

On quotients of complex surfaces under complex conjugation

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Recall that all nonsingular complex surfaces of a fixed degree d in $\mathbb{CP}^3$ are diffeomorphic to each other, even though they might have different real structures. If $m_i(x)$, $i = 1, \dots, k$ are the list of all homogeneous monomials of degree d in variables

$$x = (x_0, x_1, x_2, x_3),$$

then the set of all complex surfaces of degree d in $\mathbb{CP}^3$ are parametrized by the nonsingular algebraic set:

$$\mathcal{Z} = \{(x, \lambda) \in \mathbb{CP}^3 \times \mathbb{CP}^{k-1} \mid \sum \lambda_i m_i(x) = 0\}.$$

The singular values Y of the projection map $\pi: Z \rightarrow \mathbb{CP}^{k-1}$, being a complex algebraic set, has codimension at least two. Hence $\mathbb{CP}^{k-1} - Y$ is path connected. So any nonsingular complex surface of degree d is in the form $Z(\lambda) = \pi^{-1}(\lambda)$ with $\lambda \in \mathbb{CP}^{k-1} - Y$, and $Z(\lambda) \approx Z(\mu)$ for $\lambda, \mu \in \mathbb{CP}^{k-1} - Y$.

The set of degree d nonsingular complex surfaces in $\mathbb{CP}^3$ defined over $\mathbb{R}$, that is surfaces $Z(\lambda)$ for $\lambda \in \mathbb{RP}^{k-1} - Y$, are invariant under complex conjugation. Let us denote the quotient of $Z(\lambda)$ under the complex conjugation by $\bar{Z}(\lambda)$, then the quotient $Z(\lambda) \rightarrow \bar{Z}(\lambda)$ is a 2-fold branched covering map. In general $\bar{Z}(\lambda)$ is not necessarily diffeomorphic to $\bar{Z}(\mu)$ for $\lambda, \mu \in \mathbb{CP}^{k-1} - Y$, since $Z(\lambda)$ and $Z(\mu)$ might have different real structures. Put another way, $\mathbb{RP}^{k-1} - Y$ may not be path connected. We can connect any two points λ, μ from different components of $\mathbb{RP}^{k-1} - Y$ by a generic path in $\mathbb{RP}^{k-1}$ intersecting Y finitely many times. Each intersection has an affect of changing the diffeomorphism type of $\bar{Z}(\lambda)$. This problem is addressed in [W]. According to [W] the diffeomorphism type of $\bar{Z}(\lambda)$ changes by the following steps and their inverses:

1. Connected summing $\bar{Z}(\lambda)$ with $-\mathbb{CP}^2$.
2. Cutting out an Euler class -4 disc bundle over S^2 from $\bar{Z}(\lambda)$ and gluing an Euler class -1 disc bundle over $\mathbb{RP}^2$ instead.

If we represent $\bar{Z}(\lambda)$ by a handlebody, that is by a framed link and 1-handles. The first step corresponds to introducing or erasing an unknotted circle with -1 framing, and the second step corresponds to replacing an unknotted circle with -4 framing with a 1- and 2-handle pair as in Figure 1 (reader can verify that both manifolds of Figure 1 have the same boundary). Notice that applying step 2 to a simply connected manifold would result a manifold with fundamental group $\mathbb{Z}_2$.

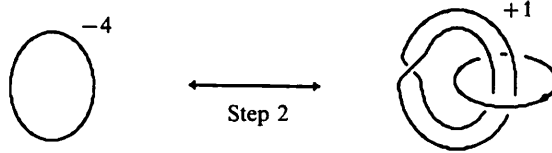


Figure 1

Hence to understand all the quotients $\bar{Z}(\mu)$ it suffices to identify the diffeomorphism type, or better yet the handlebody, of a single example $\bar{Z}(\lambda)$ for some $\lambda \in \mathbb{R}\mathbb{P}^n - Y$, then connect λ and μ by a generic path and apply the steps 1 and 2. This is similar to Kirby's theorem [K] of characterizing diffeomorphic 3-manifolds by elementary moves on their framed links, or the theorem of [AKi] characterizing rationally equivalent 3-manifolds by elementary moves on their framed links.

The purpose of this paper is to identify diffeomorphism types of some of $\bar{Z}(\lambda)$ and some other related quotients. Recall that the quotient of $\mathbb{C}\mathbb{P}^2$ under complex conjugation is S^4 , e.g. [Ku], [M]. Furthermore, handlebody structures of $Z(\lambda)$ are well understood, in fact there is an algorithm to describe the corresponding framed links, [AK]. This paper owes much to Shuguang Wang for suggesting and motivating us to prove the main result.

