Math. Proc. Camb. Phil. Soc. (1977), **82**, 99
Printed in Great Britain

On 2-dimensional homology classes of 4-manifolds

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(*Received* 8 *November* 1976)

Let K^r be a framed knot in S^3 representing the 4-manifold which is obtained by attaching a 2-handle onto B^4 along K with the framing r (see (1)). Tristram (2) proved the following non-imbedding theorem for the generator of $H_2(K^r; \mathbb{Z}) \cong \mathbb{Z}$.

Theorem (*Tristram*). If the p-signature $\sigma_p(K)$ of K is non-zero for a prime number p dividing r, then the generator of $H_2(K^r)$ cannot be represented by a smoothly imbedded 2-sphere.

A sketch of the proof. Suppose the generator of $H_2(K^r)$ is represented by a smoothly imbedded 2-sphere $S^2 \subset K^r$. Then by making S^2 transversal to the cocore of the 2-handle of K^r (i.e. the dual 2-handle) we can make $S^2 \cap \partial B^4$ to be the link

$$L = \{K, \underbrace{K_+, \ldots, K_+}_{m}, \underbrace{K_-, \ldots, K_-}_{m}\},$$

where K_+ and K_- are oppositely oriented parallel copies of K linking K r-times. Since the link L bounds a disc with holes in B^4: $|\sigma_p(L)| + n_p(L) \leqslant \mu(L)$ ((2), p. 253) where $n_p(L)$, $\mu(L)$ are the p-nullity and the number of components of L. But $\sigma_p(L) = \sigma_p(K)$ and $n_p(L) = n_p(K) + 2m = 1 + 2m$ for $p|r$ ((2), p. 260). Hence $\sigma_p(K) = 0$ for $p|r$.

We would like to complement this theorem with the following theorem:

THEOREM. *There exist knots K, J such that:*

(i) $\sigma_p(K) \neq 0$ *for all primes p, yet the generator of $H_2(K^{+1})$ is represented by a smoothly imbedded S^2.*

(ii) $\sigma_p(J) \neq 0$ *for $p = 3$, yet the generator of $H_2(J^{+2})$ is represented by a smoothly imbedded S^2.*

As a corollary to the proof of the Theorem we will get:

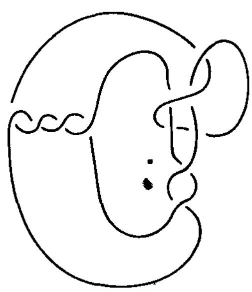
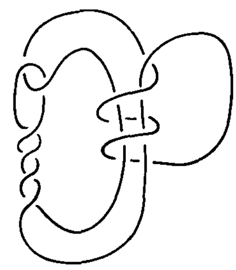
PROPOSITION. *There are non-slice knots K, J and a slice knot R such that $K^{+1} \approx R^{+1}$, and $J^{+2} \approx R^{+2}$.*

Remark 1. In (ii) we cannot hope to make $\sigma_p(J) \neq 0$ for all odd primes, in the view of Tristram's theorem and the fact that $\lim_{p\to\infty} \sigma_p(J) = \sigma_2(J)$ (see (2), p. 258).

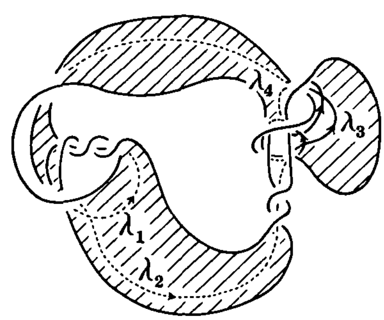
Remark 2. Let M^4 be a 4-manifold and F^2 be a homology class of M^4 of self intersection ± 1 which is represented by a P.L. imbedded 2-sphere. Then the theorem asserts that the knot cobordism invariants of the link of the singularity of F^2 in M^4 cannot possibly be an obstruction to representing F^2 by a smoothly imbedded 2-sphere.

