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ALGEBRAICITY OF IMMERSIONS IN $\mathbb{R}^n$

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SUPPOSE $M \subset \mathbb{R}^n$ is a smooth compact submanifold of Euclidean space. In [6], Nash conjectured that there is an algebraic set $X \subset \mathbb{R}^n$ and a union of components X' of X which is isotopic to M. This was proven in [1]. But then the question arises, whether we can take $X' = X$, i.e., whether M is isotopic in $\mathbb{R}^n$ to a nonsingular algebraic set. A proof of this was presented in [9] which unfortunately had a number of gaps which we detailed in [1]. So this stronger Nash conjecture still remains open. This paper addresses this problem. We show that, as usual in this subject, if something is cobordant to an algebraic situation then it is isomorphic to an algebraic situation. As it turns out, our results apply equally well to immersed submanifolds so we state and prove our results in this context.

Before stating our results we must make some definitions. Let X be a real algebraic set. We say x is an **almost nonsingular** point of X if a neighborhood of x in X is a union of analytic manifolds with dimension equal to the dimension of X and furthermore, the complexifications of these analytic manifolds form a neighborhood of x in the algebraic complexification $X_{\mathbb{C}}$ of X. Thus all nonsingular points are almost nonsingular. Furthermore, if X is normal and almost nonsingular at x then x is a nonsingular point of X. This is because normality implies there is only one analytic branch at x. We say X is **almost nonsingular** if it is almost nonsingular at all of its points. Thus if we have an algebraic set which is the image of an immersion, the strongest nonsingularity condition we can impose is to ask that it be almost nonsingular.

We say an immersion $f: M \to N$ is **degree one** if there is an open dense subset U of M so that $f|_U$ is an imbedding. The idea is that we wish to exclude immersions which are the composition of a covering projection and an immersion. Of course any immersion with $\dim(M) < \dim(N)$ may be approximated by a degree one immersion.

We say two immersions f and f' are ε-regularly homotopic if there is a small homotopy of f to f' through immersions.

The notion of immersion cobordism was defined in [10]. We say that two immersions $f_i: M_i \to N$, $i = 0, 1$ are **immersion cobordant** if there is a compact smooth manifold W and a proper immersion $g: W \to N \times [0, 1]$ so that $\partial W = M_0 \cup M_1$ and $g(x) = (f_i(x), i)$ for $x \in M_i$, $i = 0, 1$. The properness of g means that $g^{-1}(N \times \{0, 1\}) = \partial W$ and g is transverse to $N \times \{0, 1\}$.

Note that immersion cobordism classes of immersions to N form a semigroup, the operation being union. They form a group if $N = \mathbb{R}^n$, since inverses are obtained by reflection through a hyperplane. A null cobordism is obtained by revolution around this hyperplane.

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In [10], this group was shown to be $\pi_n(\Omega^\infty\Sigma^\infty MO(k))$ for immersions of smooth compact $n-k$ dimensional manifolds to $\mathbb{R}^n$, where $MO(k)$ is the Thom space of the universal $\mathbb{R}^k$ bundle and $\Omega^\infty\Sigma^\infty$ is the infinite loop-suspension. We let $\pi_n^{alg}(\Omega^\infty\Sigma^\infty MO(k))$ denote the subgroup of $\pi_n(\Omega^\infty\Sigma^\infty MO(k))$ generated by degree one immersions onto codimension k almost nonsingular algebraic subsets of $\mathbb{R}^n$.

We may now state our main theorem precisely.

THEOREM 1. *Let $f:M\to\mathbb{R}^n$ be a smooth immersion of a smooth closed manifold M. Then f is ε-regularly homotopic to an immersion onto an almost nonsingular real algebraic subset of $\mathbb{R}^n$ if and only iff is immersion cobordant to a degree one immersion onto an almost nonsingular real algebraic set in $\mathbb{R}^n$.*

COROLLARY 2. *A codimension k immersion $f:M\to\mathbb{R}^n$ is ε-regularly homotopic to an immersion onto an almost nonsingular real algebraic set in $\mathbb{R}^n$ if and only if its immersion cobordism class is in $\pi_n^{alg}(\Omega^\infty\Sigma^\infty MO(k))$.*

There are some useful algebraic conditions which imply that an immersed cobordism class lies in $\pi_n^{alg}(\Omega^\infty\Sigma^\infty MO(k))$. For example the image of the suspension map followed the natural inclusion

$$\pi_{n-1}(MO(k-1))\to\pi_n(MO(k))\to\pi_n(\Omega^\infty\Sigma^\infty MO(k))$$

is contained in $\pi_n^{alg}(\Omega^\infty\Sigma^\infty MO(k))$. This means that if an immersed (or imbedded) submanifold of $\mathbb{R}^n$ deforms through an immersed cobordism onto an imbedded submanifold of $\mathbb{R}^{n-1}$, then it can be isotoped to an almost nonsingular (nonsingular in the imbedded case) algebraic subset of $\mathbb{R}^n$. This is because by [1] every closed smooth submanifold of $\mathbb{R}^{n-1}$ can be ε-isotoped to a nonsingular algebraic subset of $\mathbb{R}^n$. Corollary 2 also implies a solution of the Nash conjecture, that is any closed smooth submanifold of $\mathbb{R}^n$ is isotopic to a nonsingular component of an algebraic subset of $\mathbb{R}^n$ [1]. Theorem 1 has a number of consequences which we will explore in forthcoming papers

It is worthwhile pointing out the consequences of Theorem 1 for imbeddings. If $M\subset\mathbb{R}^n$ is a smooth codimension k submanifold, then it gives rise to an element $\tau(M)\in\pi_n(MO(k))$ via the Thom map. Its immersion cobordism class is just $\theta\tau(M)$ where

$$\theta:\pi_n(MO(k))\to\pi_n(\Omega^\infty\Sigma^\infty MO(k))$$

is the natural map. Thus if $\pi_n^{alg}(MO(k))$ is the subgroup $\theta^{-1}(\pi_n^{alg}(\Omega^\infty\Sigma^\infty MO(k)))$ we have:

COROLLARY 3. *If $M\subset\mathbb{R}^n$ is a smooth codimension k submanifold, then M is isotopic to a nonsingular real algebraic subset of $\mathbb{R}^n$ if and only if $\tau(M)\in\pi_n^{alg}(MO(k))$.*