Let Z_{2k} be a degree $2k$ complex nonsingular surface in $\mathbb{C}\mathbb{P}^3$ in the following form:

$$Z_{2k} = \{(x_i) \in \mathbb{C}\mathbb{P}^3 \mid f(x_0, x_1, x_2) + x_3^{2k} = 0\}$$

where $f(x_0, x_1, x_2)$ is a real homogeneous polynomial of degree $2k$. The projection to the first three coordinates induces a $2k$ -fold branched covering map $Z_{2k} \rightarrow \mathbb{C}\mathbb{P}^2$ branched along the nonsingular complex curve:

$$L_{2k} = \{(x_i) \in \mathbb{C}\mathbb{P}^2 \mid f(x_0, x_1, x_2) = 0\}.$$

Now let G be the group of k th roots of unity acting holomorphically on Z_{2k} , by

$$(x_0, x_1, x_2, x_3) \rightarrow (x_0, x_1, x_2, \theta x_3),$$

$\theta \in G$, let

$$X_{2k} = Z_{2k}/G.$$

Hence X_{2k} is the 2-fold branched covering of $\mathbb{C}\mathbb{P}^2$ along L_{2k} . Our main result is: For an appropriate choice of $f(x_0, x_1, x_2)$, the quotient $\bar{X}_{2k}$ of X_{2k} under complex conjugation is diffeomorphic to a connected sum of projective planes. Notice that, our theorem applies to some K3 surfaces, since the 2-fold branched covering of $\mathbb{C}\mathbb{P}^2$ along a nonsingular curve of degree 6 is diffeomorphic the Kummer surface.

Up to deformation nonsingular real algebraic curves in $\mathbb{R}P^2$ are classified by their topological classes indicating how they separate the projective plane (e.g. [V], [Wi]). We call a complex curve L in $\mathbb{C}P^2$ is of class α if its real part $L_{\mathbb{R}}$ is of class α . Likewise we call the complex surface Z_{2k} is of class α if the corresponding L_{2k} is of class α . Let $\langle 1 \rangle^k$ denote the class consisting of k concentric circles in $\mathbb{R}P^2 - \mathbb{R}P^1$. In the standard literature the class $\langle 1 \rangle^k$ is denoted by $\langle 1 \langle 1 \langle 1 \dots \rangle \rangle \dots \rangle$, where the number of 1's are k .

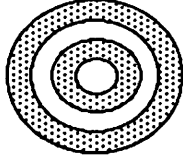


Figure 2. A curve of type $\langle 1 \rangle^4$

In Figure 2 $\mathbb{R}P^1$ lies in the outside component of all the circles. $L_{\mathbb{R}}$ separates $\mathbb{R}P^2$ into $+$ and $-$ components, W_+ and W_- respectively. They correspond to the points $f \geq 0$ and $f \leq 0$ respectively. In our case of the even degree curves, $\mathbb{R}P^1$ lies entirely in one of the components; by convention we assume that this is a component of W_- . Then the rest of W_+ and W_- are determined alternatively. For example, the shaded regions of Figure 2 are the W_+ components of $\langle 1 \rangle^k$. For real $\lambda_j, \mu_j > 0$ with $\lambda_j > \lambda_{j+1}$ and $\mu_j > \mu_{j+1}$, and for a small $\varepsilon > 0$ the following equation gives a class $\langle 1 \rangle^k$ curve in $\mathbb{C}P^2$:

$$(*) \quad f(x, y, z) = \prod_{j=1}^k (\lambda_j x^2 + \mu_j y^2 - z^2) - \varepsilon z^{2k} = 0.$$

Let $X_{2k} \rightarrow \bar{X}_{2k}$ and $\mathbb{C}P^2 \rightarrow S^4$ be the quotients by the complex conjugation. Let

$$X_{2k} \rightarrow \mathbb{C}P^2$$

be the 2-fold branched covering map along L_{2k} as above. Now notice that $\bar{X}_{2k}$ is the 2-fold branched covering of S^4 branched along the closed surface $S = L_{2k}/\sigma \cup W_+$, where σ is the complex conjugation and the union is taken along the boundaries. Notice that the inverse image of a point of W_+ consists of two complex conjugate points in X_{2k} which map to a single point in $\bar{X}_{2k}$. All these branched covering maps fit into a commuting diagram:

$$\begin{array}{ccc} & X_{2k} & \\ \swarrow & & \searrow \\ \bar{X}_{2k} & & \mathbb{C}P^2 \\ \searrow & & \swarrow \\ & S^4 & \end{array}$$

Theorem. If L_{2k} is the curve given by (*), then $\bar{X}_{2k} = p(\mathbb{C}P^2) \# q(-\mathbb{C}P^2)$, with $p + q = (2k - 1)(k - 1) + \delta_k$ where $\delta_k = 1$ or 0 if k even or odd, respectively.