For the rest of the paper we adapt the terminology of (1). We use the notation $\approx$ for diffeomorphisms, and $\underset{\approx}{\partial}$ for diffeomorphisms between the boundaries of manifolds. We denote a 1-handle ($= S^1 \times B^3$) by putting a dot on a circle; this means that we first attach a 2-handle with 0-framing onto B^4 along the unknot and get $B^2 \times S^2$, then surger S^2 from this manifold. Another convenient way of picturing this is by pushing the interior of the spanning disc D^2 of the unknot in S^3 into B^4, and by removing an open tubular neighbourhood of D^2 from B^4. We would like to thank R. Kirby for many helpful discussions on this subject.

Proof of the Theorem.

Let $K =$  $J =$ 

One can compute the signature of K by choosing a suitable Seifert surface, e.g.



Then the Seifert matrix of $K = L = (\lambda_{ij})$, where $\lambda_{ij} =$ linking number $(\lambda_i, i^*\lambda_j)$, and λ_i's are curves generating the homology of the Seifert surface, and i is a normal vector field to the Seifert surface. We get

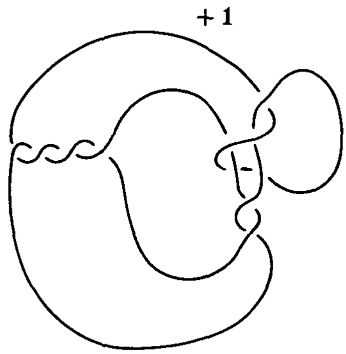
$$L = \left(\begin{array}{cc|cc} -2 & 0 & & \\ -1 & 0 & -1 & \\ \hline & & 0 & 1 \\ & -1 & 0 & -1 \end{array}\right)$$

then $\sigma_p(K) =$ signature $\frac{1}{2}(1-\bar{\omega}_p)(L - \omega_p L^T)$ (see (2)) where $\omega_p = \exp\{2m\pi i/(2m+1)\}$ for odd primes $p = 2m+1$, and $\omega_2 = -1$. An easy computation gives:

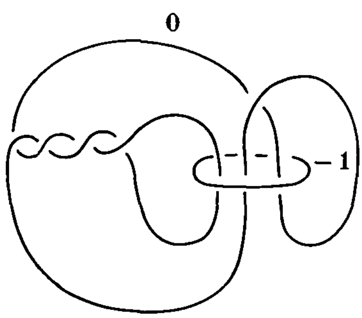
$$\sigma_p(K) = -1 + \text{Sign of } (\tfrac{1}{4} + Re\,\omega_p(1-\omega_p) - Re\,(1-\omega_p)).$$

Hence $\sigma_p(K) = -2\ \forall p$. Similarly one can show $\sigma_3(J) = -2$. Recall 'Blow up'

K^{+1} =



$\overset{\partial}{\approx}$ Blow up



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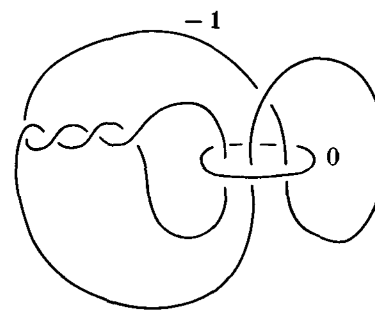
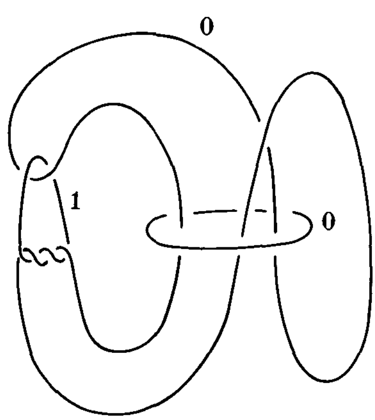