We will now sketch a proof of Theorem 1. Suppose f is immersion cobordant to a degree one immersion onto an almost nonsingular real algebraic set Q. The immersion to Q must be the normalization $h:P\to Q$ of Q, (P is nonsingular). Let $\pi:\mathbb{R}^n\to\mathbb{R}^n$ be reflection through a hyperplane. Then $[f]+[\pi h]$ is zero in the immersion cobordism group. So there is a null cobordism of $f\cup\pi h$, i.e., a proper immersion $g':V'\to\mathbb{R}^n\times[0,1]$ so that $\partial V'=M\cup P$ and $g'|_M=f$ and $g'|_P=\pi h$. Double this immersion, obtaining an immersion $g:V\to\mathbb{R}^n\times\mathbb{R}$ with $g(V)\cap\mathbb{R}^n\times 0=f(M)\cup\pi(Q)$. (See Fig. 1.) By Lemma 10 (a relative version of the proof of Lemma 6 of [1]) we may assume V is the nonsingular points of a real algebraic set Z, g is a polynomial and the complexification of g composed with projection $\mathbb{C}^n\times\mathbb{C}\to\mathbb{C}^n$ is a finite map. By Lemma 9 (a relative version of Theorem 8 of [1]) we are able to assume that $g_{\mathbb{C}}^{-1}(\mathbb{R}^n\times 0)=M\cup P$. Hence $g(M)\cup g(P)=g(M)\cup\pi(Q)$ is an almost nonsingular real algebraic set and consequently $g(M)$ is an almost nonsingular real algebraic set. But $g:M\to\mathbb{R}^n$ is ε-regularly homotopic to f.

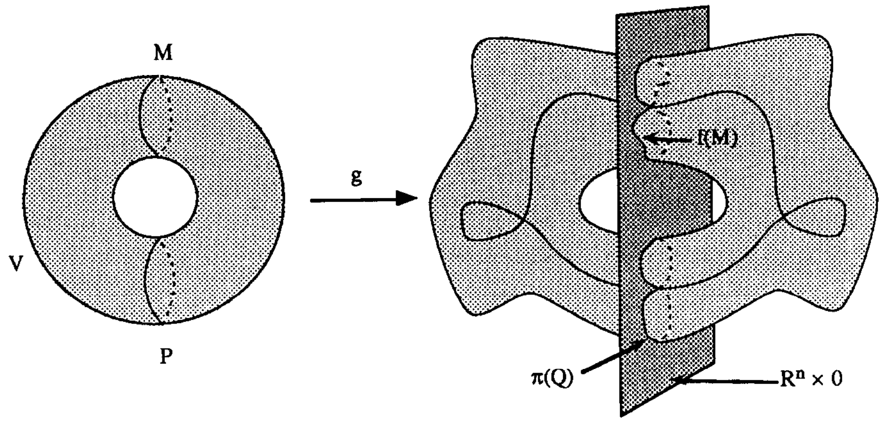

Fig. 1. The immersion g.

ALGEBRAIC GEOMETRY PRELIMINARIES

We have need of a number of notions from algebraic geometry. Many of them are ordinarily used only for irreducible algebraic sets over algebraically closed fields, so we will make explicit their obvious extensions to the real or reducible case. Our standard algebraic geometry reference is [8].

We say that a complex algebraic set $Z \subset \mathbb{C}^n$ is **defined over** $\mathbb{R}$ if it is the set of zeroes of polynomials with real coefficients. Equivalently, it is invariant under complex conjugation, $\bar{Z} = Z$. Likewise, if Z and V are defined over $\mathbb{R}$ then a rational function $f: Z - V \to \mathbb{C}^k$ is **defined over** $\mathbb{R}$ if it is locally given as the quotient of polynomials with real coefficients. Equivalently it is equivariant, $\overline{f(z)} = f(\bar{z})$.

If $X \subset \mathbb{R}^n$ is a real algebraic set then $X_{\mathbb{C}} \subset \mathbb{C}^n$ is its algebraic complexification, the smallest algebraic set containing X. We use without reference many properties of the complexification shown in [12]. If $X \subset \mathbb{C}^n$ is a complex algebraic set then we denote $X_{\mathbb{R}} = X \cap \mathbb{R}^n$, a real algebraic set. We have similar local analytic notions also. Thus if X is a germ of a real analytic set, its analytic complexification $X_{\mathbb{C}^{an}}$ is the germ of a subset of $\mathbb{C}^n$, namely the intersection of all complex analytic set germs containing X. If X is the germ of a real algebraic set then clearly $X_{\mathbb{C}^{an}}$ is contained in the germ of $X_{\mathbb{C}}$.

We say a rational function f from X to Y is a **birational equivalence** if there is a one to one correspondence between the irreducible components of X and Y so that f restricts to a birational isomorphism (in the sense of [8]) from each irreducible component of X to its corresponding irreducible component in Y.

Let $f: Y - V \to X$ be a rational function. Then we say that f is **dominating** if for every irreducible component Y' of Y, either $Y' \subset V$ or $f(Y' - V)$ is Zariski dense in some irreducible component of X. This implies that if $W \subset X$ contains no irreducible components of X then $f^{-1}(W) \cup V$ contains no irreducible components of Y which were not already contained in V. Dominating rational functions have the following easily proven unique lifting property.

LEMMA 4. *Suppose $f: Y - V \to X$ is a dominating rational function, and $g: Z \to X$ is a birational equivalence and $h_i: Y - V \to Z$, $i = 0, 1$ are rational functions so that $gh_i = f$. Then $h_0 = h_1$.*

We say that an algebraic set Y is **normal** at y if exactly one irreducible component Y' of Y contains y and Y' is normal at y. If $V \subset Y$ is an algebraic set we say that $Y - V$ is normal if Y is normal at all points of $Y - V$.

We have need of a construction from algebraic geometry called the normalization. Let $X \subset \mathbb{C}^m$ be a complex algebraic set. Suppose $X = \cup X_i$ is the irreducible decomposition of X. Let $\theta_i: Z_i \to X_i$ be the normalizations of the irreducible algebraic sets X_i, c.f. [8]. Then the normalization of X is $\theta: Z \to X$ where Z is the disjoint union $Z = \cup Z_i$ and $\theta|_{Z_i} = \theta_i$.

The following Lemma gives several useful properties of the normalization.

