1992 PROCEEDINGS OF KAIST MATHEMATICS WORKSHOP

LECTURES ON REAL ALGEBRAIC SPACES

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Real algebraic geometry is one subject where the flexible methods of topology and the rigid methods or algebraic geometry nicely complement each other. Often, failure of topological methods signal existence of algebraic invariants. For example, the cohomology rings of smooth submanifolds of $\mathbf{RP}^n$ can give rise to obstructions to isotoping them to real parts of nonsingular complex algebraic subsets of $\mathbf{CP}^n$. Here we will discuss real algebraic structures on topological spaces, by first surveying algebraic structures on smooth manifolds and then reviewing the general case of algebraic structures on topological spaces. Our discussion is mostly based on the joint work with H.King ([**AK1**]). We would like to thank to KAIST for kind invitation to deliver these lectures.

1. Algebraic Spaces:

Let k be a field. Recall that (cf. [**AK2**]) a **k-algebraic space** $(X, \{\varphi_\alpha\})$ is a topological space X, with a collection of imbeddings $\varphi_\alpha : S_\alpha - S'_\alpha \to X$ such that $S'_\alpha \subset S_\alpha$ are k-algebraic sets, the images of φ_α cover X and

(i) Each $T_{\alpha\beta} = S'_\alpha \cup \varphi_\alpha^{-1}(X - \text{image}\varphi_\beta)$ is a k-algebraic set

(ii) Each $\varphi_\beta^{-1}\varphi_\alpha : S_\alpha - T_{\alpha\beta} \to S_\beta - T_{\beta\alpha}$ is a birational isomorphisim.

We say $Y \subset X$ is a closed algebraic subspace if each $S'_\alpha \cup \varphi_\alpha^{-1}(Y)$ is an algebraic set. This describes a topology on X which is usually called the Zariski topology. Condition (i) says that the charts are open in this topology. If each $S_\alpha - S'_\alpha$ is contained in the set of nonsingular points of S_α we say $(X, \{\varphi_\alpha\})$ is **nonsingular**. In particular this makes X a smooth manifold. If

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each $S_\alpha = k^m$ we call $(X, \{\varphi_\alpha\})$ a **rational algebraic space.** An algebraic set is an example of an algebraic space with a single chart S_α and $S'_\alpha = \emptyset$.

A morphisim or a **rational function** between two algebraic spaces $(X, \{\varphi_\alpha\})$ and $(Y, \{\psi_\lambda\})$ is a continuous map $f : X \to Y$ in this topology which is locally rational. More precisely: For each $x \in X$ there are charts $\varphi_\alpha : S_\alpha - S'_\alpha \to X$ and $\psi_\lambda : V_\lambda - V'_\lambda \to Y$ based at x and $f(x)$ respectively, and an algebraic set $Z \subset S_\alpha$ with $x \notin Z$ such that: $\psi_\lambda^{-1} f \varphi_\alpha$ restricts to a rational function on $S_\alpha - (S'_\alpha \cup Z) \to V_\lambda - V'_\lambda$.

In these lectures we are mainly interested in real algebraic spaces, i.e. the case $k = \mathbf{R}$. In the rest of these lectures unless otherwise stated all algebraic spaces are real algebraic spaces Projective spaces are examples of nonsingular algebraic spaces. More generally toric manifolds [O] are nonsingular algebraic spaces where the gluing functions $\varphi_\beta^{-1}\varphi_\alpha$ are monomials. A fundamental question is to decide which topological spaces can be realized as an algebraic space. We approach to this problem by looking at special kinds of algebraic spaces.

2.Nonsingular Algebraic Spaces:

The two most important kind of nonsingular algebraic spaces are nonsingular algebraic sets and rational algebraic spaces.

(a) Nonsingular algebraic sets:

Ideally nonsingular algebraic sets should fit in the diagram:

$$\begin{array}{ccc} \{\text{Nonsingular algebraic sets}\} & \Longrightarrow & \left\{\begin{array}{c}\text{Interiors of compact}\\ \text{smooth manifolds}\end{array}\right\} \\ \cup & & \cup \\ \left\{\begin{array}{c}\text{Compact nonsingular}\\ \text{algebraic subsets of } \mathbf{R}^n\end{array}\right\} & \overset{?}{\Longrightarrow} & \left\{\begin{array}{c}\text{Closed smooth}\\ \text{submanifolds of } \mathbf{R}^n\end{array}\right\} \end{array}$$

where the horizontal double arrows are surjective forgetful maps from algebraic sets onto topological spaces. In the case of closed smooth manifolds the surjectivity of the top horizontal arrow is due to Nash and Tognoli ([N], [T]) , in the case of noncompact manifolds it is a result of [AK3]. So we still don't know whether the bottom horizontal arrow is onto.