Proof. Since S^4 is the quotient of $\mathbb{C}P^2$ under the complex conjugation, $\mathbb{C}P^2$ is the two fold branched cover of S^4 branched along the standardly imbedded $\mathbb{R}P^2$. This can be

seen as follows: By taking the usual handlebody decomposition we can view $\mathbb{C}P^2$ as the union of three 4-balls consisting of 0, 2, and 4 handles: $\mathbb{C}P^2 = B_0^4 \cup B_1^4 \cup B^4$

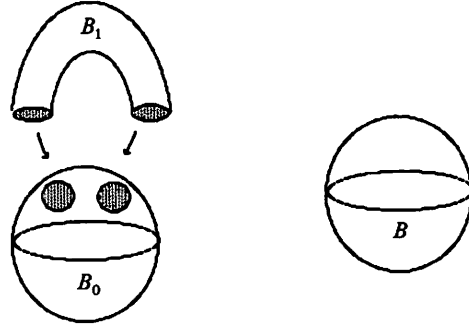


Figure 3

The balls B_0 and B_1 (i.e. the 0 and 2 handles) meet along an unknotted solid torus $S^1 \times B^2$ along their boundaries. The gluing diffeomorphism $S^1 \times B^2 \rightarrow S^1 \times B^2$ is given by the +1 framing map. The canonical $\mathbb{R}P^2 \subset \mathbb{C}P^2$ can be seen as follows: Consider the real projective plane as the union of a Möbius band M and a 2-disc glued along their boundaries $\mathbb{R}P^2 = M \cup_\partial D^2$, then M lies in the solid torus $S^1 \times B^2$ with $\partial M \subset S^1 \times S^1$ as the (2, 1) torus knot, and $D^2 \subset B$ as the properly imbedded unknotted disc.

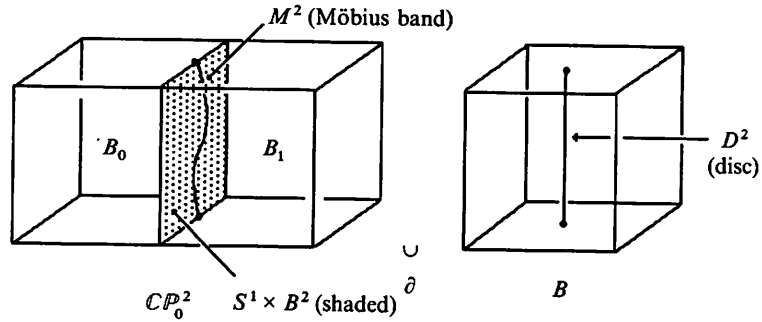
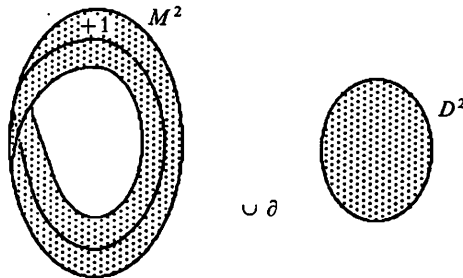


Figure 4

The complex conjugation map of $\mathbb{C}P^2$ restricts to the obvious involution map on $B_0 \cup B_1$ switching B_0 and B_1 by fixing M , and it rotates B around D^2 .

Figure 5. The actual picture of the imbedded $\mathbb{R}P^2$

The picture of the nonsingular curve L_{2k} of degree $2k$ in $\mathbb{CP}^2$ is described in [AK]. It consists of the union along their boundaries of two properly imbedded surfaces L_{2k}^+ and L_{2k}^- inside of B_0 and B_1 , respectively.