LEMMA 5. *Let X be a complex algebraic set and let $\theta: Z \to X$ be its normalization. Then:*

(a) *Z is a normal affine algebraic set,* $\dim_{\mathbb{C}}(X) = \dim_{\mathbb{C}}(Z)$ *and Z is locally analytically irreducible.*

(b) *θ is a proper map (i.e. θ^{-1} (compactum) is a compactum) and θ has finite fibers. In fact θ is a finite map in the sense of algebraic geometry. Also, θ is a birational equivalence.*

(c) *If X is defined over $\mathbb{R}$ then:*

- (i) *Z and θ are defined over $\mathbb{R}$.*
- (ii) *If $(X_{\mathbb{R}})_{\mathbb{C}} = X$ then $(Z_{\mathbb{R}})_{\mathbb{C}} = Z$.*
- (iii) *If the germ $(X_{\mathbb{R}})_{\mathbb{C}^{an}}$ is the germ of X at all points of $X_{\mathbb{R}}$ (for example if $X = (X_{\mathbb{R}})_{\mathbb{C}}$ and $X_{\mathbb{R}}$ is almost nonsingular) then $\theta^{-1}(X_{\mathbb{R}}) = Z_{\mathbb{R}}$.*
- (iv) *If $X_{\mathbb{R}}$ is almost nonsingular then $Z_{\mathbb{R}}$ is nonsingular. Furthermore, if $f: M \to \mathbb{R}^m$ is a proper C^∞ degree one immersion with $f(M) = X_{\mathbb{R}}$ then there is a unique diffeomorphism $h: M \to Z_{\mathbb{R}}$ so that $\theta h = f$.*

(d) *Suppose $f: Y - V \to X$ is a dominating rational function and $Y - V$ is normal, for example if $V \supset \mathrm{Sing}(Y)$. Then there is a unique rational function $g: Y - V \to Z$ so that $\theta g = f$. Furthermore, if Y, V and X are defined over $\mathbb{R}$ then g is defined over $\mathbb{R}$.*

Proof. This is all fairly standard knowledge except possibly for conclusions c-(iii) and c-(iv) which we prove explicitly. Since θ is a birational equivalence on each irreducible component, we may pick a nowhere dense algebraic subset $W \subset X$ and a rational function $\mu: X - W \to Z$ so that $\theta \circ \mu$ is the inclusion $X - W \hookrightarrow X$.

Let us first prove c-(iii). We may as well suppose X is irreducible. In general $\theta^{-1}(X_{\mathbb{R}})$ is strictly bigger than $Z_{\mathbb{R}}$. For instance, let $X = \{(x, y) \in \mathbb{C}^2 | y^2 = x^3 - x^2\}$. Then $Z = \{(x, y, u) \in \mathbb{C}^3 | y = ux,\ u^2 = x - 1\}$ but $(0, 0, \sqrt{-1}) \in \theta^{-1}(X_{\mathbb{R}})$. However, suppose the germ of X at each point of $X_{\mathbb{R}}$ is the analytic complexification of $X_{\mathbb{R}}$. If $x \in X_{\mathbb{R}} - \mathrm{Cl}(\mathrm{Nonsing}(X_{\mathbb{R}}))$ then the germ of $X_{\mathbb{R}}$ at x is the germ of $\mathrm{Sing}(X_{\mathbb{R}})$ so its analytic complexification has dimension smaller than that of X. This is a contradiction since the local dimension of the complex algebraic set X is constant. Consequently, $\mathrm{Nonsing}(X_{\mathbb{R}})$ is dense in $X_{\mathbb{R}}$. Then since $\dim(W_{\mathbb{R}}) \leq \dim_{\mathbb{C}}(W) < \dim_{\mathbb{C}}(X) = \dim(X_{\mathbb{R}})$, we know $X_{\mathbb{R}} - W$ must be dense in $X_{\mathbb{R}}$. So pick $x \in X_{\mathbb{R}}$ and $z \in \theta^{-1}(x)$. Take a germ U of Z at z. Now $\dim_{\mathbb{C}}(X) = \dim_{\mathbb{C}}(U)$, U is analytically irreducible and θ is finite so $\theta(U)$ is an analytically irreducible component of X at x. We must have $(\theta(U)_{\mathbb{R}})_{\mathbb{C}^{an}} = \theta(U)$, otherwise we would not have $(X_{\mathbb{R}})_{\mathbb{C}^{an}} = X$ as germs at x. So $\theta(U)_{\mathbb{R}} - W$ is dense in $\theta(U)_{\mathbb{R}}$ and thus $z \in \mathrm{Cl}(\mu(\theta(U)_{\mathbb{R}} - W))$ which is contained in $Z_{\mathbb{R}}$ since μ is defined over $\mathbb{R}$.

Let us now prove c-iv. We must have h restricted to $M - f^{-1}(W)$ is μf. Now $\dim(M) > \dim(W_{\mathbb{R}})$ so $f^{-1}(W)$ is nowhere dense in M. We claim μf extends to a diffeomorphism h. Since nowhere density implies this extension is unique, we only need to show this locally.

So pick any $x \in M$ and pick a small open neighborhood U of x in M which is imbedded by f. By Lemma 6 below, the germ A of $f(U)$ at $f(x)$ is analytic and irreducible. Let $A' \subset X$ represent the germ of the complexification $A_{\mathbb{C}}$ of A. We may as well assume $f(U) = A' \cap \mathbb{R}^m$ and $\mathrm{Cl}(A')$ is compact. By finiteness of θ we know that $\theta^{-1}(\mathrm{Cl}(A'))$ is compact, and hence $\mu|_{A'-W}$ is bounded. Hence μ extends to an analytic function $\mu' : A' \to Z$ by, say [11] or [4]. Note $\mu'(A')$ is an analytic manifold (since μ' has inverse θ). By local analytic irreducibility of Z, we then know that $\mu'(A')$ is an open subset of Z. In fact $\mu'(A) \subset \mathrm{Nonsing}(Z)$ since $\mu'(A')$ is an analytic manifold. Since $\mu(f(U) - W) \subset Z_{\mathbb{R}}$ we know that $\mu'(f(U)) \subset Z_{\mathbb{R}}$. So μf extends to $\mu' f$ defined on all of U and $\mu' f(U) \subset \mathrm{Nonsing}(Z_{\mathbb{R}})$.

