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# On Algebraic Structures of Manifolds

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Moduli spaces of algebraic structures on manifolds have long been a source of curiosity. Apart from studying them to classify algebraic structures on manifolds there is always an exciting possibility of using them as functors to understand the topology of manifolds themselves as in [**D**]. Here we review the moduli spaces of nonsingular algebraic structures in an elementary way, and then discuss some related facts along with some problems and speculations. The stated propositions are joint work with with H. King.

For any nonsingular (real or complex) algebraic set $V$ let $A(V)$ be the set of nonsingular algebraic subsets of $V$, and let $A^d(V)$ to be the degree $d$ elements of $A(V)$. If $V$ is real let $A_o(V)$ denote the set of nonsingular components of algebraic subsets of $V$, and similarly let $A_o^d(V)$ be the degree $d$ elements of $A_o(V)$. If $V$ is complex define $A_{\mathbb{R}}(V)$ to be the elements of $A(V)$ defined over $\mathbb{R}$ and $A_{\mathbb{R}}^d(V) = A_{\mathbb{R}}(V) \cap A^d(V)$. We will also denote the real part of a complex algebraic set $V$ by $V_{\mathbb{R}}$, and denote the complexification of a real algebraic set $W$ by $W_{\mathbb{C}}$. Up to isomorphisim we can always assume that if $W$ is nonsingular then $W_{\mathbb{C}}$ is a nonsingular projective algebraic set [**AK6**, Lemma 2.2.15].

For a compact smooth manifold $M$ as in [**AK4**] we can define:

$$\mathcal{S}\,(M) = \left\{ (V,f)| \begin{array}{l} V \text{ is a nonsingular algebraic set,} \\ f\colon M \to V \text{ is a diffeomorphisim} \end{array} \right\} / \sim$$

where $\sim$ is the equivalence relation $(V,f) \sim (V',f')$ if there is a birational isomorphisim $\varphi\colon V \to V'$ with $\varphi \circ f = f'$. If this space is too big, the standard procedure is to let the group of diffeomorphisim Diff$(M)$ act on $\mathcal{S}(M)$ by composition on the left and take the quotient $Alg(M) = \mathcal{S}(M)/\text{Diff}(M)$. To be

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consistent with the usual definition of moduli spaces, alternatively we can first divide $\mathcal{S}(M)$ by the identity component $\text{Diff}_o(M)$ of $\text{Diff}(M)$ (this would be the analogue of the Teichmüller space) and then divide the resulting space by the mapping class group $\text{Diff}(M)/\text{Diff}_o(M)$ to get $Alg(M)$.

To topologize $Alg(M)$ we can require that $M \subset \mathbb{R}^n$ and the elements $V$ of $Alg(M)$ be nonsingular algebraic subsets of $\mathbb{R}^n$. In this case we define

$$Alg(M, \mathbb{R}^n) = \{ \, [V, f] \in Alg(M) \mid V \in A(\mathbb{R}^n) \, \}$$

where the square bracket denotes the equivalence class of $(V, f)$ in $Alg(M)$. Here we take the quotient topology induced by the space of imbeddings $M \hookrightarrow \mathbb{R}^n$. This space has many components corresponding to each isotopy class of imbedding of $M$ into $\mathbb{R}^n$. By fixing the isotopy class of an imbedding $f : M \to V \subset \mathbb{R}^n$ we can study a single component $Alg(M, \mathbb{R}^n; f)$ of this space. Hence, by identifying $f$ by the inclusion $M \subset \mathbb{R}^n$ we naturally arrive at the definitions :

$$\begin{aligned} J(M) &= \{ \, V \in A(\mathbb{R}^n) \mid V \text{ is isotopic to } M \, \}/\sim \\ J_o(M) &= \{ \, V \in A_o(\mathbb{R}^n) \mid V \text{ is isotopic to } M \, \}/\sim \\ J_{\mathbb{R}}(M) &= \{ \, V_{\mathbb{R}} \mid V \in A_{\mathbb{R}}(\mathbb{CP}^n) \; \textit{and} \; V_{\mathbb{R}} \text{ is isotopic to } M \, \}/\sim \end{aligned}$$

