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KNOTS AND EXOTIC SMOOTH STRUCTURES ON 4-MANIFOLDS

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Abstract

In this paper we give a short survey of exotic smooth structures of 4-manifolds. Various methods of constructing exotic structures and their relation to "Shake-Sliceness" and the Zeeman conjecture is discussed.

Keywords: 4-manifold, smooth, fake, exotic, framed link.

A fake or an exotic structure on a smooth manifold M is a smooth manifold N which is homeomorphic but not diffeomorphic to M. Dimension four is the lowest dimension where exotic smooth structures occur. By their handlebody structures, 4-manifolds can be represented with framed links and balls in S^3. Hence we can hope to "see" the exotic structures. This exciting possibility gives 4-manifolds a special significance in the smoothing theory of manifolds. The current emerging picture is that the exotic smooth structures on 4-manifolds are obtained by cutting up standard 4-manifolds along certain simple 3-manifolds and gluing them back in by different diffeomorphisims.

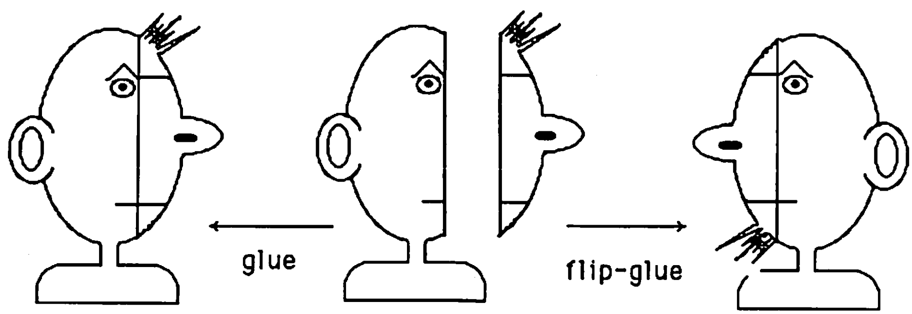

Figure 1: different pictures obtained from the same pieces by different gluing

A particular case of this is to twist 4-manifold M along an imbedded surface $F \subset M$, That is to cut open the manifold along F and glue back in by a different diffeomorphisim. When $F = S^2$ this process called a **Gluck-twist**, and when

$F = T^2$ it is called a **Logorithmic-transform**. We can also twist a manifold with boundary along a properly imbedded subsurfaces with boundary.

The first such example is the fake RP^4 of Capplell and Shaneson [8]. Recall that we can write $RP^4 = B^2\tilde{\times}RP^2 \cup B^3\tilde{\times}S^1$, where $B^2\tilde{\times}RP^2$ is a B^2-bundle over RP^2 and $B^3\tilde{\times}S^1$ is the twisted B^3-bundle over S^1, and the union is taken along the boundary. We can obtain this fake RP^4 by first twising $B^2 \times RP^2$ along a properly imbedded $D^2 \subset B^2 \times RP^2$ then taking the union with $B^3\tilde{\times}S^1$ [2]. The fake $S^3\tilde{\times}S^1\sharp S^2 \times S^2$ of [3] is obtained by Gluck-twisting a standard copy of this manifold. The non-complex fake Kummer surfaces of Gompf and Mrowka [11] is obtained from the standard Kummer Surface by the operations of Logorithmic-transform. Finally in [5] a fake smooth structure on a manifold homotopy equivalent to the punctured CP^2 is constructed; and it is shown that it can be obtained from the standard copy by cutting out a codimension zero contractible submanifold $W \subset M$ and gluing W back in by an involution of ∂W. This gives a fake smoothing of W relative to boundary. Here we will discuss this along with the related notion of "Shake sliceness" of knots in S^3. We would like thank the Middle East Technical University for hospitality and TUBITAK for their kind invitation to deliver these lectures.

