.

Example 1 We now present an example which illustrates the above phenomenon. Consider the algebraic sets $W = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 = 1\}$ and $W' = \{(x, y) \in \mathbb{R}^2 | (x^2 + y^2)^2 - 2xy^2 - 6x^3 = 0\}$. Let $\pi: W \to W'$ be $\pi(x, y) = ((x-2)(x-1), (x-2)y)$. If $(x, y) \in W'$ then $x \geq 0$ and $y^2 = x(1 - x + \sqrt{4x+1})$ so $(x, y) = \pi(u, v)$ where $u = 3/2 - \sqrt{x + 1/4}$ and $v = y/(u-2)$. Thus $W' = \pi(W)$ and by checking the Jacobian we see that π imbeds W onto W'. So W' is a real algebraic set and a smooth analytic manifold. However, W' is a singular algebraic set, since (0, 0) is a singular point of W'. A similar noncompact example appears in [M].

We now look at a more serious gap in the proof. It was asserted that there is an algebraic set $|C(\theta)^{-1}(C(S))|$ so that $|C(\theta)^{-1}(C(S))| \supset \theta^{-1}(S - \pi(W))$ and $|C(\theta)^{-1}(C(S))| \cap |C(\theta)^{-1}(\pi(W))|$ is empty. (This will give the weak approximation result. For the stronger approximation result one needs $|C(\theta)^{-1}(C(S))| \supset \theta^{-1}(W') - |C(\theta)^{-1}(\pi(W))|$.) Example 2 shows that $|C(\theta)^{-1}(C(S))|$ with the required properties might not exist.

Example 2 Let $W \subset \mathbb{R}^3$ be the nonsingular real algebraic set

$$W = \{(x, y, z) \in \mathbb{R}^3 \,|\, (x^3 - 2)^2 + y^2 + z^2 = 1\}.$$

Let $c(W) \subset \mathbb{C}^3$ be the complexification of W,

$$c(W) = \{(x, y, z) \in \mathbb{C}^3 \,|\, (x^3 - 2)^2 + y^2 + z^2 = 1\}.$$

Let $\pi: W \to \mathbb{R}^3$ be the polynomial map $\pi(x, y, z) = (x^3 + \varepsilon xy - \varepsilon x^2 z, y, z)$ for some small $\varepsilon \neq 0$. We claim π imbeds W onto an analytic surface $\pi(W) \subset \mathbb{R}^3$. Let $W' = \mathrm{Cl}_{\mathbb{R}}(\pi(W))$. Then $W' = \pi(W) \cup S$ where $S = \mathrm{Sing}(W')$ (see Proposition 4 below). Let $c(\pi): c(W) \to \mathbb{C}^3$ be the complexification of π. For convenience let θ denote $c(\pi)$. Let $Z = \mathrm{Cl}_{\mathbb{C}}(\theta^{-1}(S - \pi(W)))$. We claim that $Z \cap W$ is nonempty.

Note $\theta\colon c(W) \to c(W')$ is the normalization of $c(W')$ since $c(W)$ is normal and θ is finite, regular, dominating and degree one. Then $|C(\theta)^{-1}(\pi(W))| = W$ and $|C(\theta)^{-1}(C(S))| \supset Z$ so $|C(\theta)^{-1}(C(S))| \cap |C(\theta)^{-1}(\pi(W))|$ is nonempty. Consequently, the method in [T1] is faulty.

We will now proceed to prove our claims. It is easy to see by checking the Jacobian that π is an immersion. On the other hand, if $\varepsilon = 0$ then π would be an imbedding onto the sphere $\{(x, y, z) \in \mathbb{R}^3 | (x-2)^2 + y^2 + z^2 = 1\}$. Hence, for ε small, π is still an imbedding. Now let us prove that $Z \cap W$ is nonempty. Let $\omega = (-1 + \sqrt{-3})/2$, a cube root of unity. Consider the following set

$$E = \{(t\omega, -ts, s) \,|\, (t^3 - 2)^2 + (t^2 + 1)s^2 = 1,\ s, t \in \mathbb{R}\}\,.$$

Then it is easy to check that E is a circle in $c(W)$. Furthermore,

$$\theta(t\omega, -ts, s) = (t^3 + \varepsilon st^2, -ts, s) \in \mathbb{R}^3\,,$$