QUESTION 1. *Is every closed smooth submanifold of* $\mathbf{R}^n$ *isotopic to a nonsingular algebraic subset of* $\mathbf{R}^n$ *?*

On the positive side we have the following result:

THEOREM. [AK4]: *Every closed smooth submanifold* $M \subset \mathbf{R}^n$ *is isotopic to the set of nonsingular points* V_0 *of an algebraic subset* $V \subset \mathbf{R}^n$. *In particular* M *is isotopic to a nonsingular component of an algebraic subset of* $\mathbf{R}^n$.

Suppose that the polynomial equations $f(x) = 0$ and $g(x) = 0$ describe V and the singular points $V - V_0$ of V, respectively. Then in $\mathbf{R}^n \times \mathbf{R}$ the equations $f(x) = 0$ and $tg(x) = 1$ describe an algebraic set diffeomorphic to V_0 . So the theorem implies that M is isotopic to a nonsingular algebraic subset of $\mathbf{R}^n \times \mathbf{R}$. This theorem generalizes to immersions $f : M \to \mathbf{R}^n$; in this case the word nonsingular in the conclusion of the theorem is replaced by almost nonsingular. Almost nonsingular algebraic sets are analogues of immersed manifolds in the algebraic category. In particular if an imbedded closed smooth submanifold is almost nonsingular algebraic set then it is a nonsingular algebraic set.

In [AK5] the Question 1 is reduced to a bordisim problem, that is : Image of any degee one immersion $f : M \to \mathbf{R}^n$ is isotopic to an almost nonsingular algebraic subset if and only if through immersed cobordisims f can be deformed onto an almost nonsingular algebraic subset. Hence by the above discussion if an imbedding $f : M \hookrightarrow \mathbf{R}^n$ deforms through immersed cobordisim onto a smooth submanifold of $\mathbf{R}^{n-1}$ then f is isotopic to a nonsingular algebraic subset of $\mathbf{R}^n$. To explain this result in more fancy terms , recall that the group of immersed cobordisim classes of codimension k immersed submanifolds of $\mathbf{R}^n$ can be identified by the group $\pi_n(\Omega^\infty\Sigma^\infty MO(k))$. So if $\pi_n^{alg}(\Omega^\infty\Sigma^\infty MO(k))$ denotes the group generated by the immersions onto almost nonsingular algebraic subsets, then f is isotopic to an algebraic subset of $\mathbf{R}^n$ if and only if the cobordisim class $[f]$ lies in $\pi_n^{alg}(\Omega^\infty\Sigma^\infty MO(k))$. So the natural questions are:

QUESTION 2. *Is every closed smooth submanifold of* $\mathbf{R}^n$ *cobordant through* *immersions to to an almost nonsingular algebraic subset of* $\mathbf{R}^n$ *?*

QUESTION 3. *When a closed smooth submanifold of* $\mathbf{R}^n$ *cobordant through immersions to to a smooth submanifold of* $\mathbf{R}^{n-1}$ *?*

By Hirsh-Smale theorem a closed smooth submanifold $M \subset \mathbf{R}^n$ immerses into $\mathbf{R}^{n-1}$ when the normal bundle splits a trivial line bundle. It would be nice to find computable algebraic topological conditions (e.g. conditions on the characteristic classes of the normal bundle) which would imply that the normal bundle of M splits a trivial line bundle.

On the other hand if we strengthen the notion of nonsingularity (i.e. the projective complexification is nonsingular) we can answer Question 1 negatively. That is by identifying $\mathbf{R}^n \subset \mathbf{RP}^n$ we can show:

THEOREM. [AK6]: *There are closed smooth submanifolds of* $M \subset \mathbf{R}^n$ *which can not be isotoped to the real parts of nonsingular complex algebraic subsets of* $\mathbf{CP}^n$.

In fact in the examples of this theorem some of M can be chosen to be nonsingular algebraic subsets of $\mathbf{R}^n$. So either these algebraic subsets are not isotopic to nonsingular algebraic subsets of $\mathbf{RP}^n$ (i.e. they are not projectively closed) or we have examples of nonsingular algebraic subset of $\mathbf{RP}^n$ which don't admit nonsingular complexification in $\mathbf{CP}^n$, even after an isotopy. The obstructions of this theorem comes from the cohomology of M. To understand this theorem we first need to talk about algebraic cohomology groups.