1.Framed Links: Let M be a smooth 4-manifold, and fix a handlebody structure of M. By placing ourselves on the boundary S^3 of the zero handle B^4, we can "see" the attaching $S^0 \times B^3$ of 1-handles $B^1 \times B^3$ as a pair of balls. We also see the attaching $S^1 \times B^2$ of the two handles $B^2 \times B^2$ as framed links on the boundary of 4-ball with 1-handles ,i.e. in $\overset{k}{\sharp}\,(S^1 \times S^2)$. Fortunately we do not have to draw three and four handles since they are attached a unique way. This last statement follows because any self diffeomorphisim of $\overset{k}{\sharp}\,(S^1 \times S^2)$ extends to $\overset{k}{\sharp}\,(S^1 \times B^3)$ by [13]. Forexample, the following describes a 4-manifold

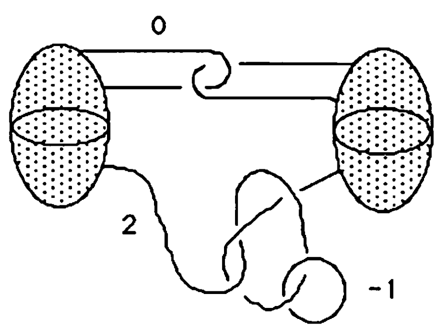

Figure 2:

The alternatively we can use the notation of [1] to represent 1-handles as "dotted" circles; i.e. going through these dotted circles corresponds to going through 1-handles. Forexample Figure 2 can be drawn as Figure 3. The framings on the link in Figure 3 are induced from S^3. A nice feature of the "dotted circle" notation for a 1-handles is that, putting a dot on an unknotted circle α is equivalent to the any of the following statements :

(i) Pushing the interior of disc D^2, which α bounds in S^3, into the interior of B^4 and removing an open tubular neighborhod of D^2 from B^4.

(ii) Attaching a two handle with 0-framing to α then surgering the imbedded S^2 with the trivial normal bundle induced from this 2-handle.

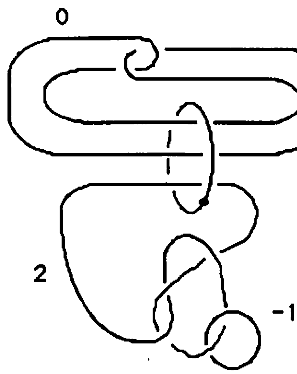

Figure 3:

Sliding one 2-handle over another 2-handle corresponds to a band connecting summing the corresponding framed knot of this 2-handle to a parallel copy (pushed off by the framing) of the framed knot corresponding the other 2-handle, as in the example of Figure 4. The framing of the new knot is uniquely determined. In literature any sequence of this 2-handle sliding operations is called " Kirby Calculus".

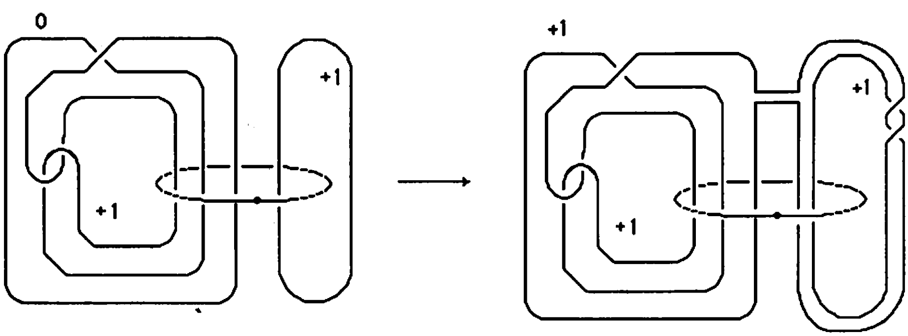

Figure 4:

If a 2-handle, corresponding a framed knot K, goes over a 1-handle a (a circle with a dot) geometrically once, we can cancell it by the 1-handle a. In the framed link notation, we perform this cancelling operation by sliding the other framed links over K until they no longer go over the 1-handle a, then simply erasing K and a. For example Figure 5 shows how to cancell +1-framed 2-handle and the 1-handle of Figure 4.