where $\sim$ denotes the equivalence class under birational isomorphisim. Here we identify $\mathbb{R}^n \subset \mathbb{RP}^n$. Clearly $J_{\mathbb{R}}(M) \subset J(M) \subset J_o(M)$. As above we define the degree $d$ elements of $J_{\mathbb{R}}(M)$, $J(M)$, $J_o(M)$ by $J^d_{\mathbb{R}}(M)$, $J^d(M)$, $J^d_o(M)$ respectively. For a fixed $d$ there are strong restrictions on topological types of manifolds $M$ satisfying $J^d(M) \neq \phi$, e.g. [**V1**], [**V2**]. On the other hand [**AK2**] implies that $J_o(M)$ is nonempty; and $J(M)$ is nonempty if the immersed cobordisim class of $M$ contains an algebraic representative, [**AK3**]. Also in some special cases there are elementary methods available to show that $J_{\mathbb{R}}(M)$ is nonempty, for example:

PROPOSITION 1. *If $M$ is a topological complete intersection, that is if it is an intersection $\cap L_i$ of smooth codimension one submanifolds in general position in $\mathbb{RP}^n$, then $J_{\mathbb{R}}(M) \neq \phi$.*

PROOF. First isotop each $L_i$ to a nonsingular algebraic hypersurface $V_i$ by a small isotopy (this can be done since the group $H_{n-1}(\mathbb{RP}^n; \mathbb{Z}_2)$ is algebraic, e.g. [**AK5**], then change the coefficients of the defining equations of each $V_i$ a little so that the complex solutions become nonsingular and transverse without affecting the isotopy type of $\cap V_i \approx M$. □

For example, this proposition implies that any knot or a link in $\mathbb{RP}^3$ is a real part of a nonsingular complex algebraic curve in $\mathbb{CP}^3$. In general the quotient topology makes these moduli spaces hard to understand. Existence of nonalgebraic homology classes (see below) imply that $J(M)$ is nontrivial in general, [**AK4**]. We now know by [**BK**], and [**B**] that when $M^m \subset \mathbb{R}^{2m+1}$ the set $J(M)$ is uncountable; in fact it contains an imbedded arc.

Even though $A^d(\mathbb{CP}^n)$ is connected $J^d(M)$ is not necessarily connected [**R**]. In the hypersurface case this can be visualized by taking the Verenose imbedding $\lambda : \mathbb{RP}^n \hookrightarrow \mathbb{RP}^N$ which sends a point $[x_0, .., x_n]$ to a point whose coordinates consist of all possible monomials of degree $d$ in $(x_0, .., x_n)$. Then every element $V_d \in A^d(\mathbb{RP}^n)$ corresponds to $\lambda(\mathbb{RP}^n) \cap H_d$ where $H_d$ is a linear subspace of $\mathbb{RP}^N$. Then for example the homology of $\mathbb{RP}^n$ prevents us connecting two submanifolds $\lambda(\mathbb{RP}^n) \cap H_d^i$ diffeomorphic to $M$, $i = 0, 1$ by a path of submanifolds $\lambda(\mathbb{RP}^n) \cap H_d^t$, $t \in [0, 1]$, each of which is diffeomorphic to $M$. However any two elements of $J^d(M)$ can be connected inside $J^D(M)$, for some large $D$ [**Na**].

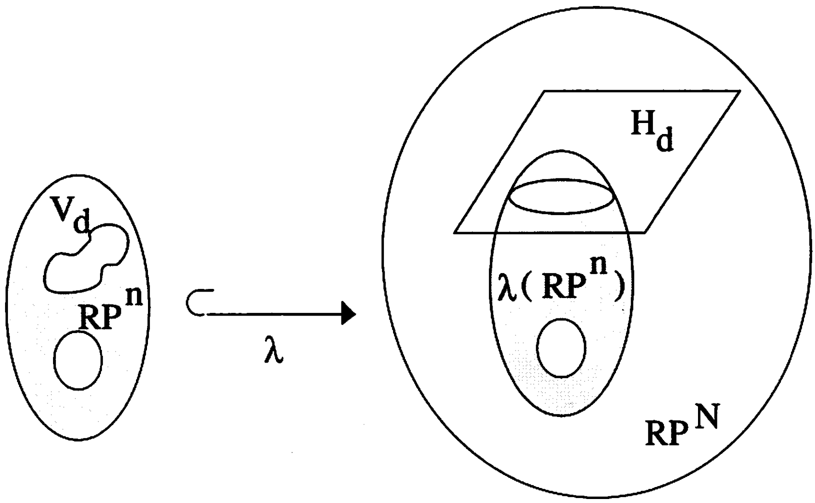


Figure 1.

