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On Algebraic Vector Bundles

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ABSTRACT. We discuss elementary properties of strongly algebraic vector bundles, and give a proof that the classifying map of such bundles to Grassmann variety could be algebraically deformed to an algebraic imbedding.

1. Introduction

Let X be a compact nonsingular real algebraic set. Recall that an $\mathbb{R}^k$ vector bundle $E \to X$ is called a strongly algebraic if E is a nonsingular real algebraic set containing X as the zero section. Let us review some of the useful properties of totally algebraic bundles: Perhaps, the most important totally algebraic vector bundle is the universal bundle over the Grassmanian variety of k planes in $\mathbb{R}^n$:

$$\begin{array}{ccl} E(k,n) & = & \{(A,v) \in \mathcal{M}_{\mathbb{R}}(n) \times \mathbb{R}^n \mid A \in G(k,n),\ Av = v\ \} \\ \downarrow & & \\ G(k,n) & = & \{A \in \mathcal{M}_{\mathbb{R}}(n) \mid A^t = A,\ A^2 = A,\ \text{trace}(A) = k\ \} \end{array}$$

Here $\mathcal{M}_{\mathbb{R}}(n)$ denotes $n \times n$ real matrices $\cong \mathbb{R}^{n^2}$. A useful property of a totally algebraic vector bundle $E \to X$ is that its classifying (Gauss) map

$$\rho_E : X \to G(k,n)$$

(for some large n) is an entire rational map, (cf [**1**]). Conversely any entire rational map from a nonsingular real algebraic set X to $G(k,n)$ induces a totally algebraic bundle over X by pulling back $E(k,n)$. Hence a vector bundle $E \to X$ over a nonsingular algebraic set is isomorphic to a totally algebraic vector bundle if and and only if its classifying map is homotopic to an entire rational map.

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Another useful property of a totally algebraic bundle $E \to X$ is that generically any smooth manifold $L \subset X$ representing the Euler class, can be approximated by a nonsingular algebraic subset of X. This is because a transverse section $s : X \to E$ corresponds to a map $\sigma : X \to E(k,n)$, where $\sigma(x) = (\rho_E(x), f(x))$ and $f : X \to \mathbb{R}^n$ is a smooth map with $\rho_E(x)f(x) = f(x)$. We can obtain a rational approximation $\tilde{\sigma}$ of σ by first approximating f by a polynomial map F and letting $\tilde{\sigma}(x) = (\rho_E(x), \rho_E(x)F(x))$. Then $\tilde{\sigma}^{-1}(E(k,n))$ approximates L.

The following Theorem 1.1 is due to M. Buchner who produced a proof at this conference by using resolution of singularities. Here we offer a direct elementary proof as a corollary to an observation on imbedding Grassmannianns:

THEOREM 1.1. *Let $E \to X$ be a totally algebraic $\mathbb{R}^k$-bundle over a compact nonsingular algebraic set. Then for some large n we can find an entire rational map $\rho : X \to G(k,n)$ classifying this bundle, which is an imbedding onto its image.*

We noticed that Theorem 1.1 could be proven easily because of the following elementary result. One can imbed the product of two Grassmannians in another via the tensor product. When $k = \ell = 1$, this is the well known Segre imbedding. The reason this helps is that it gives many linearly independent algebraic sections of the normal bundle of one Grassmanian in another. These sections can then be used to approximate a rational map to the Grassmanian by an imbedding.

THEOREM 1.2. *There is an entire rational function*

$$\beta_{k\ell mn} \colon G(k,n) \times G(\ell,m) \to G(k\ell, mn)$$

which is an isomorphism onto its image. This map is given by tensor product. If $k = 1$ then there is a $q \in G(1,n)$ so that $\beta_{1\ell mn}$ restricted to $q \times G(\ell,m)$ is the usual inclusion $G(\ell,m) \to G(\ell,mn)$.

PROOF. As notation let A_{ij} denote entries of a matrix A. Let A be an $n \times n$ matrix and let B be an $m \times m$ matrix. Then we let $A \otimes B$ denote the $mn \times mn$ matrix whose entry in the $(im+j)$-th row and $(sm+t)$-th column is $A_{is}B_{jt}$ (for convenience, we start indexing with 0 rather than 1).

$$A \otimes B = \begin{pmatrix} A_{00}B & A_{01}B & \dots & \dots & \dots & A_{0,n-1}B \\ A_{10}B & A_{11}B & \dots & \dots & \dots & A_{1,n-1}B \\ \vdots & \vdots & \ddots & & & \vdots \\ \vdots & \vdots & & A_{is}B & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ A_{n-1,0}B & A_{n-1,1}B & \dots & \dots & \dots & A_{n-1,n-1}B \end{pmatrix}.$$

Define $\beta_{k\ell mn}(K,L) = K \otimes L$. Let $V \subset G(k\ell, mn)$ be the subvariety given by the points $M \in G(k\ell, mn)$ satisfying the equations:

$$M_{im+j,sm+t}M_{i'm+j',s'm+t'} = M_{i'm+j,s'm+t}M_{im+j',sm+t'}$$

$$\ell \sum_{t=0}^{m-1} M_{im+j,im+t}M_{im+t,im+t'} = \left(\sum_{t=0}^{m-1} M_{im+t,im+t}\right) M_{im+j,im+t'}$$

for all $0 \leq j, j', t, t' < m, \quad 0 \leq i, i', s, s' < n$.