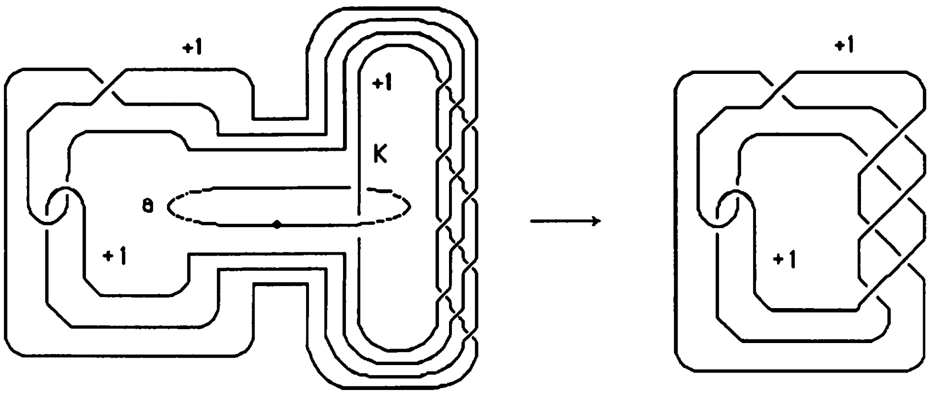

Figure 5:

Curious reader can consult [2] and [12] for a further discussion of framed links. For the rest of the paper, let us denote the 4-manifold obtained by attaching a 2-handle to B^4 along the knot $K \subset S^3$ with framing r, by K^r. We will also adapt the notation $\approx$ for a diffeomorphisim.

2.Shake Slice Knots: A first obvious attempt to find exotic smooth manifolds is to find two different knots $K, R \subset S^3$ such that there is a diffeomorphisim $f : \partial(K^r) \to \partial(R^r)$ extending to a homeomorphisim $F : K^r \to R^r$ but not extending to a diffeomorphisim $K^r \to R^r$. Then the pull-back smooth structure of K^r by F would be a fake smoothing relative to boundary. Unfortunately, it turns out that there can exist very different knots with $K^r \approx R^r$. In fact we can even choose K to be a slice knot and R to be a non-slice knot.

Theorem A ([1]) *The 4-manifolds in Figure 6 are diffeomorphic.*

Notice in Figure 6 the knot K is slice whereas R_r is not always slice (the sliceness of K can be seen by performing a slice move along the dotted line in the figure). Hence the generator of $H_2(K^r)$ is represented by a smoothly imbedded 2-sphere (namely the slice disc union the core of the 2-handle). For simplicity denote R_r by R. Therefore there is a smoothly imbedded 2-sphere $S^2 \subset R^r$ representing $H_2(R^r)$. By making S^2 transversal to the cocore of the 2-handle of R^r (i.e. the dual 2-handle) we can make $S^2 \cap \partial B^4$ to be the link

$$L = \{R, R_+, ..., R_+, R_-, ..., R_-\}$$

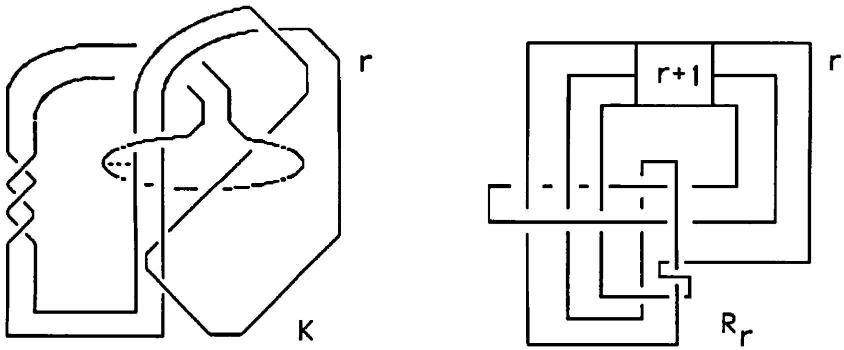

Figure 6:

where R_+ and R_- are oppositely oriented copies of R linking R r-times, and L contains equal number of R_+'s and R_-'s. The link L is affectionally called an "**r-shaking** of R". Clearly L bounds a disc with holes in B^4. We call R an **r-shake-slice** knot.