Now let us review some of properties of nonsingular real algebraic sets. First of all the Grassmanian manifold has a natural real algebraic structure:

**Grassmanian:**

Recall that the Grassmann manifold $G(n, k)$ of $k$-planes in $\mathbb{R}^n$ can be identified by the nonsingular algebraic variety (cf. [**AK6**])

$$G(n, k) = \{L \in \mathcal{M}(n) \mid L^2 = L,\ L = L^t,\ \text{trace}(L) = k\}$$

where $\mathcal{M}(n)$ denote the set of $(n \times n)$-matrices with real coefficients, and the identification is given by $k$-plane $\to$ the matrix of orthogonal projection onto that plane. One of the most important properties of compact real algebraic sets is that their Gauss maps are naturally algebraic. That is if $V \subset \mathbb{R}^n$ is a compact nonsingular real algebraic set then the Gauss map $V \to G(n, k)$, where $k = dim(V)$, which assigns every point of $V$ to the tangent plane at that point, is an entire rational map. In fact more generally if $W$ is a nonsingular real algebraic set with $V \subset W \subset \mathbb{R}^n$ then the map $\alpha : V \to G(n, k)$ given by $\alpha(x) =$ the tangent plane to $W$ which is normal to $V$ at $x$ is an entire rational function (cf. [**AK6**]), where $k = dim(W) - dim(V)$.

Let us recall the proof for $W = \mathbb{R}^n$: For any $y_s \in V$ we pick a set of generators of the ideal of polynomials vanishing at $V$, $f_1, \dots, f_k \in \mathcal{I}(V)$ so that the gradients $\nabla f_i$ are linearly independent at a Zariski open set $U_s$ containing $y_s$. Let $A(x)$ be the $n \times k$ matrix whose $i$-th column is $\nabla f_i(x)$ and let $A^t(x)$ be its transpose. Then for $x \in U_s$ we have:

$$\alpha(x) = A(x)\left(A^t(x)A(x)\right)^{-1} A^t(x).$$

By Cramer's rule it is easily seen that $\alpha(x) = P_s(x)/q_s(x)$, where $P_s(x)$ is an $n \times n$ matrix with polynomial entries in $x$ and $q_s(x)$ is a polynomial not vanishing at $U_s$. We can than extend $\alpha$ to all of $M$ by covering $M$ by open sets $U_1, ...U_m$ and defining

$$\alpha = \sum_{s=1}^{m} P_s q_s / q_s^2 .$$

Notice that $\alpha$ is independent from choices of generators of $\mathcal{I}(V)$, and if $V \in A_d(\mathbb{R}^n)$ then degree($\alpha$) $= 4k(d-1)$, where $k = codim(V)$.

PROBLEM 1. By pulling back the universal connection from $G(n,k)$ by the algebraic Gauss map find the relations between the topology, curvature, and the degree of the defining equations of a nonsingular real algebraic set.

Any diffeomorphisim $f : M \to V$ to a nonsingular algebraic set $V$ induces a map between the set of entire rational maps from $V$ to $G(n,k)$ and the set of all maps from $M$ to $G(n,k)$:

$$\theta_f : \mathrm{Rat}(V, G(n,k)) \to \mathrm{Map}(M, G(n,k)).$$

Also any two representatives $(V_0, f_0)$ and $(V_1, f_1)$ of $[V, f] \in \mathcal{S}(M)$ induces an isomorphisim $\varphi : \mathrm{Rat}(V_0, G(n,k)) \to \mathrm{Rat}(V_1, G(n,k))$ such that $\theta_{f_1} \circ \varphi = \theta_{f_0}$. This defines a map:

$$\theta : \mathcal{S}(M) \to \text{Subsets of } \mathrm{Map}(M, G(n,k))$$

given by $[M, f] \mapsto \mathrm{image}(\theta_f)$. We can define this map on the level of $Alg(M)$ if we are willing the divide the target by the action of Diff($M$). The image of all rational maps may be too large or even dense; it is more natural to look at the images of degree $d$ entire rational maps, which is finite dimensional:

$$Z_d(V) = \theta_f(\mathrm{Rat}^d(V, G(n,k)).$$

PROBLEM 2. Study the topology of $Z_d(V)$ in Map($M, G(n,k)$), e.g. study the singularity and non-compactness of this object. Is there a way of associating $Z_d(V)$ a homology cycle in Map($M, G(n,k)$)? If there is how does this homology class depend on the algebraic structure $V$?