Let $V \subset \mathbf{RP}^n$ be a compact nonsingular real algebraic set of dimension v, and $V_{\mathbf{C}} \subset \mathbf{CP}^n$ be a nonsingular projective complexification of V. Let $j : V \hookrightarrow V_{\mathbf{C}}$ denote the inclusion. Let $H^*_A(V;\mathbf{Z}_2)$ denote the subgroup of the cohomology group $H^*(V;\mathbf{Z}_2)$ generated by classes whose Poincare duals are represented by real algebraic subsets of V. Let $\bar{H}^*_A(V_{\mathbf{C}};\mathbf{Z})$ be the subgroup of $H^*(V_{\mathbf{C}};\mathbf{Z})$ generated by classes whose duals are represented by the complexification of real algebraic subsets of V. Let $\bar{H}^{2k}_{\mathbf{C}-alg}(V;\mathbf{Z}) = j^*\bar{H}^{2k}_A(V_{\mathbf{C}};\mathbf{Z})$, and let $\bar{H}^{2k}_{\mathbf{C}-alg}(V;\mathbf{Z}_2)$ be the mod 2 reduction of this group.

The real algebraic cohomology groups $H^*_A(V;\mathbf{Z}_2)$ play useful role in real algebraic geometry, they carry obstructions to isotoping submanifolds to algebraic subsets (see [AK1]). Also the groups $H^*_{\mathbf{C}-alg}(V;\mathbf{Z})$ which are defined in [BBK] appear as obstructions to algebraic approximation problems. These two groups can be related to each other in terms of the natural subgroup: $H^k_A(V;\mathbf{Z}_2)^2 = \{\ \alpha^2 \mid \alpha \in H^k_A(V;\mathbf{Z}_2)\ \}$ of $H^{2k}_A(V;\mathbf{Z}_2)$ as follows:

THEOREM [AK6]. $\bar{H}^{2k}_{\mathbf{C}-alg}(V;\mathbf{Z}_2) = H^k_A(V;\mathbf{Z}_2)^2$.

Assuming this we can find examples of smooth submanifolds $M \subset \mathbf{R}^n$ which can not be isotopic to the real parts of nonsingular complex algebraic subsets of $\mathbf{CP}^n$. First of all recall that the tangent Gauss map $\alpha : V \to G(n,v)$ of any nonsingular algebraic subset $V \subset \mathbf{R}^n$ can be represented by an entire rational map, where v dimension of V. Furthermore, the Grassmann manifold $G(n,v)$ is a nonsingular algebraic set such way that all the Poincare duals of the Steifel-Whitney classes are represented by algebraic subsets (e.g. [AK1]). Hence by pull-back, all the Steifel-Whitney classes of V are represented by algebraic subsets. This implies for example if $V = \mathbf{RP}^{10}$ then $H^1_A(V;\mathbf{Z}_2)^2 \neq 0$.

On the other hand if the projective complexification $V_{\mathbf{C}}$ is nonsingular, by a theorem of Larsen [Hr], for $i \leq 2v - n$ the restriction induces an isomorphisim

$$H^i(\mathbf{CP}^n;\mathbf{Z}) \overset{\cong}{\to} H^i(V_{\mathbf{C}};\mathbf{Z})$$

Hence $\bar{H}^i_{\mathbf{C}-alg}(V;\mathbf{Z})$ lies in the the image of the classes $H^i(\mathbf{RP}^n;\mathbf{Z})$ under restriction (this was observed in [BBK]). In particular if V lies in a chart $\mathbf{R}^n$ of $\mathbf{RP}^n$, then $\bar{H}^i_{\mathbf{C}-alg}(V;\mathbf{Z}) = 0$ for $i \leq 2v - n$. Recall that there is an imbedding $\mathbf{RP}^{10} \subset \mathbf{R}^{18}$ ([Ha]). We claim that this imbedding can not be isotoped the real part V of a nonsingular complex algebraic subsets $V_{\mathbf{C}} \subset \mathbf{CP}^{18}$. Otherwise we would have $\bar{H}^2_{\mathbf{C}-alg}(V;\mathbf{Z}_2) = 0$ which along with the first paragraph contradicts the last theorem.

(b) Rational algebraic spaces

Rational algebraic structures are hard to construct on smooth manifolds, mainly because of our lack of understanding of rational diffeomorphisims.