Proof: (of Theorem A) In Figure 7 if we cancell the 1-handle and the 0 framed 2-handle we obtain K^r. On the other hand if we slide the r framed 2-handle over the 1-handle as indicated by the dotted arrow in the figure, we obtain the handlebody on the right hand side of Figure 7. If we now cancell the $r-2$ framed 2-handle with the 1-handle we obtain R_r^r.

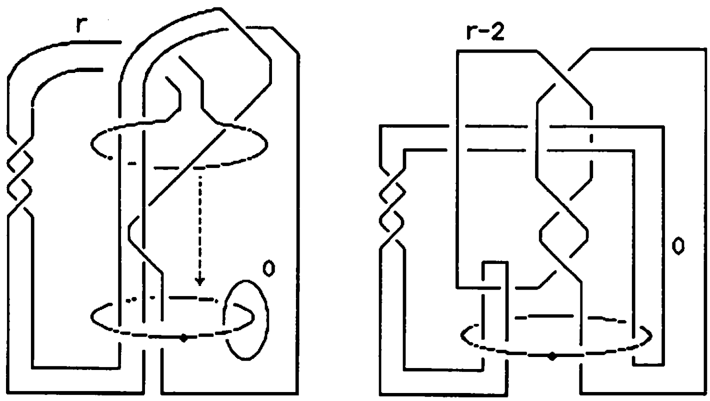

Figure 7:

3.Exotic Smooth Structures: The existance of non-slice, but r-shake slice knots at the time was a surprising; but it was a discouraging news for our hope to find fake smooth structures on 4-manifolds. Surprisingly, 13 years later the following turned out to be true:

Theorem B ([5]) *The 4-manifolds in Figure 8 are homeomorphic but not diffeomorphic to each other (even their interiors are not diffeomorphic).*

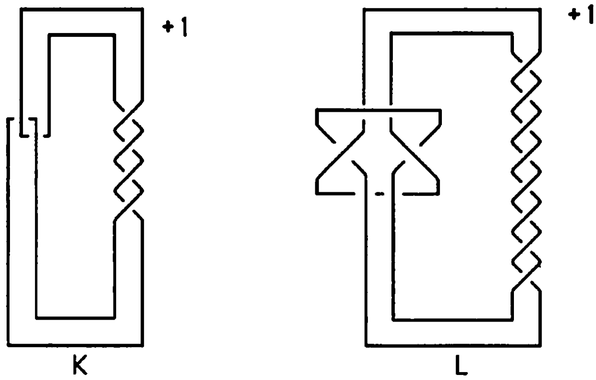

Figure 8:

Theorem 1 ([4]) *If K is the knot of Figure 8, there is no smoothly imbedded 2-sphere 2-sphere in K^{+1} representing the generator of $H_2(K^{+1}) = Z$*

Theorem 1 implies Theorem B because, L being slice, the generator of $H_2(L^{+1})$ is represented by a smoothly imbedded 2-sphere. Hence if the manifolds of Figure 8 were diffeomorphic the generator of $H_2(K^{+1})$ would be representeb by a smoothly imbedded 2-sphere

Now let W be the contractible manifold of Figure 9, and let γ be a loop in ∂W as indicated in the figure. There is an obvious diffeomorphisim (an involution) $f : \partial W \to \partial W$ obtained by first surgering $S^1 \times B^3$ to $B^2 \times S^2$ in W then surgering the other $B^2 \times S^2$ to $S^1 \times B^3$, i.e. replacing the "dot" and the 'zero" on this symetric link (e.g. [7]).