We now have a smooth local diffeomorphism $h: M \to \mathrm{Nonsing}(Z_{\mathbb{R}})$ so that $\theta h = f$ and h restricted to $M - f^{-1}(W)$ is μf. But $h(M) = \mathrm{Cl}(h(M - f^{-1}(W))) = \mathrm{Cl}(\mu(X_{\mathbb{R}} - W)) = Z_{\mathbb{R}}$, so $Z_{\mathbb{R}}$ is nonsingular. Since f has degree one, we know that h is a diffeomorphism. ■

The above proof required the following lemma:

LEMMA 6. *Let A_i, $i = 1, \ldots, k$ be germs at 0 of m dimensional analytic submanifolds of $\mathbb{R}^n$ and let B be the germ at 0 of an m dimensional C^∞ submanifold so that $B \subset \bigcup_{i=1}^k A_i$. Then $B = A_i$ for some i.*

Proof. Let A_i' and B' be closed sets representing the germs A_i and B. Let B_i be the interior of $B' \cap A_i'$ in B'. Suppose $0 \in \mathrm{Cl}(B_i)$ and $0 \in \mathrm{Cl}(B_j)$. Then A_i' and A_j' have infinite order of contact at 0. Hence $A_i = A_j$ by analyticity. So after omitting duplicates, there is only one i so that $0 \in \mathrm{Cl}(B_i)$. Hence B_i is a neighborhood of 0 in B'. Consequently, $B = A_i$. ■

We use the following well-known fact, whose proof is a slight modification of that of Theorem 10, Chapter 1, Section 5 of [8].

LEMMA 7. *Let $V \subset \mathbb{C}^n$ be a complex algebraic set defined over $\mathbb{R}$. Suppose $m \geq \dim(V)$ and $L \subset \mathbb{C}^n$ is an m-dimensional linear subspace defined over $\mathbb{R}$. Then for an open dense set of linear retractions $\varphi : \mathbb{C}^n \to L$ defined over $\mathbb{R}$, the restriction of φ to V is a finite map to $\varphi(V)$.*

Next we need an estimate on the growth of the fibers of a finite map.

LEMMA 8. *Suppose $\theta : W \to V$ is a finite map where W and V are complex algebraic sets. Suppose $g : W \to \mathbb{C}$ is a polynomial. Then there are $c, d \in \mathbb{R}$ and $n \in \mathbf{Z}$ so that*

$$|g(z)| < c + d|\theta(z)|^n$$

for all $z \in W$.

Proof. By finiteness of θ there are polynomials $h_i : V \to \mathbb{C}$, $i = 1, \ldots, k$ so that

$$g(z)^k + h_1(\theta(z))g(z)^{k-1} + \ldots + h_k(\theta(z)) = 0$$

for all $z \in W$. So

$$g(z) = -h_1(\theta(z)) - h_2(\theta(z))g(z)^{-1} - \ldots - h_k(\theta(z))g(z)^{1-k}.$$

If n is the maximal degree of the polynomials h_i we may choose $a > 0$ and $b > 0$ so $|h_i(v)| \leq a + b|v|^n$ for all $v \in V$. So if $|g(z)| \geq 1$ then

$$|g(z)| \leq |h_1(\theta(z))| + |h_2(\theta(z))|\,|g(z)|^{-1} + \ldots + |h_k(\theta(z))|\,|g(z)|^{1-k} \leq k(a + b|\theta(z)|^n).$$

So setting $c = 1 + ak$ and $d = bk$ we are done. ■

THE PROOF OF THEOREM 1

Before proving Theorem 1, we need relative versions of some results in [1]. We first prove a relative version of Theorem 8 of [1].

LEMMA 9. *Let $\theta:\mathbb{C}^m \to \mathbb{C}^n$ be a polynomial map defined over $\mathbb{R}$. Let T be a compact subset of $\theta^{-1}(\mathbb{R}^n)$ and suppose $f:T\to\mathbb{C}$ is a continuous function so that $f(\bar{z}) = \overline{f(z)}$ for all $z\in T\cap\bar{T}$. Suppose also that $\theta|_T$ is finite-to-one. Then there is a polynomial $h:\mathbb{C}^m\to\mathbb{C}$ defined over $\mathbb{R}$ so that $h|_T$ approximates f (in the C^0 topology). What is more, if $Q\subset\mathbb{C}^m$ is a complex algebraic set defined over $\mathbb{R}$ and $f=0$ in a neighborhood of $Q\cap T$ then we may conclude that $h|_Q=0$. Furthermore, suppose that $f|_S$ can be locally C^k approximated by polynomials defined over $\mathbb{R}$ which vanish on Q for some $S\subset T$ invariant under complex conjugation. Then we may also conclude that $h|_S$ is a C^k approximation to $f|_S$.*

Proof. It suffices to find for each $x\in\mathbb{R}^n$ a neighborhood U_x of x in $\mathbb{C}^n$ and a polynomial $h_x:\mathbb{C}^m\to\mathbb{C}$ so that h_x restricted to $\theta^{-1}(U_x)\cap T$ approximates f and h_x restricted to $\theta^{-1}(U_x)\cap S$ is a C^k approximation of f and h_x restricted to Q is 0. To see this, note that by compactness we may cover $\theta(T)$ with a finite number of such U_{x_i}'s, $i=1,\dots,b$. Take a partition of unity $\psi_i:\mathbb{R}^n\to[0,1]$ with $\mathrm{supp}(\psi_i)\subset U_{x_i}$ and approximate the ψ_i's by real polynomials p_i. Think of these real polynomials p_i as being complex polynomials defined over $\mathbb{R}$. Now just let $h(z)=\sum p_i(\theta(z))h_{x_i}(z)$.

So pick any $x\in\mathbb{R}^n$. For each $y\in\theta^{-1}(x)\cap Q$ let $g_y=0$. For each $y\in\theta^{-1}(x)\cap S-Q$, let g_y be a polynomial defined over $\mathbb{R}$ which is C^k close to f on some neighborhood V_y of y and which vanishes on Q. For $z\in S\cap\bar{V}_y$, $f(z)=\overline{f(\bar{z})}$ which is C^k close to $\overline{g_y(\bar{z})}=g_y(z)$. Thus we may as well assume $g_y=g_{\bar{y}}$. For $y\in\theta^{-1}(x)\cap(T-S)-Q$, pick a polynomial λ_y defined over $\mathbb{R}$ so $\lambda_y(Q)=0$ but $\lambda_y(y)\neq 0$. Let $g_y(z)=f(y)\lambda_y(z)/\lambda_y(y)$. For each $y\in\theta^{-1}(x)\cap T$ use Lemma 7 of [1] to pick a polynomial q_y defined over $\mathbb{R}$ so that q_y approximates 1 near y and $\bar{y}$ and so that q_y approximates 0 near $\theta^{-1}(x)\cap(T-y-\bar{y})$. We now just let $h_x=\sum g_yq_y$ where the sum is taken over a set A of y's so that $A\cup\bar{A}=\theta^{-1}(x)\cap(T\cup\bar{T})$ and $A\cap\bar{A}=A\cap\mathbb{R}^m$. ∎

We now prove a relative version of Lemma 6 of [1].