One way to produce rational algebraic spaces is to apply blowup and blowdown operations to a nonsingular algebraic set which already has this structure, such as $\mathbf{RP}^m$. Let us call any two algebraic spaces which are related by these operations **blowup equivalent**. In the topological category there is an analogue of of the blowup operation, i.e. one can blowup smooth manifolds along smooth submanifolds. So, we can also talk about smooth manifolds that are **topologically blowup equivalent**. Then we can hope to have the following implications:

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{Compact rational}\\ \text{algebraic spaces}\end{array}\right\} & \overset{?}{\Rightarrow} & \{\text{Closed smooth manifolds}\} \\
\bigcup & & \|? \\
\left\{\begin{array}{c}\text{Nonsingular algebraic spaces}\\ \text{blowup equivalent to } \mathbf{RP}^m\end{array}\right\} & \Rightarrow & \left\{\begin{array}{c}\text{Smooth manifolds topologically}\\ \text{blowup equivalent to } \mathbf{RP}^m\end{array}\right\}
\end{array}$$

where the horizontal double arrows are forgetful surjections from algebraic sets to manifolds. Presently, neither the surjectivity of the horizontal arrows nor the second vertical equivalence is known. So we have the questions:

QUESTION 4. *Is every smooth compact manifold diffeomorphic to a nonsingular rational space?*

QUESTION 5. *Is every smooth compact m-manifold topologically blowup equivalent to* $\mathbf{RP}^m$*?*

We can also ask a stronger version of Question 4:

QUESTION 6. *Is every smooth compact m-manifold diffeomorphic to a nonsingular algebraic set which is blowup equivalent to* $\mathbf{RP}^m$ *?*

Question 4 is a version of a question posed by Nash in [N]. Currently we have no example of a smooth manifold which does not admit a structure of a rational space. We know more about Question 5: The answer is yes for 2-manifolds since blowing up a surface at a point has the affect of connected summing with an $\mathbf{RP}^2$. The answer is yes for 3-manifolds ([AK7], [BM]). Recently, the 4-dimensional case has also been verified ([M]). The proof of [M] is inductive, so it has a good chance of being generalized to all dimensions.

3. Singular Algebraic Spaces:

Simplest kind of singular algebraic spaces are singular algebraic sets. In general algebraic sets are stratified spaces. Furthermore, they admit a certain topologically defined structure which are called resolution tower structures, or $\mathcal{TR}$ structure in short. A topological space with such a structure is called a $\mathcal{TR}$ space. Resolution tower structures completely characterize algebraic sets of dimension less than 4. In higher dimension these structures also characterize all algebraic sets in a weaker sense. Furthermore in dimensions less than 4 existence of resolution tower structures can be detected by combinatorial invariants of the underlying topological spaces (see [**AK1**]).

A resolution tower structure $\mathcal{F} = \{V_i, V_{ji}, p_{ji}\}$ on an algebraic set X roughly consists of: A collection of smooth manifolds V_i $i = 1, 2, ...k$, tansversally intersecting smooth submanifolds $V_{ji} \subset V_i$, and certain topologically defined maps $p_{ji} : V_{ji} \to V_j$ satisfying some compatibility conditions. By gluing these smooth manifolds with these maps we obtain a stratified space isomorphic to X.

$$X = \bigcup V_i / x \sim p_{ji}(x)$$

In a sense V_i encodes the information that various strata of X has resolutions, and p_{ji} encodes the information of how these resolutions put together, along with complexification.

Rather than going into details, let us describe our invariants and show how to construct these structures on some examples. Let X be a locally cone-like Euler stratified space (i.e. the Euler characteristics of the links of all strata are even). Then for every component of the codimension one stratum U of X we define $\kappa_X(U)$ to be the number of points in the link of U. For every component of the codimension two stratum W of X we define $\alpha_i(W)$ to be the number of vertices v of the link L of W so that $\kappa_L(v) = i \ mod \ 8$, where $i = 0, .., 7$. Define

$$\epsilon_0(W) = \alpha_0(W) \ \ mod \ 2$$

$$\epsilon_1(W) = \alpha_6(W) \ \ mod \ 2$$

$$\epsilon_2(W) = (\alpha_0(W) + \alpha_4(W))/2 \ \ mod \ 2$$

$$\epsilon_3(W) = (\alpha_2(W) + \alpha_6(W))/2 \ \ mod \ 2$$

and define $\epsilon_X(W) = (\epsilon_0(W), \epsilon_1(W), \epsilon_2(W), \epsilon_3(W)) \in \mathbf{Z}_2^4$. It is an easy exercise to show that $\epsilon_X(W)$ is well defined, i.e. $\alpha_0 + \alpha_4$, and $\alpha_2 + \alpha_6$ are even. For every component of the codimension three stratum V of X and $\epsilon \in \mathbf{Z}_2^4$ we define $\beta_\epsilon(V) \in \mathbf{Z}_2^4$ to be the number of vertices v of the link L of V with $\epsilon_L(v) = \epsilon$. Then we have :