so $\theta(E) \subset W'$. In fact we will now show that $\theta(E) \subset \operatorname{Sing}(W')$. Suppose $\theta(t\omega, -ts, s) = \pi(x, y, z)$ for some $(x, y, z) \in W$. Then it is elementary to check that $z = s$, $y = -st$, $x = t + \varepsilon s$ and $x^3 - 2 = \pm(t^3 - 2)$. If $x^3 - 2 = t^3 - 2$, then $x = t$ so $s = 0$ so $y = z = 0$ and $x^3 = 2 \pm 1$. If $x^3 - 2 = -(t^3 - 2)$, then $x^3 + t^3 = 4$. So s and t satisfy the two independent equations $t^3 + (t + \varepsilon s)^3 = 4$ and $(t^3 - 2)^2 + (t^2 + 1)s^2 = 1$. Hence there are only a finite number of possibilities for s and t. Thus $E \cap \theta^{-1}(\pi(W))$ is finite. (In fact it consists of just 4 points: $(\omega, 0, 0)$, $(\sqrt[3]{3}\omega, 0, 0)$ and two points with t approximately $\sqrt[3]{2}$.) In particular, $E \subset \operatorname{Cl}(\theta^{-1}(W' - \pi(W)))$ so $\operatorname{Cl}_{\mathbb{C}}(E) \subset \operatorname{Cl}_{\mathbb{C}}(\operatorname{Cl}(\theta^{-1}(W' - \pi(W)))) = Z$. But $\operatorname{Cl}_{\mathbb{C}}(E)$ is easy to compute, it is just

$$\{(x, y, z) \in \mathbb{C}^3 \,|\, y = -\omega^2 xz,\ (x^3 - 2)^2 + z^2(1 + \omega x^2) = 1\}\,.$$

But this has a real point $(1, 0, 0)$, so $(1, 0, 0) \in Z \cap W$.

Note that Example 2 is generic in the sense that any $\pi'\colon W \to \mathbb{R}^3$ near π will contradict the proof of [T1]. The reason is that $\operatorname{Cl}_{\mathbb{C}}(E)$ is an irreducible component of the double point set of θ. Also for most points $w \neq v$ with $\theta(w) = \theta(v)$, $d\theta(T_w(c(W)))$ is transverse to $d\theta(T_v(c(W)))$. Hence the double point set of a nearby θ' will be close to the double point set of θ. But $\operatorname{Cl}_{\mathbb{C}}(E)$ is transverse to W at $(1, 0, 0)$ so the corresponding irreducible component of the double point set of θ' will also intersect W. Also $\operatorname{Cl}_{\mathbb{C}}(E)$ is nonsingular near E and is imbedded by θ near E, so the same will be true for a nearby θ', which means that $\theta'^{-1}(\mathbb{R}^3)$ will contain a circle near E. For dimension reasons, the Zariski closure of this circle is the irreducible component near $\operatorname{Cl}_{\mathbb{C}}(E)$ of the double points of θ'. We have already seen that this must have a real point.

Our next counterexample is of the assertion that the image of $(\theta, 1/h - 1)$ is an algebraic set.

Example 3 Let

$$W = \{(x, y, z, w) \in \mathbb{R}^4 \,|\, x^2 + y^2 + z^2 = 1, \quad w = z(x - 2)\}\,.$$

Let $\pi\colon W \to \mathbb{R}^3$ be the map $\pi(x, y, z, w) = (x, y, w)$. We will now go through the steps of the proof in [T1] and see what happens. Now

$$W' = \{(x, y, z) \in \mathbb{R}^3 \,|\, z^2 + (x - 2)^2(x^2 + y^2 - 1) = 0\}$$

and

$$S = \{(2, y, 0) \in \mathbb{R}^3\} .$$

The algebraic set W' is the union of a sphere and a line. We have

$$C(W) = \{[x:y:z:w:s] \in \mathbb{CP}^4 \mid x^2 + y^2 + z^2 = s^2, \quad ws = z(x - 2s)\} .$$

Note $C(W)$ is nonsingular and hence normal. As we will see, $C(W) = N(C(W'))$. We have $\theta: C(W) \to \mathbb{CP}^3$, the complexification of π, given by $\theta([x:y:z:w:s]) = [x:y:w:s]$. Then $\theta(C(W))$ is the projective algebraic set

$$C(W') = \{[x:y:z:s] \mid z^2 s^2 + (x - 2s)^2 (x^2 + y^2 - s^2) = 0\} .$$

Since $\theta: C(W) \to C(W')$ is a finite, regular, dominating birational isomorphism it is the normalization map. Note $|C(\theta)^{-1}(\pi(W))| = W$ and

$$|C(\theta)^{-1}(C(S))| = \{[x:y:z:0:s] \in \mathbb{CP}^4 \mid x = 2s, y^2 + z^2 + 3s^2 = 0\} .$$

Let $H = \{[x:y:z:w:s] \mid y = 2s\}$. For h we take

$$h([x:y:z:w:s]) = (x - 2s)p(x, y, z, w, s)/(y - 2s)^d$$

where p is some homogeneous polynomial of degree $d - 1$ chosen so h approximates 1 on W. After adding some εs^{d-1} to p if necessary, we may assume $c = p(2, 2, \sqrt{-7}, 0, 1) \neq 0$. For each $\alpha \in \mathbb{C}$ consider the curve

$$u_\alpha(t) = [2 + t^d : 2 + \alpha t : \lambda(\alpha, t) : t^d \lambda(\alpha, t) : 1]$$

where $\lambda(\alpha, t) = \sqrt{-7 - 4t^d - t^{2d} - 4\alpha t - \alpha^2 t^2}$. Then $u_\alpha(t) \in C(W)$. But

$$(\theta(u_\alpha(t)), 1/h(u_\alpha(t)) - 1) = ([2 + t^d : 2 + \alpha t : t^d \lambda(\alpha, t) : 1], \alpha^d / p(u_\alpha(t)) - 1) .$$

This approaches $([2:2:0:1], \alpha^d/c - 1)$ as $t \to 0$. Since α is arbitrary, the Zariski closure of the image of $(\theta, 1/h - 1)$ must contain $[2:2:0:1] \times \mathbb{C}$. In particular, its real points contain $[2:2:0:1] \times \mathbb{R}$.