$\overset{\partial}{\approx}$ Blow up



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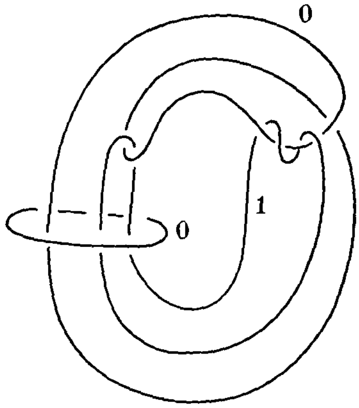

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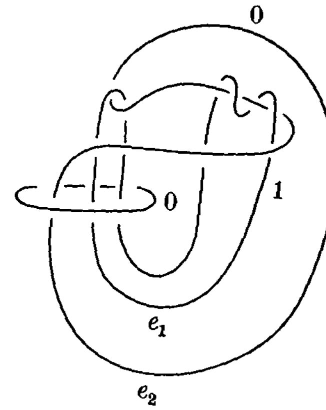
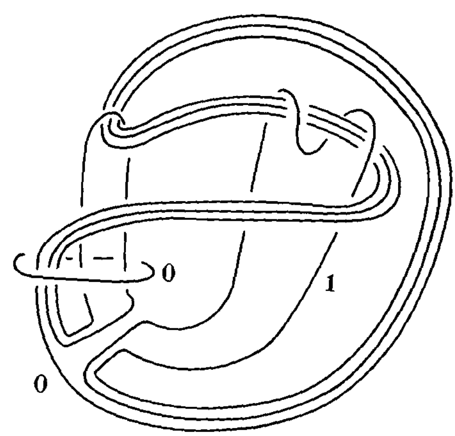


$\approx$
handle addition

$$\left\{\begin{array}{l} e_1 \longrightarrow e_1 + e_2 - e_2 \\ e_2 \longrightarrow e_2 \end{array}\right\}$$



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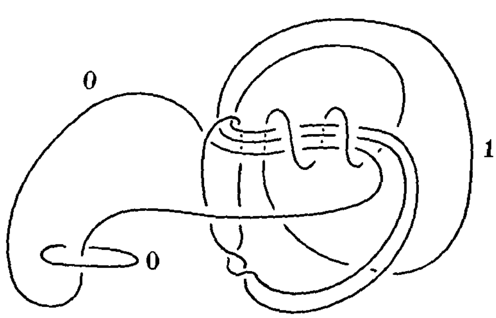
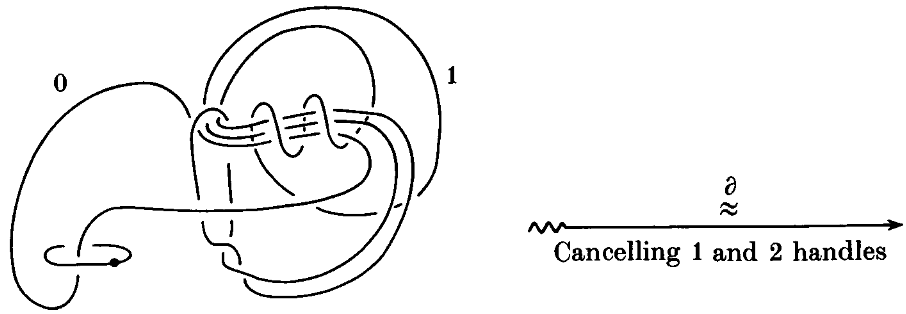
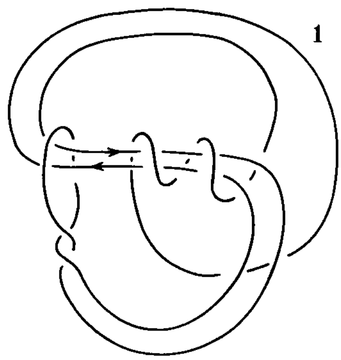
$\overset{\partial}{\approx}$
Surgering 2-handle