LEMMA 10. *Let N be any compact smooth n-dimensional manifold, let M be any (possibly empty) codimension one submanifold of N with trivial normal bundle and let $f:N\to V$ be any smooth immersion to a nonsingular algebraic set V so that the bordism class of the map f is algebraic, $f(M)$ is an almost nonsingular real algebraic set Q and $f|_M$ is a degree one immersion. Then there are a real algebraic set W, a nonsingular algebraic set $P\subset W$, a*

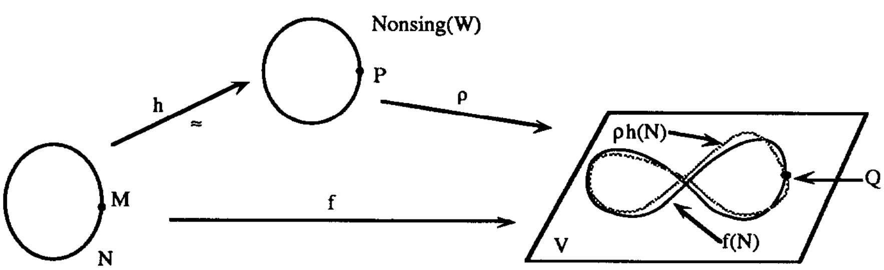

Fig. 2. Making the immersion f algebraic.

diffeomorphism $h: N \to Nonsing(W)$ *and a polynomial* $\rho: W \to V$ *so that:*

(1) $\rho h: N \to V$ *approximates* f.

(2) *If* $\rho_{\mathbb{C}} W_{\mathbb{C}} \to V_{\mathbb{C}}$ *is the complexification of* ρ, *then* $\rho_{\mathbb{C}}$ *is a finite regular map to its image.*

(3) $h(M) = P$.

(4) $\rho(P) = Q$.

Proof. After perturbing f, we may as well assume that f is degree one. Let $\varphi: T \to Q_{\mathbb{C}}$ be the normalization of $Q_{\mathbb{C}}$. By Lemma 5 we may assume that $T \subset \mathbb{C}^b$ for some b, $T_{\mathbb{R}}$ is nonsingular, $(T_{\mathbb{R}})_{\mathbb{C}} = T$ and there is a diffeomorphism $\kappa: M \to T_{\mathbb{R}}$ so that $\varphi\kappa = f|_M$.

ASSERTION 10.1. *After perhaps increasing b, we may find a nonsingular real algebraic set* $S \subset \mathbb{R}^b$, *a diffeomorphism* $g: N \to S$ *and a polynomial* $r: S \to V$ *so that rg approximates f,* $T_{\mathbb{R}} \subset S$, $g(M) = T_{\mathbb{R}}$ *and* $r(x) = \varphi(x)$ for all $x \in T_{\mathbb{R}}$.

Proof. After perhaps increasing b a bit we may imbed N in $\mathbb{R}^b$ in such a way that the imbedding restricts to κ on M. By Theorem 2.8.4 of [3] we then obtain our assertion. ∎

Let $Y = \mathrm{Cl}_{\mathbb{R}}(r(S))$ and let $Y_{\mathbb{C}} \subset V_{\mathbb{C}}$ be its complexification. Let $\theta: Z \to Y_{\mathbb{C}}$ be the normalization of $Y_{\mathbb{C}}$. By Lemma 5 we may assume $Z \subset \mathbb{C}^k$, Z and θ are defined over $\mathbb{R}$ and if $W = Z_{\mathbb{R}}$ then $Z = W_{\mathbb{C}}$. Let $S' = \mathrm{Nonsing}(S_{\mathbb{C}})$. By Lemma 5(d), there is a rational function $\lambda: S' \to Z$ defined over $\mathbb{R}$ so that $\theta\lambda = r_{\mathbb{C}|S'}$. As in the proof of Lemma 6 of [1] we have $\lambda(S) = \mathrm{Nonsing}(W)$, and in fact λ restricts to a diffeomorphism from S to Nonsing(W).

Let $B = \mathrm{Cl}_{\mathbb{C}}(\lambda(T \cap S'))$ and $P = B_{\mathbb{R}}$.

ASSERTION 10.2. $P = \lambda(T_{\mathbb{R}})$.

Proof. First, $\lambda(T_{\mathbb{R}}) \subset B$ by definition and $\lambda(T_{\mathbb{R}}) \subset \mathbb{R}^k$ since λ is defined over $\mathbb{R}$. So $\lambda(T_{\mathbb{R}}) \subset P$. Now take any $x \in P$. Because $\lambda(T \cap S')$ is constructable, we know that $B = \mathrm{Cl}(\lambda(T \cap S'))$, the closure in the usual topology. Hence there is a sequence $y_i \in T$ so that

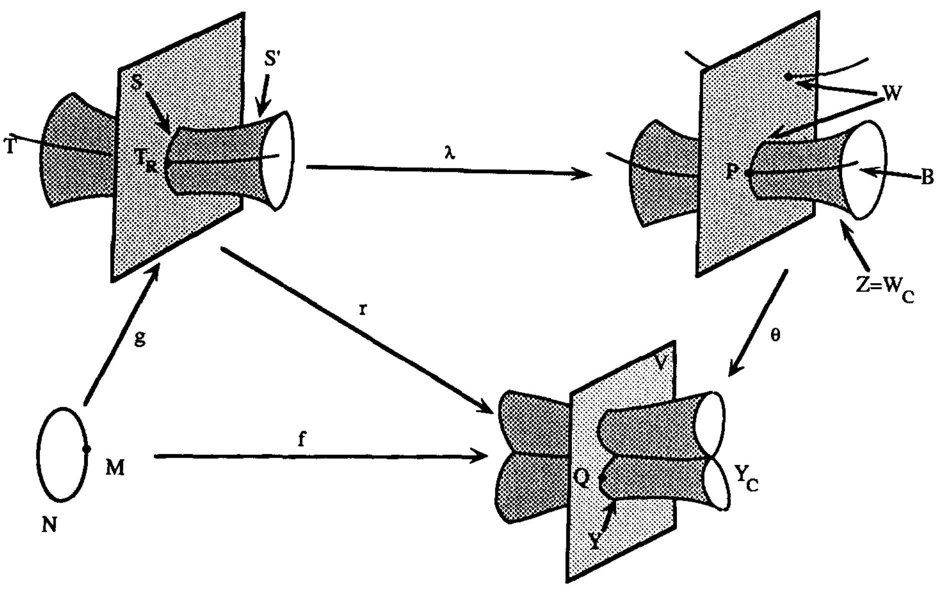

Fig. 3.