So there are a number of reasons why the proof in [T1] fails. Examples 1 and 3 illustrate correctable gaps, but Example 2 demonstrates a serious gap in the method.

Miscellaneous lemmas and a complex approximation theorem

Suppose $f: X \to Y$ is a regular map between complex algebraic sets and we have $Y = \mathrm{Cl}_{\mathbb{C}}(f(X))$. Then $f(X)$ is dense in Y and if f is proper, we even get $Y = f(X)$. However, a regular map $f: X \to Y$ between real algebraic sets with $Y = \mathrm{Cl}_{\mathbb{R}}(f(X))$ does not behave so nicely. Think, for example, of the projection of a circle to a line. A useful notion for dealing with this problem is the concept of the degree of f (essentially introduced in [BT], see also [AK3, Lemma 5.2] for the case of unequal dimensions). If $f: X \to Y$ is a regular map between real algebraic sets, Y is irreducible and $\dim(X) = \dim(Y)$ then there is a real algebraic set $Z \subsetneq Y$ and a $d \in \mathbb{Z}/2\mathbb{Z}$ so that for each $x \in Y - Z$, the number of points in $f^{-1}(x)$ is finite and congruent to d mod 2. Suppose $d = 1$, (for example if f imbeds an open dense subset of X). Then $Y - f(X) \subset Z$ since if $x \in Y - f(X)$, then $f^{-1}(x)$ has 0 points and 0 is not odd. Then $\dim(Y - f(X)) \leqq \dim(Z) < \dim(Y)$. One can then conclude for example that $\mathrm{Nonsing}(Y) \subset f(X)$ if f is proper. We also get the following amusing result:

Proposition 4 *Let $\rho: W \to V$ be a regular function between real algebraic sets which is one to one. Suppose $X \subset V$ is any irreducible real algebraic subset. Then* $\dim(X \cap \rho(W)) \neq \dim(X - \rho(W))$.

Proof. Let $Y = \rho^{-1}(X)$. Now $\rho|: Y \to X$ is a one to one rational function. Let d be the degree of $\rho|_Y$. Then there is an algebraic set $Z \subset X$ so that $\dim(Z) < \dim(X)$ and the number of points in $\rho^{-1}(x)$ mod 2 is d for all $x \in X - Z$. If $d = 0$ we must then have $\rho(Y) \subset Z$, so $X \cap \rho(W) \subset Z$, so $\dim(X \cap \rho(W)) < \dim(X) = \dim(X - \rho(W))$. If $d = 1$ we must have $X - \rho(Y) \subset Z$, so $X - \rho(W) \subset Z$, so $\dim(X - \rho(W)) < \dim(X) = \dim(X \cap \rho(W))$. □

The complexification of a nonsingular real algebraic set need not be nonsingular (e.g. $y^3 = (x^2 + 1)^2$) and the complexification of a regular function need not be regular (e.g. $1/(1 + x^2)$). However, the following lemma shows that up to isomorphism they are.

Lemma 5 *Let $\rho: W \to V$ be a regular function from a nonsingular affine real algebraic set. Then there is a nonsingular algebraic set $W' \subset \mathbb{R}^n$ and a regular function $\eta: W \to W'$ so that η is a diffeomorphism, η^{-1} is regular, the complexification $W'_{\mathbb{C}}$ of W' is nonsingular and $\rho\eta^{-1}: W' \to V$ extends to a regular function from $W'_{\mathbb{C}}$* to $V_{\mathbb{C}}$.

Proof. Suppose $W \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^k$. It is a well-known fact (first used in [K]) that there are polynomials $p: W \to \mathbb{R}^k$ and $q: W \to \mathbb{R}$ so that $q^{-1}(0)$ is empty and $\rho = p/q$. (Just take local expressions p_i/q_i for ρ, then $\rho = \sum p_i q_i / \sum q_i^2$.) Let $W_{\mathbb{C}}$ be the complexification of W and let $p_{\mathbb{C}}$ and $q_{\mathbb{C}}$ be the complexifications of p and q. Pick a polynomial h defined over $\mathbb{R}$ so that $\mathrm{Sing}(W_{\mathbb{C}}) \subset h^{-1}(0)$ and $h^{-1}(0) \cap W$ is empty. (For example, h could be the sum of squares of real and imaginary parts of generators of the ideal of polynomials vanishing on $\mathrm{Sing}(W_{\mathbb{C}})$.) Let $W' = \{(x, t) \in W \times \mathbb{R} \mid th(x)q(x) = 1\}$ and $\eta(x) = (x, 1/(h(x)q_{\mathbb{C}}(x)))$. Then $W'_{\mathbb{C}} = \{(x, t) \in W_{\mathbb{C}} \times \mathbb{C} \mid th(x)q_{\mathbb{C}}(x) = 1\}$ is nonsingular since it is isomorphic to $W_{\mathbb{C}} - (h^{-1}(0) \cup q_{\mathbb{C}}^{-1}(0)) \subset \mathrm{Nonsing}(W_{\mathbb{C}})$. Also $\rho\eta^{-1}(x, t) = th(x)p(x)$ which extends to the polynomial map $th(x)p_{\mathbb{C}}(x)$. □