$$\left\{\begin{array}{c}\text{Compact real algebraic} \\ \text{sets of dimension } \leq 3\end{array}\right\} \equiv \left\{\begin{array}{c}\text{Compact } \mathcal{TR}-\text{spaces} \\ \text{of dimension } \leq 3\end{array}\right\}$$

$$\equiv \left\{\begin{array}{c}\text{Compact locally cone-like Euler stratified spaces } X \\ \text{of dimension} \leq 3 \text{ with} \\ \beta_{1110}(x) + \beta_{1111}(x) = 0 \;,\; \beta_{0100}(x) + \beta_{0101}(x) = 0 \\ \beta_{1000}(x) + \beta_{1001}(x) = 0 \;,\; \beta_{1100}(x) + \beta_{1101}(x) = 0 \\ \text{for all codimension three strata x of X}\end{array}\right\}$$

Therefore by a purely combinatorial data one can decide whether a stratified space of dimension less then four is homeomorphic to an algebraic set. In **[AK1]**, a similar characterization, when homeomorphisim replaced by a stratified set isomorphisim, is also given.

Let us apply this theorem to some examples. For example the suspension of $\mathbf{R}P^2$ is not homeomorphic to an algebraic set, since algebraic sets must be Euler spaces [S] . But if X is the suspension of X_0, where X_0 is the disjoint union of $\mathbf{R}P^2$ and a point x_0 then X is homeomorphic to an algebraic set.

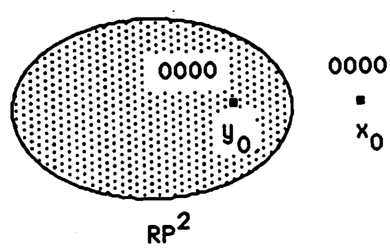

Figure 1: X_0

To see this first stratify X_0 by taking the zero strata to be x_0 and a point y_0 on $\mathbf{R}P^2$, and let the rest to be the 2-stratum. Then take the induced satratification on X. The zero stratum X^0 of X consists of the two suspension pints, and each has the link X_0. The ϵ numbers of the two vertices of X_0 is 0000 so in fact all $\beta_\epsilon(x) = 0$ for all $x \in X^{(0)}$

Hence by the above characterization X is homemorphic to an algebraic set. In particular it has a $\mathcal{TR}$ structure. **[AK1]** gives an algorithm of finding this $\mathcal{TR}$ structure. Let us indicate how to find this structure by bare hands. First we put a $\mathcal{TR}$ structure $\mathcal{F} = \{V_2, V_{02}, V_0, p_{02}\}$ on X_0 as follows:
$V_2 = \mathbf{R}P^2 \sharp \mathbf{R}P^2 \sharp F_4$, where F_4 is the surface of genus four.
$V_{02} = \{S_1, S_2, C_1, C_2, C_3, C_4\}$ are the collection of circles in $\mathbf{R}P^2 \sharp F_4 \subset \mathbf{R}P^2 \sharp \mathbf{R}P^2 \sharp F_4$ as indicated in Figure 2 . More specifically Figure 2 is the picture of the Mobius band with four 1-handles attached, which is only part of $\mathbf{R}P^2 \sharp F_4$. $V_0 = \{x_0, y_0\}$, and $p_{02} : V_{02} \to V_0$ to be the collapsing map onto y_0 .

Reader familiar with **[AK1]** will notice that V_{02} is the result of the usual "spine construction" of applied to the Mobius band. In particular $\cup S_i \cup C_j$ is the spine of the Mobius band $\sharp F_4$. It is an easy exercise to see that the realization $|\mathcal{F}|$ of this resolution tower $\mathcal{F}$ is X_0. Our goal is to extend this resolution tower structure on X_0 to a resolution tower structure on X.

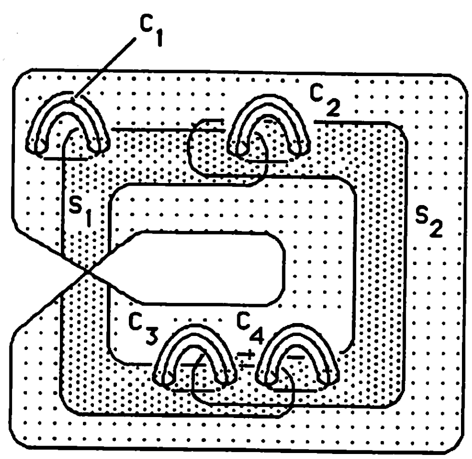

Figure 2:

We first construct a compact 3-manifold M and properly imbedded surfaces in general position $L \subset M$ with the property that $\partial M = \mathbf{R}P^2 \sharp \mathbf{R}P^2 \sharp F_4$, and $\partial L = \cup S_i \cup C_j$ as in Figure 3.