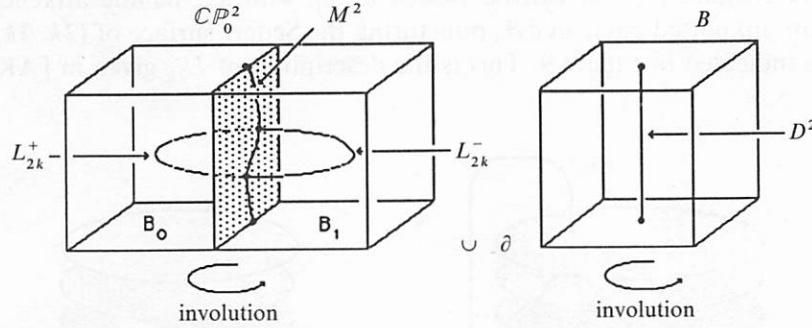


Figure 6

They are obtained by taking two copies of the unique Seifert surface (the fiber) of $(2k, k)$ -torus link in S^3 ($2k$ discs connected by k bands) as in Figure 7 and pushing their interiors into the 4-balls B_0 and B_1 , respectively.

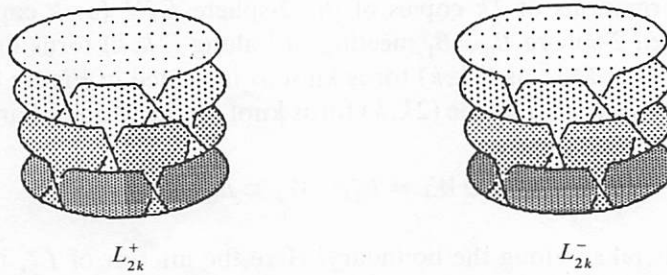


Figure 7

The involution swaps L_{2k}^- and L_{2k}^+ . Here the $(2k, k)$ torus link lies on the Möbius band M , it appears as k parallel copies of the boundary as in Figure 8 (in the figure $k = 2$). This is because, by assumption the curve L_{2k} is of class $\langle 1 \rangle^k$, i.e. in $\mathbb{RP}^2 - \mathbb{RP}^1$ it should consist of k concentric circles.

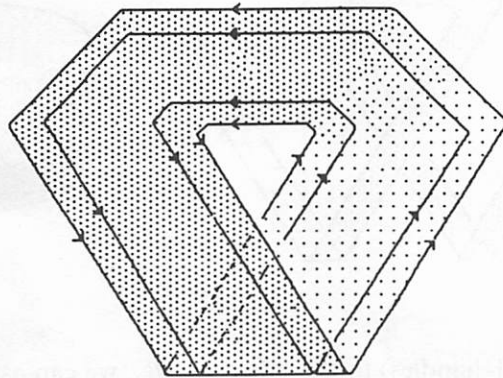


Figure 8

The description of L_{2k} in [AK] looks slightly different. To see that description is equivalent to above description we proceed as follows: By pushing the 1-handles of the surface L_{2k}^- to B_0 from B_1 (or just making the ball B_1 smaller) we can turn $L_{2k} \cap B_0$ to be the Seifert surface of $(2k, 2k)$ torus link, and $L_{2k} \cap B_1$ to $2k$ disjoint trivial discs (Figure 9). In this description $B_0 \cap B_1$ can be viewed as B_0 with a 2-handle attached with $+1$ framing to an unknotted circle in ∂B_0 puncturing the Seifert surface of $(2k, 2k)$ torus link $2k$ times, as indicated in Figure 9. This is the description of L_{2k} given in [AK].

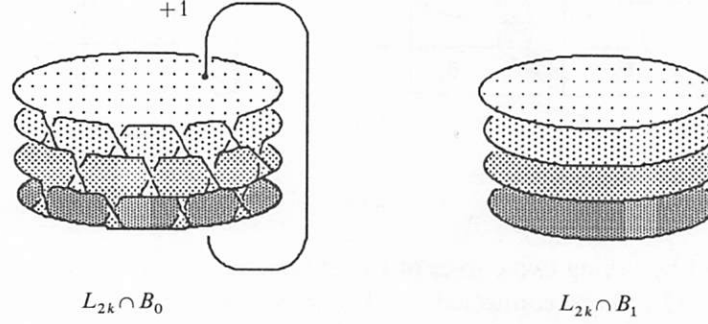


Figure 9

Another way of visualizing $L_{2k} \subset \mathbb{C}P^2$ is as follows: Think of the equation (*) of L_{2k} as small deformations of $2k$ copies of the 2-sphere $\mathbb{C}P^1$ (or k copies of the conic) inside the thickened 2-sphere $B_0 \cup B_1$ meeting M^2 along $(2k, k)$ torus link. L_{2k} intersects the Möbius band $M \subset \mathbb{R}P^2$ as $(2k, k)$ torus knot as indicated in Figure 8. W_+ appears as the parallel regions separated by the $(2k, k)$ torus knot on the Möbius band M (Figure 10). Hence