The following lemma makes an immersion algebraic. The reasons for wanting the finiteness conclusion will become apparent later. For instance, Theorem 8 requires finite point inverses and properness is required in the proofs of Theorems A through F. We presume Lemma 6 remains true if we drop the immersion assumption on f (but require $\dim(M) \leqq \dim(V)$), drop conclusion 3 and require W to be nonsingular. However, the following form is all we need here.

Lemma 6 *Let $f: M \to V$ be a smooth immersion from a compact smooth manifold M to a nonsingular real algebraic set V. Suppose the bordism class of f is algebraic. Then there are a real algebraic set W, a diffeomorphism h: $M \to$* $\mathrm{Nonsing}(W)$ *and a polynomial map ρ: $W \to V$ so that:*

(1) *ρh: $M \to V$ approximates f.*

(2) *If $\rho_{\mathbb{C}}$: $W_{\mathbb{C}} \to V_{\mathbb{C}}$ is the complexification of ρ, then $\rho_{\mathbb{C}}$ is a finite regular map to its image.*

(3) $\mathrm{Nonsing}(W)$ *is a union of connected components of $\rho_{\mathbb{C}}^{-1}(V)$.*

Furthermore, in case $V = \mathbb{R}^n$ (and hence the bordism condition is always satisfied), we may even specify that $W_{\mathbb{C}}$ be nonsingular.

Proof. Let $m = \dim(M)$. First note that conclusion (3) follows from (1) and (2). To see this, note that at any point $x \in \text{Nonsing}(W)$, $d\rho_{\mathbb{C}}$ has rank m. Thus after an analytic coordinate change defined over $\mathbb{R}$, $\rho_{\mathbb{C}}$ is locally an injective linear map defined over $\mathbb{R}$. But an injective linear map defined over $\mathbb{R}$ takes nonreal points to nonreal points. So Nonsing(W) is open in $\rho_{\mathbb{C}}^{-1}(V)$. But it is also closed since it is the image of the compact set M. Hence it is a union of connected components of $\rho_{\mathbb{C}}^{-1}(V)$.

By Proposition 2.3 of [AK2] we may find a nonsingular real algebraic set X, a diffeomorphism $g\colon M \to X$ and a polynomial $\psi\colon X \to V$ so that ψg approximates f. The only problem is that $\psi_{\mathbb{C}}$ might not be finite.

In the case $V = \mathbb{R}^n$ we may finish the proof as follows. By Lemma 5, we may as well assume $X_{\mathbb{C}}$ is nonsingular. Suppose $X_{\mathbb{C}} \subset \mathbb{C}^k$. Let $W = \{(x, y) \in X \times \mathbb{R}^n \mid y = \psi(x)\}$, then $W_{\mathbb{C}} = \{(x, y) \in X_{\mathbb{C}} \times \mathbb{C}^n \mid y = \psi_{\mathbb{C}}(x)\}$. By the proof of Theorem 10, Chap. 1, Sect. 5 of [S], if we take a generic linear retraction $\mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^n$, its restriction to $W_{\mathbb{C}}$ is finite. So we may pick a linear retraction $\pi\colon \mathbb{C}^n \times \mathbb{C}^k \to \mathbb{C}^n$ which is defined over $\mathbb{R}$, close to the standard projection and so that $\pi|\colon W_{\mathbb{C}} \to \pi(W_{\mathbb{C}})$ is finite. So we may set $h(x) = (g(x), \psi g(x))$ and $\rho = \pi|_W$. So the case $V = \mathbb{R}^n$ is finished.