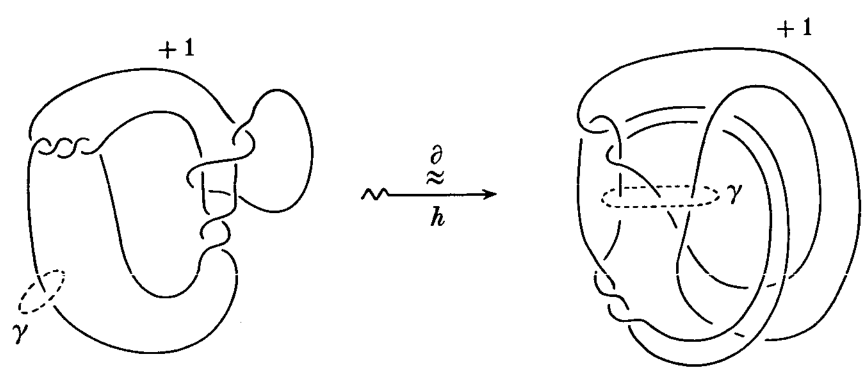




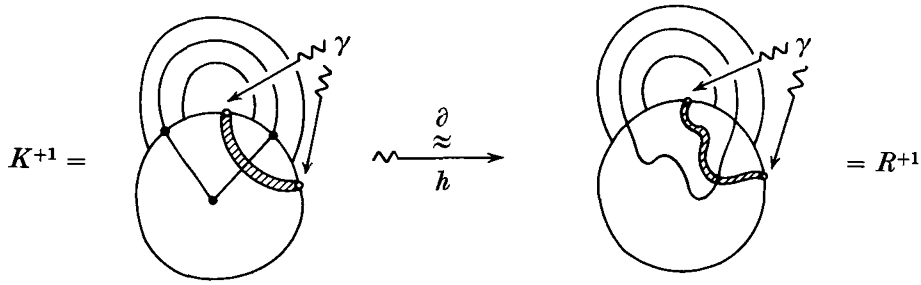


$= R^{+1}$ (definition)

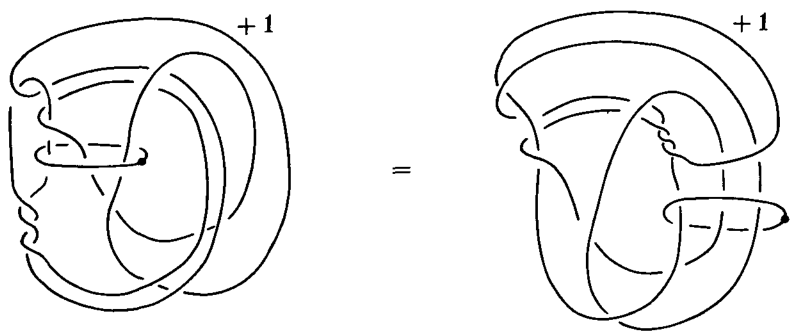
means that (1) to introduce an unknotted $+1(-1)$ framed circle and put a right (left) twist to all strands that go through the circle and increase (decrease) the framing of any framed knot, which links this circle k times, by k^2. We have constructed a diffeomorphism $h: \partial(K^{+1}) \to \partial(R^{+1})$. Since R is a slice knot, the generator of $H_2(R^{+1}; \mathbb{Z})$ is represented by a smoothly imbedded S^2, namely the slice disc in $B^4 \cup$ core of the handle of R^{+1}. Therefore, to conclude (i), it is enough to show that h extends to a diffeomorphism $K^{+1} \approx R^{+1}$. To see this, observe that h takes the boundary of the dual 2-handle of K^{+1} (denoted γ in the picture) to a loop (which we also denote by γ) in $\partial(R^{+1})$. We can