Having natural homology cycles in $\text{Map}(M, G(n,k))$ is a useful tool in obtaining invariants. For example by imitating [**D**] we can construct a bundle $\xi \longrightarrow M \times \text{Map}(M, G(n,k))$ obtained by pulling back the universal bundle by the evaluation map $ev : M \times \text{Map}(M, G(n,k)) \to G(n,k)$, $ev(x,f) = f(x)$. Then any characteristic class $c \in H^n(M \times \text{Map}(M, G(n,k)))$ of this bundle determines a map:

$$\mu : H_k(M) \to H^{n-k}(\text{Map}(M, G(n,k)))$$

given by the slant product operation $\mu(x) = c/x$. So, if we have a natural homology cycle in $H_{n-k}(\text{Map}(M, G(n,k)))$ depending only on the topology of $M$, or less ideally on the algebraic structure $V$ of $M$, we can evaluate $\mu(x)$ on this cycle and get a topological invariant of $M$, or an invariant of $V$ respectively.

This is the idea of the Donaldson invariants of four dimensional manifolds. In Donaldson's case the homology cycles are given by the subset of anti-self dual connections in the gauge equivalence classes of the space all connections on a fixed bundle over a 4-manifold $M$, which in turn homotopy equivalent to a component of $\text{Map}(M, G(n,k))$. In this context it is a curious question whether $Z_d(V)$ is related to the set of real stable bundles over $V$, [**W**]. It likely that in order to get any reasonably nontrivial result from this approach we have to involve the complexification $V_{\mathbb{C}}$. This is one of the lessons we learned from the Nash problem, [**AK2**].

Another natural property of real algebraic sets is that they enable us to define algebraic homology cycles:

**Algebraic homology**:

Recall [**AK6**], if $V$ is a Zariski open real (or complex) algebraic set and $R = \mathbb{Z}_2$ (or $R = \mathbb{Z}$ ), then we can define algebraic homology groups $H_*^A(V; R)$ to be the subgroup of $H_*(V; R)$ generated by the compact real (or complex) algebraic subsets of $V$. It is known that these groups in general do not coincide with the usual homology groups.

When $V$ is real, the resolution theorem implies that $H_*^A(V; \mathbb{Z}_2)$ can equivalently be defined to be the subgroup generated by the classes $g_*([S])$ where $g \colon S \to V$ is an entire rational function, $S$ is a compact nonsingular real algebraic set and $[S]$ is the fundamental class of $S$. Hence even when $V$ is real, we can define $H_i^A(V; \mathbb{Z})$ to be the subgroup generated by $g_*([S])$ where $g : S \to V$ is an entire rational function from an oriented compact nonsingular real algebraic set and $[S]$ is the fundamental class of $S$. We define $H_A^*(V; R)$ to be the Poincaré duals of the groups $H_*^A(V; R)$ when defined.

Finally, recall that [**BBK**], for a compact nonsingular real algebraic set $V$, $H^*_{\mathbb{C}-alg}(V; \mathbb{Z})$ is the subgroup of $H^*(V; \mathbb{Z})$ generated by the restriction of the classes of $H_A^*(V_{\mathbb{C}}; \mathbb{Z})$ by the complexification map $i : V \hookrightarrow V_{\mathbb{C}}$ (this is well defined). For convenience we define $H^*_{\mathbb{C}-alg}(V; \mathbb{Z}_2)$ to be the mod 2 reduction of $H^*_{\mathbb{C}-alg}(V; \mathbb{Z})$.

One of the nice applications of the algebraic Gauss map is that all Steifel-Whitney classes of any compact nonsingular real algebraic set $V$ are represented by algebraic subsets. This is because the tangent bundle map $\alpha : V \to G(n,m)$ is entire rational and the Steifel-Whitney classes of $G(n,m)$ are represented by algebraic subsets of $G(n,m)$ (namely the Shubert subvarieties, cf. [**AK1**]). It is well known that the Chern classes of a complex algebraic set are algebraic (e.g. [**F**]); and since $p_k(V) = (-1)^k i^* c_{2k}(V_{\mathbb{C}})$ then Pontryagin classes are in $H^*_{\mathbb{C}-alg}(V;\mathbb{Z})$. The following implies that the Ponryagin classes $p_k(V)$ are also represented by real algebraic subsets.