Now suppose we are back in the case $V \neq \mathbb{R}^n$. After a small perturbation of f we may assume that f imbeds an open dense subset of M. Let $Y = \text{Cl}_{\mathbb{R}}(\psi(X))$ and let $Y_{\mathbb{C}} \subset V_{\mathbb{C}}$ be its complexification. Let $\theta\colon Z \to Y_{\mathbb{C}}$ be the normalization of $Y_{\mathbb{C}}$ (see [S]). We may pick an affine model of the normalization, so assume $Z \subset \mathbb{C}^k$. By the construction given in [S], Z and θ are defined over $\mathbb{R}$. We know θ is proper because by definition it is finite. Likewise it is onto by definition. By Lemma 5 there are complexifications $X_{\mathbb{C}}$ of X and $\psi_{\mathbb{C}}$ of ψ so $X_{\mathbb{C}}$ is nonsingular and $\psi_{\mathbb{C}}$ is regular. In particular, $X_{\mathbb{C}}$ is normal so by the universal property of the normalization there is a regular mapping $\lambda\colon X_{\mathbb{C}} \to Z$ defined over $\mathbb{R}$ so that $\theta\lambda = \psi_{\mathbb{C}}$. Since λ is defined over $\mathbb{R}$ we know $\lambda(X) \subset W = Z \cap \mathbb{R}^k$. Since θ is a finite map, we know $\dim_{\mathbb{C}}(Z) = \dim_{\mathbb{C}}(Y_{\mathbb{C}})$. But $\dim_{\mathbb{C}}(Y_{\mathbb{C}}) = \dim(Y) = m$. So $\dim_{\mathbb{C}}(Z) = m$.

Pick any $x \in X$ and an open neighborhood U of x in $X_{\mathbb{C}}$ so that $\psi_{\mathbb{C}}$ imbeds U. Since $\psi_{\mathbb{C}} = \theta\lambda$ and $d\psi_{\mathbb{C}}$ has rank m, we know that $d\lambda$ has rank m. Hence λ imbeds U. But $\dim_{\mathbb{C}}(Z) = m$ and Z is everywhere locally analytically irreducible so the complex manifold $\lambda(U)$ is open in Z. So $\lambda(U) \subset \text{Nonsing}(Z)$ since analytically nonsingular points of a complex algebraic set are algebraically nonsingular ([M, p. 13]). So $\lambda(X) \subset \text{Nonsing}(W)$.

Now θ is one to one over Nonsing$(Y_{\mathbb{C}})$ since $\theta^{-1}(u)$ contains exactly one point for each analytically irreducible component of $Y_{\mathbb{C}}$ through u. Also $\psi\colon X \to Y$ has degree 1 since ψ approximates fg^{-1} which imbeds an open dense subset. Consequently, $\dim(Y - \psi(X)) < \dim(Y)$ so $\psi(X) \supset \text{Nonsing}(Y)$. So if $z \in W - \lambda(X)$ and $\theta(z) \in \text{Nonsing}(Y)$ then $\theta(z) = \psi(x) = \theta\lambda(x)$ for some $x \in X$ and hence we have a contradiction since $z \neq \lambda(x)$. So $\theta(W - \lambda(X)) \subset \text{Sing}(Y)$.

Since θ is finite,

$$\dim(W - \lambda(X)) = \dim(\theta(W - \lambda(X))) \leqq \dim(\text{Sing}(Y)) < \dim(Y) = m = \dim(W)\,.$$

But $W - \lambda(X)$ is open in W since X is compact. So for dimension reasons, $W - \lambda(X) \subset \text{Sing}(W)$. Consequently, $\lambda(X) = \text{Nonsing}(W)$.

Now let $h = \lambda g$ and $\rho = \theta|_W$. In fact $W_{\mathbb{C}} = Z$, but in any case $W_{\mathbb{C}} \subset Z$ and $\rho_{\mathbb{C}} = \theta|_W$ so $\rho_{\mathbb{C}}$ is finite. □

In Theorem 8 below we introduce the crucial technique used in our proof of Theorems A through F. It allows us to approximate smooth functions by complex polynomials in certain situations. First we need some preliminary results. If T is a subset of $\mathbb{C}^m$ we say that $f\colon T \to \mathbb{C}$ is a C^k function if it can be extended to a k times differentiable function on some neighborhood of T, (we think of $\mathbb{C}^m$ as $\mathbb{R}^{2m}$). We say two C^k functions $f\colon T \to \mathbb{C}$ and $g\colon T \to \mathbb{C}$ are C^k close if they have extensions to a neighborhood of T which are C^k close. If S is a subset of $\mathbb{C}^m$ we say that a C^k function $f\colon S \to \mathbb{C}$ can be locally C^k approximated by polynomials defined over $\mathbb{R}$ if for each $y \in S$ there is a polynomial $g\colon \mathbb{C}^m \to \mathbb{C}$ defined over $\mathbb{R}$ and a neighborhood U of y so that $g|_{S \cap U}$ is C^k close to $f|_{S \cap U}$. For example, if $S \subset \mathbb{R}^m$, any C^k function can be locally approximated by polynomials defined over $\mathbb{R}$.