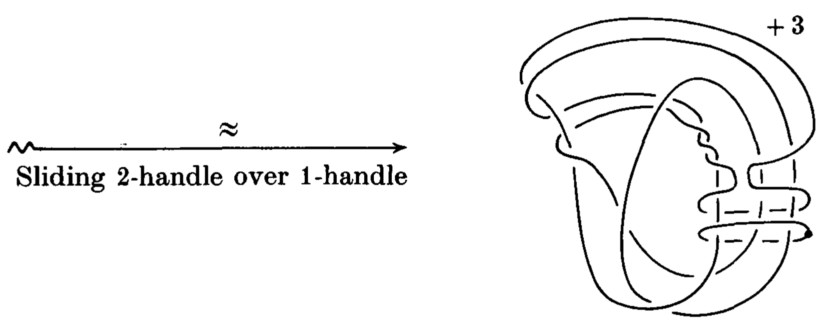


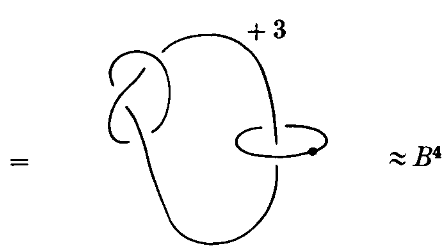
extend h over the dual 2-handle of K^{+1} and over the obvious 2-disc D^2 which γ bounds in R^{+1}. This amounts to turning γ's 1-handles (i.e. putting dots on them). Clearly, K^{+1} – dual 2-handle $\approx B^4$, and let R^{+1} – tubular neighborhood of $D^2 = W^4$. If we show



$W^4 \approx B^4$, then we can extend $h\colon \partial B^4 \overset{\approx}{\to} \partial W^4$ to $B^4 \approx W^4$ and get the desired diffeomorphism $K^{+1} \approx R^{+1}$. One can see this as follows.