In other words, V is the set of matrices M in $G(k\ell, mn)$ which when written in block form $(M(is))$ of $m \times m$ matrices $M(is)$, then all the $M(is)$ are multiples of each other and furthermore, $\ell M(ii)^2 = (trM(ii))M(ii)$ (here tr means trace).

We claim that $\beta_{k\ell mn}$ is an isomorphism to V. First of all, the image of $\beta_{k\ell mn}$ is contained in V since if $M = K \otimes L$ then M is symmetric, $trM = (trK)(trL)$, $M^2 = K^2 \otimes L^2 = K \otimes L = M$. Furthermore each $M(is)$ is a multiple of of L, in particular $M(ii) = K_{ii}L$ hence:

$$\ell M(ii)^2 = \ell K_{ii}^2 L^2 = (\ell K_{ii})K_{ii}L = (trM(ii))M(ii)$$

Now let us see that $\beta_{k\ell mn}$ is onto V and compute its inverse and see that it is rational. To see this, take any matrix M in V. Since $trM = k\ell \neq 0$ there is some i'' and j'' so that $M_{i''m+j'',i''m+j''} \neq 0$. Then

$$M_{i'm+j',s'm+t'} = M_{i'm+j'',s'm+j''}M_{i''m+j',i''m+t'}/M_{i''m+j'',i''m+j''}$$

Let L' be the $m \times m$ matrix with (i,j)-th entry $M_{i''m+i,i''m+j}/M_{i''m+j'',i''m+j''}$ and let K' be the $n \times n$ matrix with (i,j)-th entry $M_{im+j'',jm+j''}$. Then

$$M = K' \otimes L'$$

Note that $k\ell = trM = trK'trL'$ so K' and L' both have nonzero trace. Let $L = \ell L'/trL'$ and $K = kK'/trK'$, then we have $M = K \otimes L$. Note that K and L are both rational functions of M. We claim that $K \in G(k,n)$ and $L \in G(\ell, m)$. Note that K' and L' are symmetric, so K and L are symmetric also. But the second equations for V tell us that $\ell(K_{i''i''}L)^2 = (trK_{i''i''}L)K_{i''i''}L$, so $\ell K_{i''i''}^2 L^2 = \ell K_{i''i''}^2 L$ and thus $L^2 = L$. But we have $M^2 = M$ so $K \otimes L = (K \otimes L)^2 = K^2 \otimes L^2 = K^2 \otimes L$. Thus we have $K = K^2$ since $L \neq 0$. So $\beta_{k\ell mn}$ has a right inverse:

$$\alpha \colon V \to G(k,n) \times G(\ell, m)$$

But α is a left inverse also since if $\beta_{k\ell mn}(K,L) = \beta_{k\ell mn}(K', L')$ we have $K \otimes L = K' \otimes L'$ so there is a number b so that $K' = bK$ and $L' = (1/b)L$. But $k = trK' = btrK = bk$ so $b = 1$.

Notice if $\ell = 1$ we may take q to be the $n \times n$ matrix which is zero everywhere except the entry in the zeroeth row and zeroeth column where it is 1. Then $q \otimes L$ is the matrix which is zero everywhere except the first $m \times m$ block where it is L. This is the usual inclusion $G(\ell, m) \subset G(\ell, mn)$ $\square$

PROOF. (of Theorem 1.1) Let $\pi : E \to X$ be a strongly algebraic vector bundle, and $\rho\colon X \to G(\ell, m)$ be an entire rational map classifying this bundle. Now let $\gamma\colon X \to G(1,n)$ be a regular map whose image is a projectively closed algebraic subset, i.e., its image is a nonsingular algebraic variety and is contained in an affine chart (see [**1**], Theorem 2.5.13). Consider the map $\sigma\colon X \to G(k, mn)$ given by

$$\sigma(x) = \beta_{1\ell mn}(\gamma(x), \rho(x))$$

Then σ also classifies the bundle since it is homotopic to the map:

$$x \mapsto \beta_{1\ell mn}(q, \rho(x)) = \iota\rho(x)$$

where $\iota\colon G(\ell, m) \to G(\ell, mn)$ is the usual inclusion. Clearly σ is an algebraic imbedding onto its image $\square$

REFERENCES

1. Akbulut and H.King, *Topology of real algebraic sets*, MSRI book series no. 25, Springer-Verlag (1992).
2. ———, *The topology of real algebraic sets with isolated singularities*, Annals of Math. 113 (1981) 425-446.
3. Benedetti and R.Tognoli, *On real algebraic vector bundles,*, Bull. des Sci. Math. 2 serie' 104 (1980) 89-112.

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