Lemma 7 *Suppose $F \subset \mathbb{C}^m$ is a finite set and suppose $y \in \mathbb{C}^m$ is a point so that $y \notin F$ and $\bar{y} \notin F$. Then for any $k = 0, 1, \ldots$ there is a polynomial $q\colon \mathbb{C}^m \to \mathbb{C}$ defined over $\mathbb{R}$ so that q is C^k close to 0 near F and so that q is C^k close to 1 near y and $\bar{y}$.*

Proof. We may pick some linear projection $\pi\colon \mathbb{C}^m \to \mathbb{C}$ defined over $\mathbb{R}$ so that $\pi(y) \notin \pi(F)$ and $\pi(\bar{y}) \notin \pi(F)$. Thus by taking $q\pi$ it suffices to consider the case $m = 1$. Let $F = \{\alpha_i + \beta_i\sqrt{-1},\ i = 1, \ldots, b\}$. Let $\sigma_i(z) = (z - \alpha_i)^2 + \beta_i^2$ and let $\sigma(z) = \prod (\sigma_i(z))^{k+1}$. Then σ is C^k close to 0 near F and $\sigma(y) \neq 0$. Suppose $\sigma(y) = \alpha + \beta\sqrt{-1}$. Let

$$q(z) = 1 - ((\sigma(z) - \alpha)^2 + \beta^2)^{k+1}(\alpha^2 + \beta^2)^{-k-1} .$$

Then q has the required properties. □

Theorem 8 *Let $\theta\colon \mathbb{C}^m \to \mathbb{C}^n$ be a polynomial map defined over $\mathbb{R}$. Let T be a compact subset of $\theta^{-1}(\mathbb{R}^n)$ and suppose $f\colon T \to \mathbb{C}$ is a continuous function so that $f(\bar{z}) = \overline{f(z)}$ for all $z \in T \cap \bar{T}$. Suppose also that $\theta|_T$ is finite-to-one. Then there is a polynomial $h\colon \mathbb{C}^m \to \mathbb{C}$ defined over $\mathbb{R}$ so that $h|_T$ approximates f (in the C^0 topology). Furthermore, suppose that $f|_S$ can be locally C^k approximated by polynomials defined over $\mathbb{R}$ for some $S \subset T$ invariant under complex conjugation. Then we may also conclude that $h|_S$ is a C^k approximation to $f|_S$.*

Proof. It suffices to find for each $x \in \mathbb{R}^n$ a neighborhood U_x of x in $\mathbb{C}^n$ and a polynomial $h_x\colon \mathbb{C}^m \to \mathbb{C}$ so that h_x restricted to $\theta^{-1}(U_x) \cap T$ approximates f and h_x restricted to $\theta^{-1}(U_x) \cap S$ is a C^k approximation of f. To see this, note that by compactness we may cover $\theta(T)$ with a finite number of such U_{x_i}'s, $i = 1, \ldots, b$. Take a partition of unity $\psi_i\colon \mathbb{R}^n \to [0, 1]$ with $\operatorname{supp}(\psi_i) \subset U_{x_i}$ and approximate the ψ_i's by real polynomials p_i. Think of these real polynomials p_i as being complex polynomials defined over $\mathbb{R}$. Now just let $h(z) = \sum p_i(\theta(z))h_{x_i}(z)$.

So pick any $x \in \mathbb{R}^n$. For each $y \in \theta^{-1}(x) \cap S$, let g_y be a polynomial defined over $\mathbb{R}$ which is C^k close to f on some neighborhood V_y of y. For $z \in S \cap \bar{V}_y$, $f(z) = \overline{f(\bar{z})}$ which is C^k close to $\overline{g_y(\bar{z})} = g_y(z)$. Thus we may as well assume $g_y = g_{\bar{y}}$. For $y \in \theta^{-1}(x) \cap (T - S)$ let g_y be the constant $f(y)$. For each $y \in \theta^{-1}(x) \cap T$ use Lemma 7 to pick a polynomial q_y defined over $\mathbb{R}$ so that q_y approximates 1 near y and $\bar{y}$ and so that q_y approximates 0 near $\theta^{-1}(x) \cap (T - y - \bar{y})$. We now just let $h_x = \sum g_y q_y$ where the sum is taken over a set A of y's so that $A \cup \bar{A} = \theta^{-1}(x) \cap (T \cup \bar{T})$ and $A \cap \bar{A} = A \cap \mathbb{R}^m$. □

We now recall the following result used in [I] and [T2].

Lemma 9 *If X is a real algebraic set and* Nonsing(X) *is compact then there are arbitrarily small analytic functions φ:* Nonsing$(X) \to \mathbb{R}$ *so that the graph of φ, $\{(x, \varphi(x)) \in \text{Nonsing}(X) \times \mathbb{R}\}$ is a nonsingular real algebraic set.*

Proof. Pick a polynomial λ: $X \to \mathbb{R}$ so that $\lambda^{-1}(0) = \text{Sing}(X)$. Pick a polynomial μ: $X \to \mathbb{R}$ which approximates $1/\lambda$ on Nonsing(X). Let $\varphi = (1/\lambda(x)) - \mu(x)$. Then the graph of φ is $\{(x, t) \in X \times \mathbb{R} | (t + \mu(x))\lambda(x) = 1\}$ which is a nonsingular algebraic set. □

For many algebraic sets considered here, the nonsingular points are a union of connected components. The following result gives a criterion for this property to persist under mapping.