PROPOSITION 2. *If $V$ is a compact nonsingular real algebraic set then*

$$H^*_{\mathbb{C}-alg}(V;\mathbb{Z}_2) \subset H^*_A(V;\mathbb{Z}_2)\,.$$

PROOF. Let $V \subset V_{\mathbb{C}}$ be the nonsingular projective complexification. Let $\alpha_1 \in H^{2k}_{\mathbb{C}-alg}(V;\mathbb{Z}_2)$ be represented by the restriction of $\alpha_2 \in H^{2k}_A(V_{\mathbb{C}};\mathbb{Z}_2)$. Let $\beta_2 \in H^A_{2m}(V_{\mathbb{C}};\mathbb{Z}_2)$ be the Poincaré dual of $\alpha_2$ in $V_{\mathbb{C}}$, where $2m = 2n - 2k$ and $n$ is the complex dimension of $V_{\mathbb{C}}$. So we can represent $\beta_2$ by a complex algebraic subset of $V_{\mathbb{C}}$. In particular if $W$ denotes the underlying real algebraic structure of $V_{\mathbb{C}}$ then $\beta_2 \in H^A_{2m}(W;\mathbb{Z}_2)$. Hence $\beta_2$ is represented by $g_*([S])$, where $g : S \to V_{\mathbb{C}}$ is an entire rational function from a compact nonsingular real algebraic set and $[S]$ is the fundamental class.

We can isotop $g$ to a smooth function $g_0 : S \to V_{\mathbb{C}}$ such that it is transverse to $V$ in $V_{\mathbb{C}}$. Then by [**AK6**, Proposition 2.8.8], we can find a nonsingular algebraic set $S'$ and a rational diffeomorphisim $\pi : S' \to S$ and a rational map $g_1 : S' \to V_{\mathbb{C}}$ such that $g_o \circ \pi$ is $\epsilon$-close to $g_1$, hence $g_1$ is transverse to $V$. Clearly $h_*([T])$, where $T = g_1^{-1}(V)$ and $h : T \to V$ is the restriction of $g_1$, represent the Poincaré dual of $\alpha_1$ . Hence $\alpha_1 \in H^{2k}_A(V;\mathbb{Z}_2)$. □

*Remark*. One can show that $H^*_{\mathbb{C}-alg}(V;\mathbb{Z}) \subset H^*_A(V;\mathbb{Z})$, if $V$ is orientable. Also the subgroup of $H^*_{\mathbb{C}-alg}(V;\mathbb{Z}_2)$, consisting of restrictions of the complex algebraic cycles coming from $A_{\mathbb{R}}(V_{\mathbb{C}})$, is the the subgroup of $H^*_A(V;\mathbb{Z}_2)$ generated by cup product squares of the cycles coming from $A(V)$ (this was observed jointly with G. Mikhalkin).

Consider the map:

$$\psi : S(M) \to \text{ Subgroups of } H^*(M;\mathbb{Z}_2)$$

defined by $(V,f) \longmapsto f^* H^*_A(V;\mathbb{Z}_2)$. By [**BD**] this map is not onto in general, and it depends on the smooth structure of $M$ because by [**AK7**] we can always find a homeomorphisim $f : M \to V$ to a possibly singular algebraic set $V$ with $H^*(M,\mathbb{Z}_2) = f^* H^*_A(V;\mathbb{Z}_2)$. As in the case of $\theta$ the map $\psi$ descends to $Alg(M)$ if we divide the image by the action of $Aut(H^*(M;\mathbb{Z}_2))$.

PROBLEM 3. Obtain topological invariants of $M$ from the map $\psi$.

NOTE ADDED IN PROOF. In **[AK8]** we have shown in general that

$$\bar{H}^{2k}_{\mathbb{C}-alg}(V;\mathbb{Z}_2) = \{\alpha^2 \mid \alpha \in H^k_A(V;\mathbb{Z}_2)\}$$

where $\bar{H}^*_{\mathbb{C}-alg}(V;Z_2)$ is the subgroup of $H^*(V;Z_2)$ generated by the restrictions of cohomology classes of $V_{\mathbb{C}}$, which are the duals of complex algebraic subsets defined over $\mathbb{R}$. One of the amusing corollaries of this result is that there exists closed smooth submanifolds $M \subset \mathbb{R}^n \subset \mathbb{RP}^n$ which can not be isotoped to the real parts of nonsingular complex algebraic subsets of $\mathbb{CP}^n$. □

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