$\lambda(y_i) \to x$. So $\varphi(y_i) = \theta\lambda(y_i) \to \theta(x)$. Hence $\theta(x) \in \varphi(T) = Q_{\mathbb{C}}$ and by properness of φ we know that after taking a subsequence, we may assume $y_i \to y$ for some $y \in T$. But $\varphi(y) = \theta(x) \in Q$ and $\varphi^{-1}(Q) = T_{\mathbb{R}}$ by Lemma 5c so $y \in T_{\mathbb{R}}$. Hence $\lambda(y) = x$ so we have shown that $P = \lambda(T_{\mathbb{R}})$. ▌

Now let $h = \lambda g$ and $\rho = \theta|_W$. Then $W_{\mathbb{C}} = Z$ and $\rho_{\mathbb{C}} = \theta$ so $\rho_{\mathbb{C}}$ is finite. Also $\rho h = \theta\lambda g = rg$ which approximates f. We have $h(M) = \lambda g(M) = \lambda(T_{\mathbb{R}}) = P$ by Assertion 10-2. Finally, $\rho(P) = \theta\lambda(T_{\mathbb{R}}) = r(T_{\mathbb{R}}) = \varphi(T_{\mathbb{R}}) = Q$. ■

To prove Theorem 1, we have need of the following property $\mathscr{C}_n$ of a nonsingular real algebraic set V. It is similar to the conclusion of Lemma 10, but asks a little more – namely finiteness after composing with projection $V_{\mathbb{C}} \times \mathbb{C} \to V_{\mathbb{C}}$. We presume every nonsingular real algebraic set satisfies $\mathscr{C}_n$ but do not quite see the proof. In any case, Proposition 11 below shows that affine space $\mathbb{R}^m$ satisfies $\mathscr{C}_n$ for all n.

We say that a nonsingular real algebraic set V has property $\mathscr{C}_n$ if the following holds: Let X be any compact smooth $n+1$-dimensional manifold. let $N = \partial X$ and let M be any (possibly empty) codimension one submanifold of N with trivial normal bundle and let $f: X \to V \times \mathbb{R}$ be any smooth map so that $f|_N$ is an immersion, $f(M) \subset V \times 0$, $f(M)$ is an almost nonsingular real algebraic set, and $f|_M$ is a degree one immersion. Let $\pi: V \times \mathbb{R} \to V$ be projection. Then V has property $\mathscr{C}_n$ if whenever the above holds, there are a real algebraic set W, a nonsingular algebraic set $P \subset W$, a diffeomorphism $h: N \to \text{Nonsing}(W)$ and a polynomial ρ: $W \to V$ (see Fig. 4) so that:

(1) $\rho h: N \to V$ approximates πf.
(2) If $\rho_{\mathbb{C}}: W_{\mathbb{C}} \to V_{\mathbb{C}}$ is the complexification of ρ, then $\rho_{\mathbb{C}}$ is a finite regular map to its image.
(3) $h(M) = P$.
(4) $\rho(P)$ is an almost nonsingular real algebraic subset of V.

Proposition 11. *Affine space $\mathbb{R}^m$ has property $\mathscr{C}_n$ for all n.*

Proof. Let N, M and f be as in the definition of $\mathscr{C}_n$. By Lemma 10 there are a real algebraic set W, a nonsingular algebraic set $P \subset W$, a diffeomorphism $h: N \to \text{Nonsing}(W)$

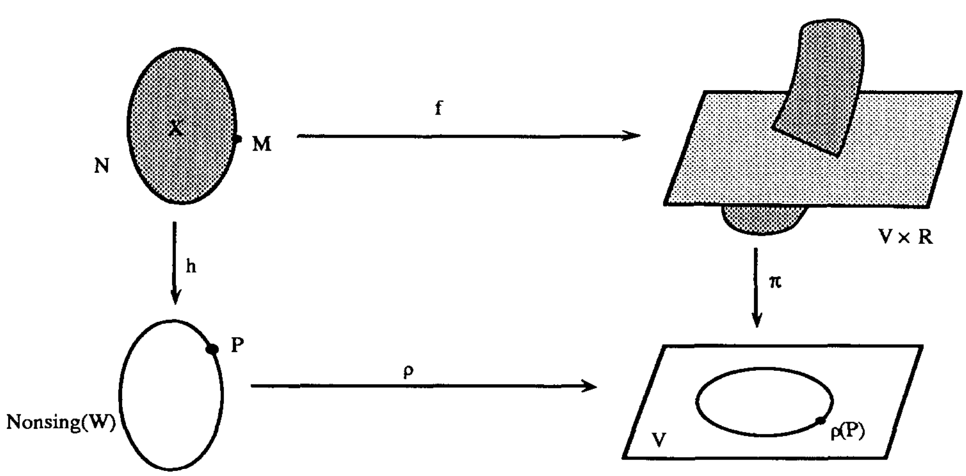

Fig. 4.

and a polynomial $\rho': W \to \mathbb{R}^m \times \mathbb{R}$ so that:

(1) $\rho' h: N \to \mathbb{R}^m \times \mathbb{R}$ approximates f.
(2) If $\rho'_{\mathbb{C}}: W_{\mathbb{C}} \to \mathbb{C}^m \times \mathbb{C}$ is the complexification of ρ', then $\rho'_{\mathbb{C}}$ is a finite regular map to its image.
(3) $h(M) = P$.
(4) $\rho'(P) = f(M)$.

By Lemma 7 there is a small real vector $u \in \mathbb{R}^m$ so that if $\kappa: \mathbb{C}^m \times \mathbb{C} \to \mathbb{C}^m$ is the map $\kappa(x, t) = x + tu$ then κ restricted to $\rho'_{\mathbb{C}}(W'_{\mathbb{C}})$ is a finite map to its image. Letting $\rho = \kappa\rho'$ we are done. ∎

We are now ready to prove Theorem 1. In light of Proposition 11 it suffices to prove the following more general result.