Lemma 10 *Let $Z \subset \mathbb{R}^m$ be a real algebraic set and let $Z_{\mathbb{C}} \subset \mathbb{C}^m$ be its complexification. Let ψ: $Z_{\mathbb{C}} \to \mathbb{C}^n$ be a proper polynomial map defined over $\mathbb{R}$. Let $Y = \text{Cl}_{\mathbb{R}}(\psi(Z))$. Suppose*

(1) Nonsing(Z) *is closed.*

(2) ψ *restricted to* Nonsing(Z) *is a smooth immersion which imbeds an open dense subset of* Nonsing(Z).

(3) $\psi^{-1}\psi(\text{Nonsing}(Z)) = \text{Nonsing}(Z)$.

Then $\psi(\text{Nonsing}(Z))$ *is the set of almost nonsingular points of Y.*

If in addition we know that ψ restricted to Nonsing(Z) *is an imbedding, then we get* $\psi(\text{Nonsing}(Z)) = \text{Nonsing}(Y)$.

Proof. Let $k = \dim(Z)$ and let Z' denote Nonsing(Z). Arguing one irreducible component at a time, it suffices to assume Z is irreducible. Since ψ is proper, we know that $X = \psi(Z_{\mathbb{C}})$ is a complex algebraic set.

Now $\psi(Z) \subset X$ so $Y_{\mathbb{C}} = \text{Cl}_{\mathbb{C}}(\psi(Z)) \subset X$. But $Z' \subset \psi^{-1}(Y_{\mathbb{C}})$ so $Z_{\mathbb{C}} = \text{Cl}_{\mathbb{C}}(Z') \subset \psi^{-1}(Y_{\mathbb{C}})$. So $X = \psi(Z_{\mathbb{C}}) \subset \psi\psi^{-1}(Y_{\mathbb{C}}) \subset Y_{\mathbb{C}} \subset X$. So $X = Y_{\mathbb{C}}$ and thus $Y = X \cap \mathbb{R}^n$.

Take any $z \in Z'$. Now $\psi^{-1}\psi(z)$ is zero dimensional by (2) and (3), hence it is finite. Let $\psi^{-1}\psi(z) = \{z_1, \ldots, z_b\}$. Note $z_i \in Z'$. Now $d\psi$ has rank k at z_i by (2), so there are open neighborhoods U_i of z_i in Nonsing$(Z_{\mathbb{C}})$ such that each $\psi|_{U_i}$ is a complex imbedding. By properness of ψ, $\psi(Z_{\mathbb{C}} - \bigcup_{i=1}^b U_i)$ is closed. Hence $\psi(\bigcup_{i=1}^b U_i)$ is a (perhaps nonopen) neighborhood of $\psi(z)$ in $X = Y_{\mathbb{C}}$. But each $\psi(U_i)$ is a complexification of the real analytic manifold $\psi(U_i \cap Z)$. So $\psi(z)$ is an almost nonsingular point of Y.

If ψ restricted to Z' is an imbedding, then $b = 1$ above and thus $\psi(z) \in \text{Nonsing}(Y)$.

Suppose now $x \in Y - \psi(Z')$ is an almost nonsingular point of Y. So x has a neighborhood U in Y which is k dimensional. But $\psi(Z')$ is closed, so we may assume $U \subset Y - \psi(Z')$. Since ψ: $Z \to Y$ imbeds an open dense subset of Z', it has degree one. Hence $\dim(Y - \psi(Z)) < \dim(Y) = k$. But $\dim(\psi(\text{Sing}(Z)) \leqq \dim(\text{Sing}(Z)) < k$ and $Y - \psi(Z') = (Y - \psi(Z)) \cup \psi(\text{Sing}(Z))$, so $\dim(Y - \psi(Z')) < k$. This contradicts the k dimensionality of U. So $\psi(Z')$ is the set of almost nonsingular points of Y. □

Proofs of Theorems A through F

We will prove Theorems D, E and F simultaneously.

Proof. By Lemma 6 we may find a real algebraic set W, a diffeomorphism h: $M \to \text{Nonsing}(W)$ and a polynomial map $\rho: W \to V$ so that if $Z = W_{\mathbb{C}}$, $\theta = \rho_{\mathbb{C}}$, $Y = \theta(Z)$ and $W' = \text{Nonsing}(W)$ then:

(1) ρh: $M \to V$ approximates f.

(2) θ: $Z \to Y$ is a finite regular map.

(3) W' is a union of connected components of $\theta^{-1}(V)$.

Let $K \subset V$ be a compact set containing a neighborhood of $\rho(W') = \rho h(M)$. Let $T = \theta^{-1}(K)$. Then T is a compact subset of Z by properness of θ. Also W' is a union of connected components of T by (3).