This can be seen as follows: Call $M_0 = (\mathbf{R}P^2 \sharp \mathbf{R}P^2 \sharp F_4) \times [0,1]$. Let M_1 be the manifold obtained from M_0 by adding 1-handles to M_0 along

$(\mathbf{R}P^2\sharp\mathbf{R}P^2\sharp F_4)\times 1$ by pairwise identifying the eight points of intersection of S_i's and C_j's. We can extend the codimension one manifolds $L_0 = (\cup S_i \cup C_j)\times[0,1]$ in M_0 to the codimension one manifolds L_1 in M_1 by adding the two transversally intersecting annuli in the 1 -handles as in Figure 4.

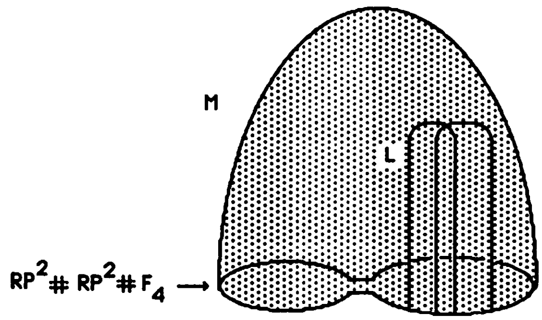

Figure 3:

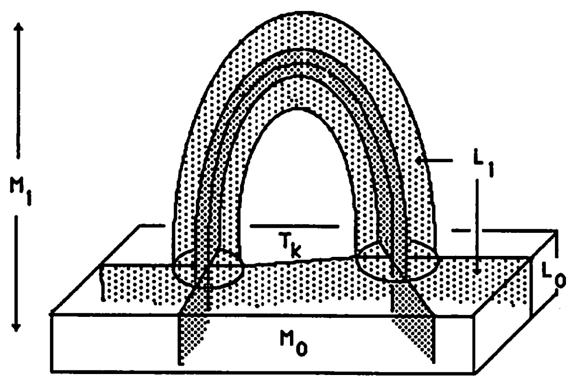

Figure 4:

$\partial L_1\cap(\partial M_1 - \mathbf{R}P^2\sharp\mathbf{R}P^2\sharp F_4\times 0)$ consists of disjoint union of circles $\cup T_k$. Then by attaching 2-handles to M_1 along T_k's we get a new 3-manifold M_2. Again we can extend the codimension one submanifolds L_1 of M_1 to the codimension one manifolds L_2 of M_2 by adding the cores of the 2-handles as in Figure 5

Now $\partial L_2\cap(\partial M_2-\mathbf{R}P^2\sharp\mathbf{R}P^2\sharp F_4\times 0)$ is empty. Since $\partial M_2-\mathbf{R}P^2\sharp\mathbf{R}P^2\sharp F_4\times 0$ is cabordant to $\mathbf{R}P^2\sharp\mathbf{R}P^2\sharp F_4\times 0$ it bounds, i.e. it is a boundary of a compact manifold W. Hence we can cap M_2 off with W to get the required compact manifold $M = M_2\cup W$ with $L = L_2$.

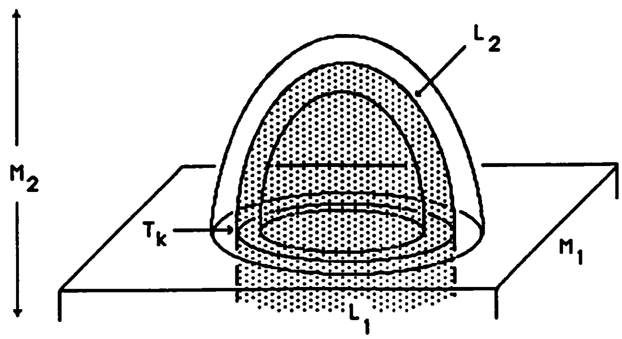

Figure 5:

By applying the "spine construction" to M (e.g. Proposition 3.1 of [**AK8**]) we can assume that there exists a union of codimension one closed submanifolds (i.e. ticos) $Z \subset \text{interior}(M)$ such that Z is a spine of M, and $Z \cap L$ is a spine of L. We can now extend the $\mathcal{TR}$-structure $\mathcal{F}$ on X_0 to a $\mathcal{TR}$-structure $\mathcal{G} = \{W_3, W_{13}, W_{03}, W_1, W_{01}, W_0\}$ on cone(X_0), by letting $W_3 = M$, $W_{13} = L$, $W_{03} = Z$, $W_1 = [0,1]$ (identify 0 by x_0, and 1 by y_0) , $W_{01} = 1/2$, W_0 is a point, p_{03}, and p_{01} be the obvious collapse maps, and p_{13} to be the fold map mapping $(L, L \cap Z)$ onto $([0,1/2], 1/2)$. By doubling $\mathcal{G}$ we get a resolution tower structure on the suspension of X_0 i.e. on X.

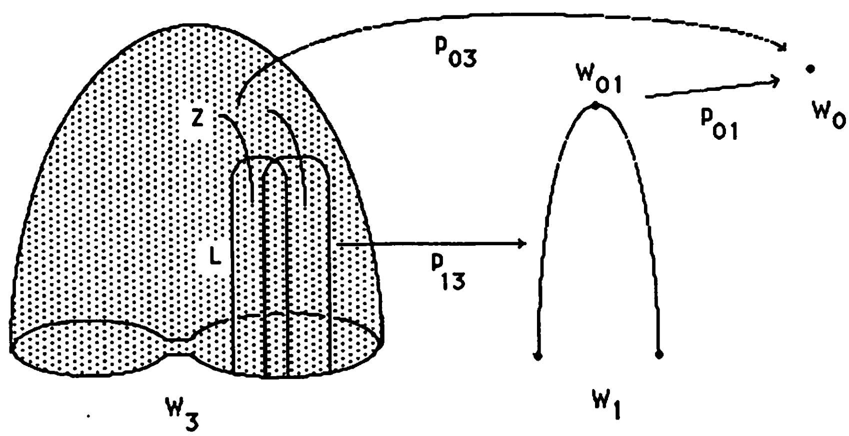

Figure 6:

For another example let Y to be the suspension of Y_0, where Y_0 is the disjoint union of $S^2 \vee S^1$ and a point x_0. We stratify Y_0 by taking the zero strata to be the four points consisting of the wedge point, the disjoint point, and two extra points : one on S^2 and one on S^1.

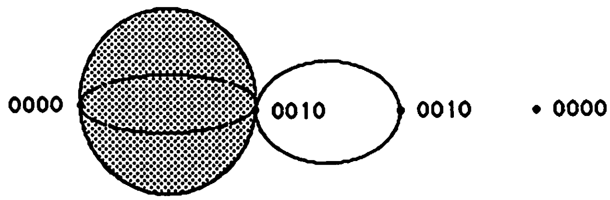

Figure 7: Y_0

We then take the induced stratum on the suspension Y. Again the zero strata Y^0 of Y consists of the two suspension points, and their links are Y_0.

As in the first example we have $\beta_\epsilon(x) = 0$ for all $x \in Y^{(0)}$. So by above Y is homeomorphic to an algebraic set. The homeomorphisim is obtained by constructing a resolution tower structure on Y. Let us again indicate how to see this resolution tower. As above we first construct a resolution tover structure $\mathcal{F} = \{V_2, V_{02}, V_1, V_{01}, V_0, p_{02}, p_{01}\}$ on Y_0 as follows:

$V_2 = (\mathbf{R}P^2\sharp F_4)\sharp(\mathbf{R}P^2\sharp F_4)$, where F_4 is the surface of genus four.

$V_{02} = \{S_i, C_j, S_i', C_j' \mid i = 1,2 \textit{ and } j = 1,2,3,4\}$, are the collection of circles in $(\mathbf{R}P^2\sharp F_4)\sharp(\mathbf{R}P^2\sharp F_4)$ obtained by taking the circles of Figure 2 in each $\mathbf{R}P^2\sharp F_4$ factor of the connected sum. $V_1 = S^1$, $V_{01} = \{a, b\}$, two points on the circle. $V_0 = \{x_0, y_0, x_1, y_1\}$, four points. $p_{02} : V_{02} \to V_0$ is the collapsing map $\cup C_i \cup S_j$ to x_0, and $\cup C_i' \cup S_j'$ to x_1. $p_{01} : V_{01} \to V_0$ is the map $p_{01}(a) = x_0$ and $p_{01}(b) = y_0$. It is easily seen that $|\mathcal{F}|$ gives Y_0.