THEOREM 12. *Let $f: N \to V \times [0, \infty)$ be a proper smooth immersion from a compact smooth manifold N of dimension n. (In particular, $f^{-1}(V \times 0) = \partial N$.) Suppose V is a nonsingular real algebraic set satisfying condition $\mathscr{C}_n$. Suppose that $\partial N = M \cup M'$, that $f(M')$ is an almost nonsingular real algebraic set and $f|_{M'}$ is degree one. Then $f|_M$ is ε-regularly homotopic to an immersion onto an almost nonsingular real algebraic subset of V.*

Proof. Let N' be the double of N which we may think of as the boundary of $N \times [-1, 1]$. (Actually N' will go a little inside $N \times [-1, 1]$ near $\partial N \times \{-1, 1\}$ to round off the corners.) Let $\pi_0: V \times \mathbb{R} \to V$ and $\pi_1: V \times \mathbb{R} \to \mathbb{R}$ be projections. We may define a map $g': N \times [-1, 1] \to V \times \mathbb{R}$ by $g'(x, t) = (\pi_0 f(x), t + t\pi_1 f(x))$. By condition $\mathscr{C}_n$, there are a real algebraic set W, a nonsingular algebraic set $P \subset W$, a diffeomorphism $h: N' \to \text{Nonsing}(W)$ and a polynomial $\rho: W \to V$ so that:

(1) $\rho h: N' \to V$ approximates $\pi_0 g'$.
(2) If $\theta: W_{\mathbb{C}} \to V_{\mathbb{C}}$ is the complexification of ρ, then θ is a finite regular map to its image.
(3) $h(M' \times 0) = P$.
(4) If $Q = \rho(P)$ then Q is an almost nonsingular real algebraic subset of V.

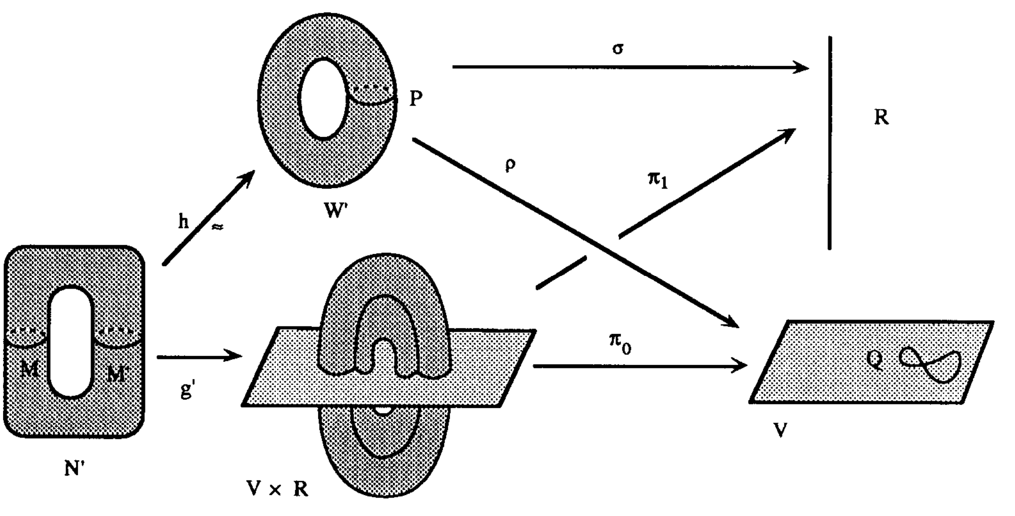

Fig. 5.

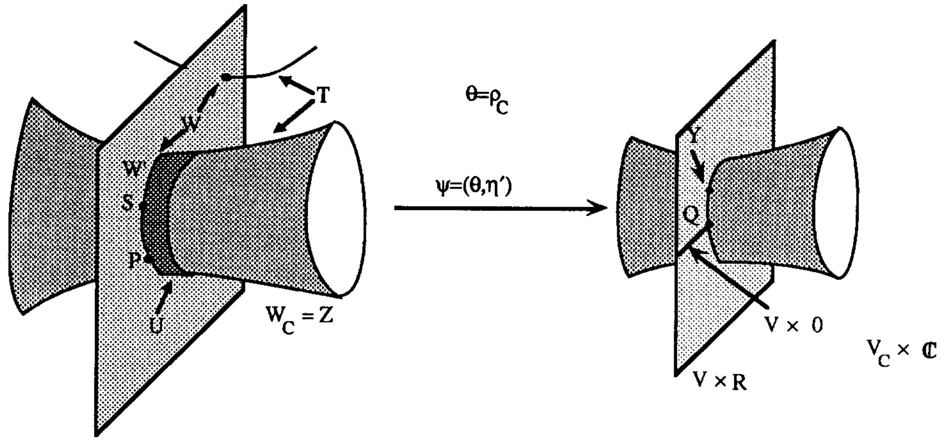

Fig. 6.

Let $W' = \text{Nonsing}(W)$ and $Z = W_{\mathbb{C}}$. Suppose $V \subset \mathbb{R}^k$. Pick e so that $|\rho(x)| < e$ for all $x \in W'$. Let $K \subset \mathbb{R}^k$ be the closed ball about 0 with radius $2e$. Let $T = \theta^{-1}(K)$. Then T is a compact subset of Z by properness of θ.

Let $\sigma: W' \to \mathbb{R}$ be a close polynomial approximation to $\pi_1 g' h^{-1}$. Since $\pi_1 g' h^{-1}(z) = 0$ for all $z \in P$ we may specify that $P \subset \sigma^{-1}(0)$. We know that for some neighborhood U of W' in Z,

$U \cap W = W'$,
$\theta(U) \subset$ the ball in $\mathbb{C}^k$ about 0 with radius e,
$(\theta, \sigma_{\mathbb{C}})|_U$ is a complex analytic immersion to $V_{\mathbb{C}} \times \mathbb{C}$,
$(\theta, \sigma_{\mathbb{C}})(U - W')$ has no real points.

Let b the maximum of $|\sigma_{\mathbb{C}}(z)|$ for $z \in T$. Pick some smooth $\alpha: T \to [0, 1]$ so that α is 0 on a neighborhood U' of W' in $T \cap U$, α is 1 on $T - U$ and $\alpha(z) = \bar{z}$ for all $z \in T$. Define $\tau: T \to \mathbb{C}$ by $\tau(z) = \alpha(z)(2b + 2) + \sigma_{\mathbb{C}}(z)$. By Lemma 9 there is a polynomial $\eta: Z \to \mathbb{C}$ defined over $\mathbb{R}$ so that $P \subset \eta^{-1}(0)$ and so that $\eta|_T$ approximates τ and this approximation is C^1 on U'. By Lemma 8, we may find $c > 0$, $d > 0$ and r so that $|\eta(z)| \leq c + d|\theta(z)|^r$ for all $z \in Z$. Pick some large integer $B > \log_2(d(2e)^r + 2c)$. Define $\beta: \mathbb{C}^k \to \mathbb{C}$ by $\beta(z_1, \ldots, z_k) = \sum_{i=1}^k z_i^2$. Pick a proper polynomial $q: \mathbb{R}^k \to \mathbb{R}$ so that $Q = q^{-1}(0)$. After multiplying by a constant we may as well assume $q(x) > 1$ if $|x| > 2e$. Let a be the maximum of $|q(x)|$ for $|x| \leq 2e$. Let $\eta'(z) = \eta(z) + (\beta(\theta(z))/(2e^2))^B q_{\mathbb{C}}(\theta(z))$ and define $\psi: Z \to V_{\mathbb{C}} \times \mathbb{C}$ by $\psi(z) = (\theta(z), \eta'(z))$.