Define a continuous function g: $T \to \mathbb{C}$ by setting $g(z) = 0$ for all $z \in W'$ and $g(z) = 2$ for all $z \in T - W'$. By Theorem 8 there is a polynomial η: $Z \to \mathbb{C}$ defined over $\mathbb{R}$ so that $\eta|_T$ approximates g. Consider (θ, η): $Z \to Y \times \mathbb{C}$. Then (θ, η) is proper, hence its image $X = (\theta, \eta)(Z)$ is a complex algebraic subset of $Y \times \mathbb{C}$.

Let $\psi = (\theta, \eta)$. To prove Theorems E and F, it suffices by Lemma 10 to show that $\psi^{-1}\psi(w) \subset W'$ for each $w \in W$. So pick any $w \in W'$. Suppose $\psi(z) = \psi(w)$. Then $\theta(z) = \theta(w) \in K$, so $z \in T$. If $z \in T - W'$ then $\eta(z) \sim 2$. But $\eta(w) \sim 0$ so we could not have $\eta(z) = \eta(w)$. So $z \in W'$. Consequently, $\psi^{-1}\psi(w) \subset W'$ for all $w \in W'$.

Setting $f' = \psi h$ we have proven Theorem E. For Theorem F, note M is ε-isotopic to $\psi(W')$. Now Theorem D follows from Theorem F and Lemma 9. □

The proof of Theorems A, B and C is similar to the proof of Theorems D, E and F but a little more subtle. Again, we will prove Theorems A, B and C simultaneously.

Proof. For Theorems A and B, let f: $M \to \mathbb{R}^n$ be the inclusion map. By Lemma 6 we may find a nonsingular real algebraic set W, a diffeomorphism h: $M \to W$ and a polynomial ρ: $W \to V$ so that if $Z = W_{\mathbb{C}}$, $\theta = \rho_{\mathbb{C}}$ and $Y = \theta(Z)$ then:

(1) ρh: $M \to V$ approximates f.

(2) θ: $Z \to Y$ is a finite regular map.

(3) W is a union of connected components of $\theta^{-1}(V)$.

Take a generic projection π: $\mathbb{C}^n \to L$ to a codimension one linear subspace L defined over $\mathbb{R}$ so that $\pi|_Y$ is finite. Now $\pi(Y)$ is a complex algebraic set defined over $\mathbb{R}$. If v is a real unit vector perpendicular to L then $\theta(z) = \pi\theta(z) + (\theta(z)\cdot v)v$. Let K be a compact neighborhood of $\pi\rho(W)$ in $L \cap \mathbb{R}^n$ and let $T = \theta^{-1}\pi^{-1}(K)$. In contrast to the situation in Theorem E, we might not have W open in T. However, we know that for some neighborhood U of W in Z, $\theta|_U$ is a complex analytic immersion and $\theta(U - W)$ has no real points. Let b be the maximum of $|\theta(z)\cdot v|$ for $z \in T$. Pick some smooth α: $T \to [0, 1]$ so that α is 0 on a neighborhood U' of W in $T \cap U$, α is 1 on $T - U$ and $\alpha(z) = \alpha(\bar{z})$ for all $z \in T$. Define g: $T \to \mathbb{C}$ by $g(z) = \alpha(z)(2b + 2) + \theta(z)\cdot v$. By Theorem 8 there is a polynomial η: $Z \to \mathbb{C}$ defined over $\mathbb{R}$ so that $\eta|_T$ approximates g and so this approximation is C^1 on U'.

Define ψ: $Z \to \mathbb{C}^n$ by $\psi(z) = \pi\theta(z) + \eta(z)v$. Now $\pi\theta$ is proper since it is finite. Hence ψ is proper since $\pi\psi = \pi\theta$ is proper. Now ψ is C^1 close to θ on U' so ψ immerses W. So to prove Theorems A and C, it suffices by Lemma 10 to show that $\psi^{-1}\psi(w) \subset W$ for all $w \in W$. Theorem B will then follow from Theorem A and Lemma 9.

So suppose $\psi(z) = \psi(w)$ for $w \in W$, $z \notin W$. Note $\pi\theta(z) = \pi\psi(z) = \pi\psi(w) = \pi\theta(w) = \pi\rho(w) \in K$ so $z \in T$. Also $\psi|_{U'}$ is an immersion ε-regularly homotopic to $\theta|_{U'}$ and $\theta(U' - W)$ has no real points. Hence $\psi(U' - W)$ has no real points. So

$z \in T - U'$. If $z \in U \cap T - U'$, $\eta(z) \sim \alpha(z)(2b + 2) + \theta(z) \cdot v$ which is not real (since $\alpha(z)$ and $\pi\theta(z)$ are real and $\theta(U - W)$ has no real points), so $\psi(z)$ is not real. Consequently $z \in T - U$. But then $\eta(z) \sim 2b + 2 + \theta(z) \cdot v$ so $|\eta(z)| > b + 1$. However,

$$|\eta(z)| = |\psi(z) - \pi\psi(z)| = |\psi(w) - \pi\psi(w)| = |\eta(w)| \sim |\theta(w) \cdot v| \leqq b \,.$$

This is a contradiction, so $\psi^{-1}\psi(w) \subset W$ for all $w \in W$. □

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