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A TOPOLOGICAL CHARACTERIZATION OF REAL ALGEBRAIC VARIETIES

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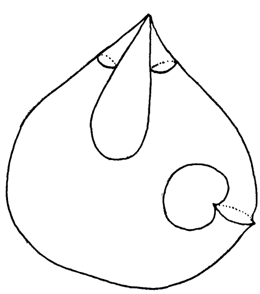
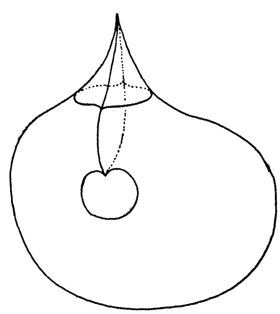
We show that if a smooth locally conelike stratified set admits a certain kind of topological resolution then it is homeomorphic to a real algebraic set, i.e. zeros of polynomial functions (this generalizes [AK_1], [AK_2]). We expect that the algebraic resolution of singularities [H] implies that every algebraic set admits such a topological resolution, hence it is reasonable to suspect that we have a complete topological characterization of real algebraic sets.

Examples of some stratified sets admitting such a topological resolution are spaces which we call A_k-spaces, $k = 0, 1, 2, \cdots$. We define A_k-spaces inductively by saying that A_0-spaces are smooth compact manifolds, and an A_k-space is a compact smooth stratified set X with a trivialization of a neighborhood of each stratum X_i, h_i: $X_i \times \operatorname{cone}(\Sigma_i) \longrightarrow X$ where Σ_i is an A_{k-1}-space which bounds a compact A_{k-1}-space with boundary (h_i required to be compatible with the trivializations of neighborhoods of the strata of Σ_i).



an A_1-space



an A_2-space

The topological resolution of an A_k-space X is obtained by a sequence of 'blow ups' as follows: take a lowest dimensional stratum X_i (the 'center' of the 'blow ups') with trivialization h_i: $X_i \times \operatorname{cone}(\Sigma_i) \longrightarrow X$ and replace $h_i(X_i \times \operatorname{cone}(\Sigma_i))$ by $X_i \times W_i$ where W_i is a compact A_{k-1}-space which Σ_i

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bounds. We have a map from the new space to X which is the identity outside image (h_i) and which collapses $X_i \times S_i$ to $X_i \times *$ where S_i is a spine of W_i and $*$ is the vertex of cone(Σ_i). After a finite number of such blow ups we obtain a smooth manifold $\widetilde{X}$ and a 'resolution' $\widetilde{X} = Z_n \longrightarrow Z_{n-1} \longrightarrow \cdots \longrightarrow Z_0 = X$. We say X is an A-space if it is an A_k-space for some k. In particular we prove:

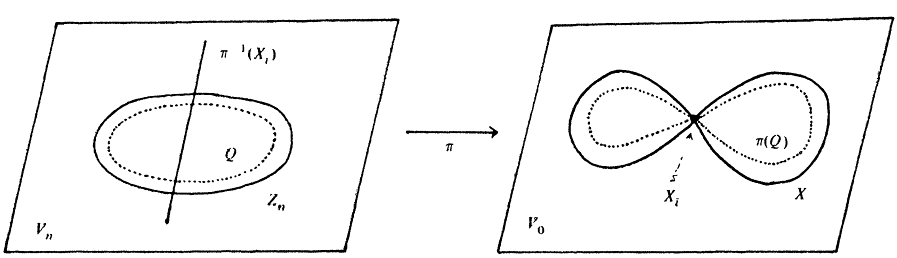
THEOREM. *The interior of any compact A-space is homeomorphic to a real algebraic set. Furthermore the natural stratification on this algebraic set coincides with the stratification of the A-structure.*

One of the reasons A-spaces are of interest is that Akbulut and Taylor have shown that any compact P.L. manifold has the structure of an A-space **[AT]**

COROLLARY. *The interior of any compact P.L. manifold is P.L homeomorphic to a real algebraic set.*

SKETCH OF PROOF. Take a resolution of a compact A_k-space X: $Z_n \longrightarrow Z_{n-1} \longrightarrow \cdots \longrightarrow Z_0 = X$ where Z_n is a smooth compact manifold. Each $Z_{i+1} \longrightarrow Z_i$ has a certain 'center' a smooth manifold $X_i \subset Z_i$ along which a topological 'blow up' occurs. We construct a tower of nonsingular varieties $V_n \longrightarrow V_{n-1} \longrightarrow \cdots \longrightarrow V_0 = \mathbf{R}^n$ with $X_i \subset V_i$ as a nonsingular subvariety and imbeddings $Z_i \subset V_i$ which commute with projections (i.e. all the above maps given by arrows) and are in some sense 'stable' over the projections.

This means that if Z_n is moved by a small isotopy in V_n the image of Z_n under the composite projection π: $V_n \longrightarrow V_0$ is isotopic to X. We then approximate the submanifold $Z_n \subset V_n$ by an algebraic set Q; and then 'blow down' Q algebraically to an algebraic set V which is homeomorphic to $\pi(Q)$, which is in turn homeomorphic to X.



Each V_{i+1} and the projection $V_{i+1} \xrightarrow{\pi_i} V_i$ is obtained by a certain algebraic 'multiblowing-up' process from V_i along X_i. X_i is a lowest dimensional stratum of Z_i. Given $Z_i \subset V_i$ the imbedding $Z_{i+1} \subset V_{i+1}$ is obtained roughly as follows: Let h_i: $X_i \times \text{cone}(\Sigma_i) \longrightarrow Z_i$ be the neighborhood trivialization of X_i, and W_i a compact A_k-space which Σ_i bounds; imbed $X_i \times W_i$ into V_{i+1} so that

(a) $X_i \times W_i$ is transverse to $\pi_i^{-1}(X_i)$,

(b) $\pi_i^{-1}(X_i) \cap (X_i \times W_i) = X_i \times$ (a spine of W_i),

(c) $\pi_i(X_i \times W_i) \approx h_i(X_i \times \operatorname{cone}(\Sigma_i))$.

Then extend this imbedding to an imbedding of Z_{i+1} into V_{i+1} by simply lifting the imbedding Z_i - image(h_i) ($\approx Z_{i+1} - X_i \times W_i$) to V_{i+1} via π_i so that $\pi_i(Z_{i+1}) \approx Z_i$. In particular, along the way we prove that the A_k-space Σ_i, which bounds, necessarily has to bound an A_k-space W_i, which has a spine consisting of transversally intersecting codimension one A_k-subspaces without boundaries. Choosing such W_i's enables us to show (a), (b), (c). Details are long and geometric in nature, they will appear in [$\mathbf{AK_3}$]. The proof applies to spaces more general than A-spaces, which leads us to believe that a satisfactory topological classification theorem for real algebraic sets is within reach.

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