Assertion 12.1. $\psi^{-1}(V \times \mathbb{R}) \subset W' \cup (Z - U)$.

Proof. Suppose $\psi(z) \in V \times \mathbb{R}$ but $z \in U - W'$. Then $|\theta(z)| < e$ so $\theta(z) \in K$ so $z \in T$. Also $\eta'(z) - \eta(z) = (|\theta(z)|^2/(2e^2))^B q(\theta(z)) < a2^{-B}$ which is very small if we pick B large enough. Moreover, for any partial derivative D,

$$|D\eta'(z) - D\eta(z)| = |\theta(z)|^{2B-2}(2e^2)^{-B}|BD\beta\theta(z)q\theta(z) + |\theta(z)|^2 Dq(\theta(z))|$$
$$< aB2^{-B}e^{-2}|D\beta\theta(z)| + 2^{-B}|Dq(\theta(z))|$$

which is small on U. So η' is C^1 close to η on U. Thus $\psi|_{U'}$ is an immersion ε-regularly homotopic to $(\theta, \sigma_{\mathbb{C}})|_{U'}$. But $(\theta, \sigma_{\mathbb{C}})(U' - W)$ has no real points. Hence $\psi(U' - W)$ has no real points. So $z \in U \cap T - U'$. Thus, $\eta'(z) \sim \eta(z) \sim \tau(z) = \alpha(z)(2b + 2) + \sigma_{\mathbb{C}}(z)$ which is not real (since $\alpha(z)$ is real and $(\theta, \sigma_{\mathbb{C}})(U - W')$ has no real points), so $\psi(z)$ is not real. ∎

ASSERTION 12.2. *$\psi^{-1}(V\times 0) = P\cup S$ where S is a submanifold of W' ε-isotopic to $h(M\times 0)$.*

Proof. Suppose $\psi(z)\in V\times 0$. Note that by Assertion 12.1, $z\in W'\cup(Z-U)$. If we had $|\theta(z)| > 2e$ then we would have

$$\begin{aligned}|\eta'(z)| &\geq |\beta\theta(z)/(2e^2)|^B|q\theta(z)| - |\eta(z)|\\ &\geq (|\theta(z)|^2/(2e^2))^B - c - d|\theta(z)|^r\\ &= |\theta(z)|^r((|\theta(z)|/2e)^{2B-r}(2e)^{-r}2^B - d) - c\\ &\geq (2e)^r((2e)^{-r}2^B - d) - c\\ &= 2^B - d(2e)^r - c \geq c > 0.\end{aligned}$$

This is a contradiction.

So $\theta(z)\in K$ and hence $z\in T$. If $z\notin W'$ then $z\in T-U$ by Assertion 12.1 so

$$\begin{aligned}0 = \eta'(z) &= \eta(z) + (|\theta(z)|^2/(2e^2))^B\\ &\geq \eta(z) = \Re(\eta(z))\\ &\sim \Re(\tau(z)) = 2b + 2 + \Re(\sigma_{\mathbb{C}}(z)) \geq b+2\end{aligned}$$

which is a contradiction. ($\Re(z)$ denotes the real part of z.)

So $z\in W'$. But on W', ψ is C^1 close to (θ,σ) which is C^1 close to $g'h^{-1}$. Now

$$(g'h^{-1})^{-1}(V\times 0) = h((g'|_{N'})^{-1}(V\times 0)) = h(M\times 0\cup M'\times 0) = P\cup h(M\times 0).$$

Furthermore, $g'|_{N'}$ is transverse to $V\times 0$. So $\psi^{-1}(V\times 0)$ is a manifold ε-isotopic to $P\cup h(M\times 0)$. However, for $z\in P$, $\eta'(z)=0$ so $P\subset\psi^{-1}(V\times 0)$. Thus this ε-isotopy of $P\cup h(M\times 0)$ to $\psi^{-1}(V\times 0)$ can fix P. We let $S=\psi^{-1}(V\times 0) - P$. ∎

ASSERTION 12.3. *$\psi^{-1}\psi(W') = W'$.*

Proof. Suppose not. Then there is a $w\in W'$ and $z\in Z-W'$ so that $\psi(z)=\psi(w)$. By Assertion 12.1 we know that $z\in Z-U$. But $\theta(z)=\theta(w)=\rho(w)\in K$ so $z\in T$. But then

$$b+2 < |2b+2+\sigma_{\mathbb{C}}(z)| \sim |\eta'(z)| = |\eta'(w)| \sim |\sigma(w)| \leq b.$$

This is a contradiction, so $\psi^{-1}\psi(w)\subset W'$ for all $w\in W$. ∎

Since ψ is transverse to $V\times 0$, we know $P\cup S=\psi^{-1}(V\times 0)$ is nonsingular by Lemma 1.4 of [2]. Hence S is a real algebraic set by Lemma 1.6 of [2] or [7]. Let $Y=\mathrm{Cl}_{\mathbb{R}}(\theta(S))$.

ASSERTION 12.4. *$Y=\theta(S)$.*

Proof. We know $Y=\theta(S_{\mathbb{C}})\cap V$ by finiteness of θ. if $Y\neq\theta(S)$ there is a $x\in S_{\mathbb{C}}-S$ so that $\theta(x)\in V$. But η' vanishes on S, hence it vanishes on $S_{\mathbb{C}}$. So $\psi(x)\in V\times 0$ which contradicts Assertion 12.2. ∎

Now by Assertion 12.3 and Lemma 10 of [1] all points of Y are almost nonsingular. By Assertion 12.2 there is a small isotopy $h_t\colon W'\to W'$ so that $h_1(h(M\times 0)) = S$. Now $f|_M$ is close to the map $f'(x)=\rho h_1 h(x,0)$ so f and f' are ε-regularly homotopic. But $f'(M) = \rho h_1 h(M\times 0) = \rho(S) = \theta(S) = Y$. ∎

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