As before it suffices to extend $\mathcal{F}$ to a resolution tower structure to $cone(Y_0)$. This gives a resolution tower structure on the suspension Y. By the same construction as before we find a compact 3-manifold M with properly imbedded surfaces in general position $L \cup L' \subset M$ such that $\partial M = (\mathbf{R}P^2\sharp F_4)\sharp (\mathbf{R}P^2\sharp F_4)$, and $\partial L = \cup S_i \cup C_j$, $\partial L' = \cup S_i' \cup C_j'$. Here the only difference of the construction is that we apply the process of getting L to the both sides of the connected sum factors of Y_0. As before by the "spine construction" of [AK8] we can assume that there exists a union of codimension one closed submanifolds $Z \subset$ interior(M) which is a spine of M and $Z \cap (L \cup L')$ is a spine of $L \cup L'$.

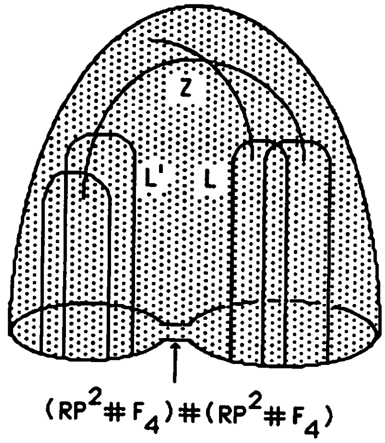

Figure 8:

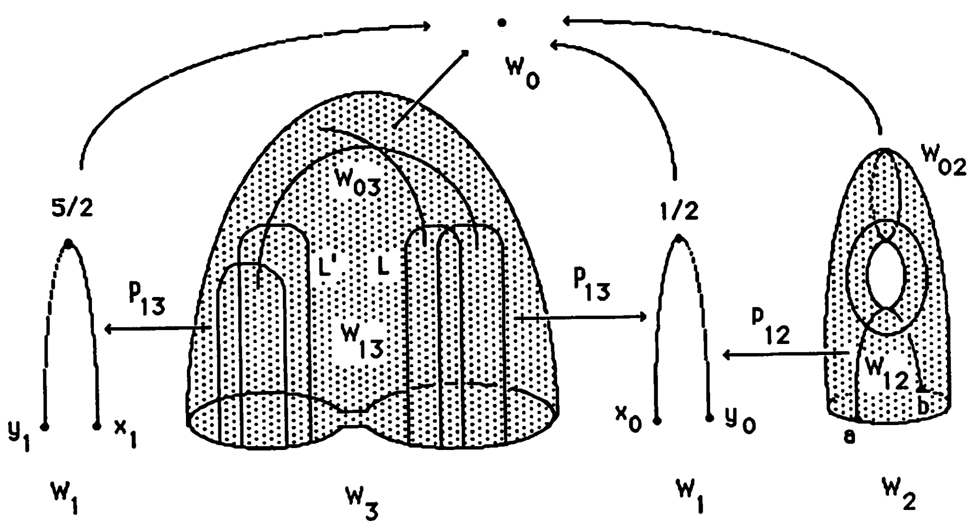

Figure 9:

Now let $\mathcal{G} = \{W_3, W_{13}, W_{03}, W_2, W_{12}, W_{02}, W_1, W_{01}, W_0\}$, where : $W_3 = M$, $W_{13} = L \cup L'$, $W_{03} = Z$, $W_2 = T^2 - D^2$ a punctured 2-torus, W_{02} are the wedge of two circles which is a spine of the punctured torus. W_{12} is a properly imbeded interval $[a, b]$ in the punctured torus meeting W_{02} at one point and meeting the boundary in $\{a, b\}$. $W_1 = [0, 1] \cup [2, 3]$ two intervals whose end points are identified by $\{x_0, y_0\}$ and $\{x_1, y_1\}$ respectively,

$W_{01} = \{1/2, 5/2\}$, W_0 is a point, p_{13} is the fold map mapping $(L, L \cap Z)$ onto $[0, 1/2]$, and mapping $(L', L' \cap Z)$ onto $[2, 5/2]$, p_{12} is the identification map $[a, b] \to [0, 1]$, p_{03}, p_{02}, p_{01} are all collapse maps to W_0. Then clearly $|\mathcal{G}| = \mathrm{cone}(Y_0)$.

Our knowledge of general algebraic spaces is very little. The most abvious questions is:

QUESTION 7. *Is every real algebraic space homeomorphic to a real algebraic set? If it is, then how canonical is the real algebraic set?*

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