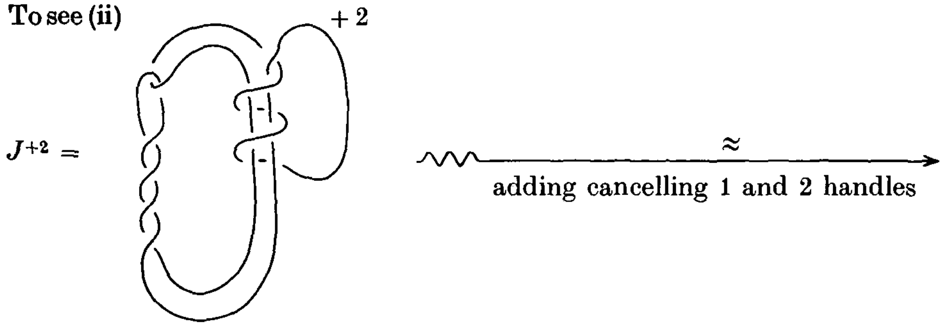


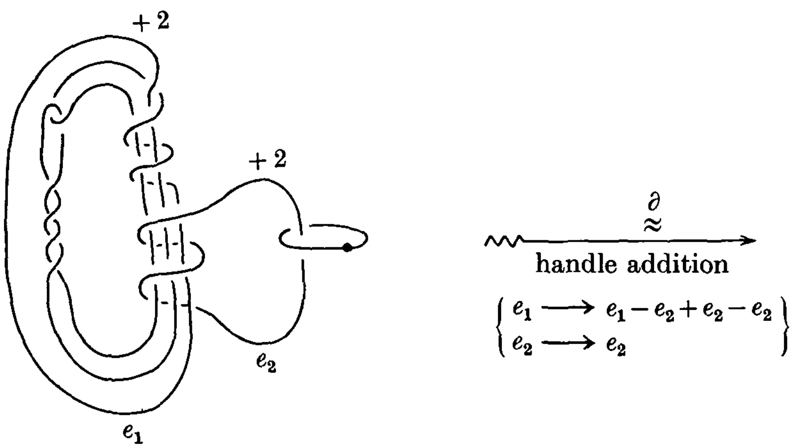


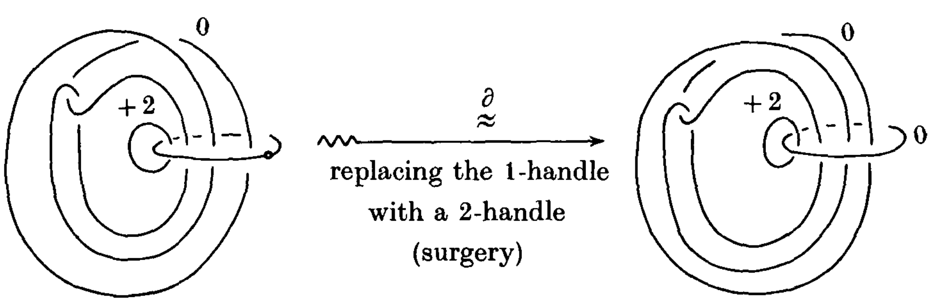


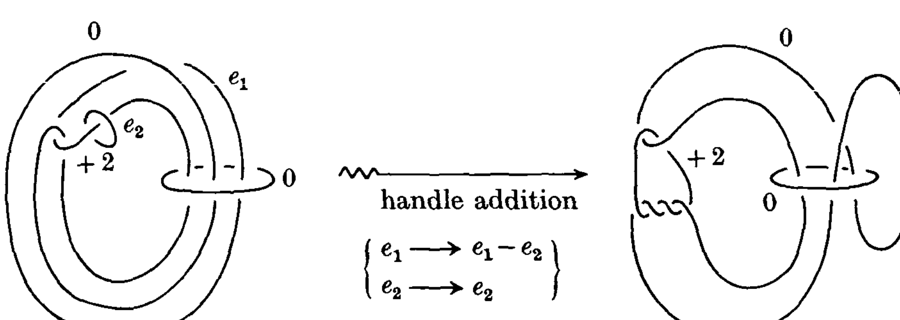
The last diffeomorphism exists because the 2-handle is attached along a knot which intersects (point) $\times S^2$ in $S^1 \times S^2 = \partial(S^1 \times B^3)$ geometrically once (see (3)).

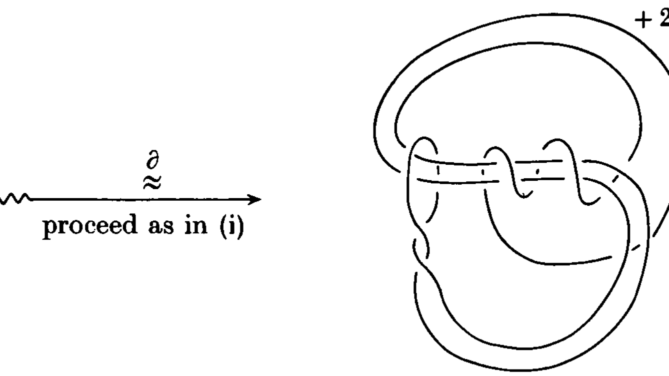
To see (ii)


+ 2
J^{+2} =
≈
adding cancelling 1 and 2 handles


+ 2
+ 2
∂
≈
handle addition
$e_1 \longrightarrow e_1 - e_2 + e_2 - e_2$
$e_2 \longrightarrow e_2$
e_2
e_1


0
+ 2
∂
≈
replacing the 1-handle
with a 2-handle
(surgery)
0
+ 2
0


0
e_1
e_2
+ 2
0
handle addition
$e_1 \longrightarrow e_1 - e_2$
$e_2 \longrightarrow e_2$
0
+ 2
0



The rest of the proof proceeds almost exactly like (i), so we leave the rest as an exercise for the reader.

Remark 3. By identifying the tubular neighbourhood of the imbedded sphere in K^{+1} with $\mathbb{C}P^2 - B^4$, we get the splitting $K^{+1} \approx W \# \mathbb{C}P^2$ where W is a contractible manifold with boundary.

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