$$S = L_{2k}/\sigma \cup W_+ = L_{2k}^+ \cup W_+ \subset B_0 \cup B/\sigma = S^4$$

where the union is taken along the boundary. Here the interior of L_{2k}^+ is pushed into B_0 and the interior of W_+ is pushed into the other 4-ball B/σ , and these surfaces meet along their boundary $(2k, k)$ torus knot in S^3 (Figure 10).

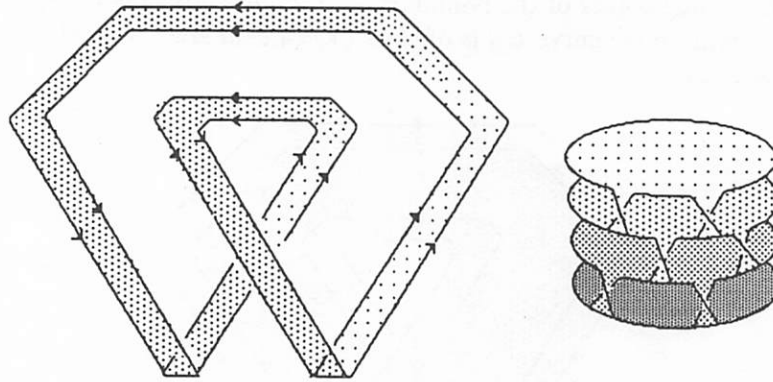


Figure 10

By pushing part of W_+ (1-handles) to the other side B_0 , we can assume that W_+ are discs and L_{2k}^+ is a Seifert surface of the unlink.

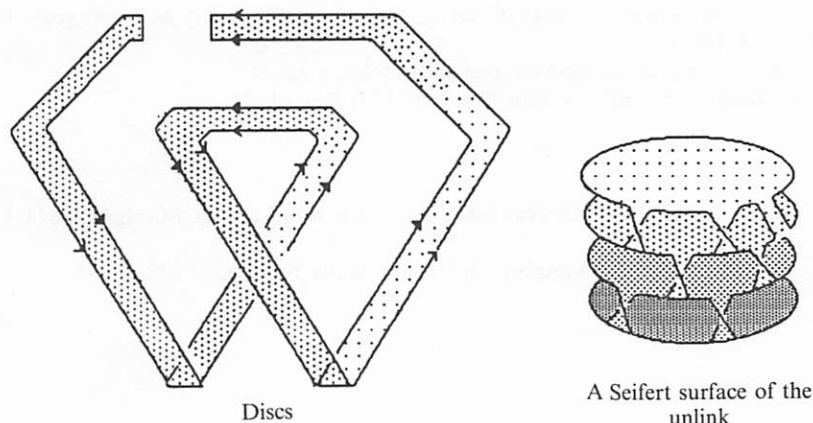


Figure 11

Hence by Theorem 4.1 of [AK], or by [L] the branched cover of S^4 branched along S is a connected sum of projective spaces as described in the statement of the theorem.

Notice that when k is odd one of the discs of W_+ is not visible in Figure 11 (the figure is drawn in case $k = 2$). This corresponds to drawing odd number of parallel circles in Figure 10, and after they are pairwise connected to each other the last remaining circle is the one whose bounding disc lies in the 4-ball B/σ and which is not visible in the figure. $\square$

Remark 1. Note that we have not shown the imbedding $L_{2k} \subset \mathbb{CP}^2$ described above corresponds exactly the curve given by the solution of (*). We have only shown that this L_{2k} is an ε -isotopic copy of (*) (since they are both small deformations $2k$ copies $\mathbb{CP}^1$), and it is also invariant under the complex conjugation. Hence its quotient S in S^4 is isotopic to the quotient of (*) by complex conjugation. Since the branched cover of S^4 along S , and along an isotopic copy of S gives diffeomorphic manifolds, it suffices to use this L_{2k} .

Remark 2. In the statement of the theorem we have not specified the values for p and q . By [W] the value of p is $b_2^+(\bar{X}_{2k}) = 1 + k(k-3)/2$. We can also calculate p by counting the number of appropriate half-twisted 1-handles of the Seifert surface L_{2k}^+ (see [AK]).

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