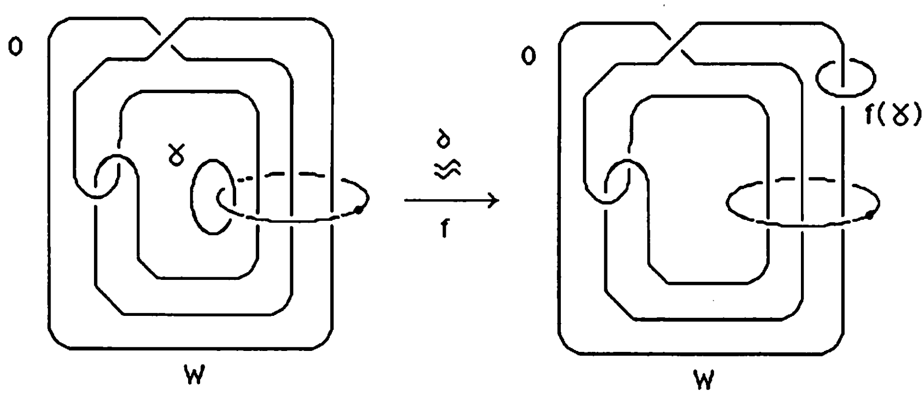

Figure 9:

We can ask whether f extends to a diffeomorphisim $W \to W$. Notice that $f(\gamma)$ is slice in W; and hence if such an extension exists γ would be slice in W. The following implies that f can not extend to a diffeomorphisim $W \to W$.

Theorem 2 ([4]) *γ can not bound a smoothly imbedded 2-disc in W.*

Proof: If γ bounds a smoothly imbedded 2-disc in W, then the 4-manifold γ^{+1}, which is obtained by attaching a 2-handle to W along γ with a +1 framing would contain an imbedded smooth 2-sphere representing the generator $H_2(\gamma^{+1}) = Z$. But attaching a 2-handle to γ and cancelling this 2-handle with the 1-handle, as in Figure 5, gives K^{+1} of Figure 8, i.e. $\gamma^{+1} \approx K^{+1}$. This contradicts Theorem 1.

From this peculiar property of f we can easily construct a fake smooth structure on W relative to the boundary:

Theorem 3 ([4]) *There is a smooth compact contractible 4-manifold V with $\partial V = \partial W$ which is homeomorphic but not diffeomorphic to W rel boundary. In other words the identity on the boundary can not extend to a diffeomorphisim $W \to V$.*

Proof: By Freedman's theorem we can always find a homeomorphisim $F : W \to W$ extending $f : \partial W \to \partial W$. Let V be the smooth structure on W pulled back by F from W. This gives a diffeomorphisim $F : V \to W$. But if there was a diffeomorphisim $G : W \to V$ extending the identity on the boundary, then $F \circ G$ would be a diffeomorphisim $W \to W$ extending f on the boundary; contradiction.

One can hope to constuct an exotic S^4 by doubling W, or by gluing two copies of W along their boundaries by f. Unfortunately these attempts fail; in both cases we get S^4, [4]. However we can use this to our advantage to solve a conjecture of Zeeman [14], namely:

Theorem 4 ([6]) *γ can not bound a P.L. imbedded 2-disc (not necessarily locally flat) in W.*

Proof: We prove this by contradiction. Assume that γ bounds a PL 2-disc D in W. This means that D is locally flat except a finitely many isolated points. By pushing these points together we can assume that there is only one nonlocally flat point x_0 on D. Hence x_0 has a 4-ball neighborhood V in W such that $V \cap D$ is a cone on a knot $R \subset \partial V$. Now we glue another copy of -W to W by f to obtain $S^4 = W \cup_f (-W)$. Then since f identifies $\gamma \subset \partial W$ with $f(\gamma) \subset \partial(-W)$ which is slice in $-W$, the knot R bounds a slice disc in the ball $S^4 - V$. Hence R is a slice knot, therefore it bounds a smooth disc in V. Hence γ is smoothly slice, contradicting Theorem 2. (Figure 10).

We now turn to the theme in begining of our survey. We claim that the two manifolds K^{+1} and L^{+1}, which are exotic copies of each other, differ from each other by a diffeomorphisim of a three manifold. More specifically they both are obtained by gluing W to another manifold H along ∂W and a component of ∂H one by identity, the other by the diffeomorphisim f (Figure 11).

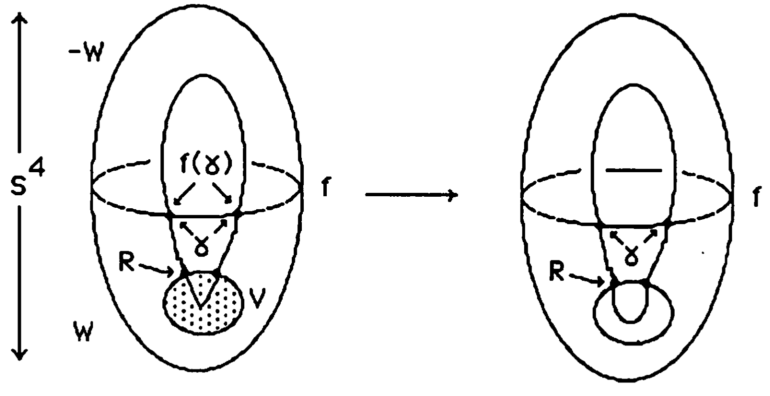

Figure 10:

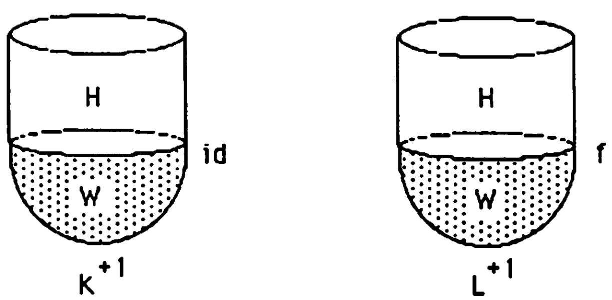

Figure 11:

To see this observe that the involution f induces a diffeomorphisim between the boundaries of the two manifolds of Figure 12, and the loop γ on the boundary is mapped to the indicated loop in the figure. Now observe that these two manifolds of Figure 10 are diffeomorphic to the manifolds K^{+1} and L^{+1}, respectively. This is done in [5]. The reader could verify this herself as follows:
(a) In the left handlebody of Figure 12 cancelling the +1-framed 2-handle and the 1-handle results K^{+1} (as in Figure 5).
(b) In the right handledody of Figure 12 sliding 0-framed 2-handle over +1-framed 2-handle, along the dotted lines as indicated in the figure, gives a 3-framed 2-handle which goes over the 1-handle geometrically only once. Then cancelling this 3-framed 2-handle with the 1-handle gives L^{+1} .

Now, in Figure 12, let $H \subset K^{+1}$ be the collar on the boundary union the tubular neighborhood of the disc in the interior which γ bounds. Then also $H \subset L^{+1}$, since it is the collar on the boundary union the tubular neighborhood of the disc in the interior which $f(\gamma)$ bounds. The complements of H are clearly W. Recall the complements are obtained by putting dots on the circles γ and $f(\gamma)$ respectively , i.e. turning them to 1-handles. In both cases the dotted circle cancell +1-framed 2-handle. Now observe that the induced diffeomorphisim on ∂W is the involution f .

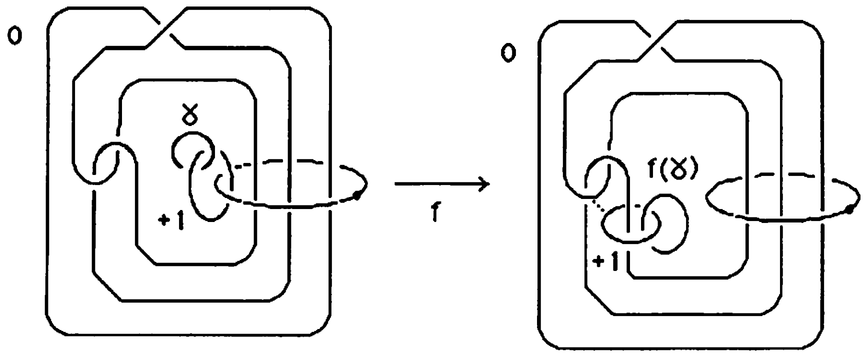

Figure 12:

Finally we refer the reader to [4] for proof of Theorem 1 which we have not discussed here. The proof uses a combination of handlebody techniques along with some Gauge theoretical results of Donaldson (cf. [8]) and a calculation of Fintushel and Stern [10]

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