LEGENDRIAN CONTACT HOMOLOGY IN CLOSED CONTACT MANIFOLDS

SELMAN AKBULUT AND M. FIRAT ARIKAN

Abstract. We study Legendrian embeddings of a compact Legendrian submanifold L sitting in a closed contact manifold (M,ξ) whose contact structure is supported by a (contact) open book $\mathcal{OB}$ on M. We prove that if $\mathcal{OB}$ has Weinstein pages, then there exist a contact structure ξ' on M, isotopic to ξ and supported by $\mathcal{OB}$, and a contactomorphism $f:(M,\xi)\to(M,\xi')$ such that the image $f(L)$ of any such submanifold can be Legendrian isotoped so that it becomes disjoint from the closure of a page of $\mathcal{OB}$. As a consequence, we conclude that for every compact embedded Legendrian submanifold of a closed contact manifold, there exists a well-defined Legendrian contact homology which is invariant under Legendrian isotopies.

1. Introduction

A *contact manifold* is a pair (M^{2n+1},ξ) where M is a smooth manifold and $\xi\subset TM$ is a totally non-integrable $2n$-plane field distrubution on M. The distrubution ξ is called a *contact structure* on M, and is said to be *co-oriented* if it is the kernel of a globally defined 1-form α with the property $\alpha\wedge(d\alpha)^n\neq 0$. Such a 1-form is called a *contact form* on M. Here we always assume that ξ is a co-oriented *positive* contact structure, that is, $\xi=\mathrm{Ker}(\alpha)$ and $\alpha\wedge(d\alpha)^n>0$ with respect to a pre-given orientation on M. We say that two contact manifolds (M,ξ) and (M',ξ') are *contactomorphic* if there exists a diffeomorphism $f:M\longrightarrow M'$ such that $f_*(\xi)=\xi'$. Two contact structures ξ_0,ξ_1 on a M are said to be *isotopic* if there exists a 1-parameter family ξ_t $(0\leq t\leq 1)$ of contact structures joining them.

A submanifold $L\subset(M^{2n+1},\xi)$ is said to be *isotropic* if $TL\subset\xi$, and an isotropic submanifold is called *Legendrian* if $\dim(L)=n$. A *Legendrian embedding* is an embedding $\phi:\Sigma^n\hookrightarrow(M^{2n+1},\xi)$ of a smooth manifold Σ^n such that the image $\phi(\Sigma^n)$ is Legendrian. A smooth 1-parameter family of Legendrian submanifolds is called a *Legendrian isotopy*. Equivalently, a Legendrian isotopy is a smooth 1-parameter family $\phi_t:\Sigma^n\hookrightarrow(M^{2n+1},\xi)$ of Legendrian embeddings.

An *open book* (*decomposition*) $\mathcal{OB}$ on a closed manifold M is determined by a pair (B,φ) where $B\hookrightarrow M$ is codimension 2 submanifold with trivial normal bundle, and $\varphi:M-B\to S^1$ is a fiber bundle projection. The neighborhood of B should have a trivialization $B\times D^2$, where the angle coordinate on the disk agrees with the fibration map φ. The manifold B is called the *binding*, and for any $t_0\in S^1$ a fiber $X=\varphi^{-1}(t_0)$ is called a *page* of the open book.

Date: February 27, 2017.

2000 *Mathematics Subject Classification.* 57R65, 58A05, 58D27.

Key words and phrases. contact, convex symplectic, Weinstein, Liouville, Lefschetz fibration, open book.

The first author is partially supported by NSF FRG grant DMS- 0905917.

The second author is partially supported by NSF FRG grant DMS-1065910, and also by TUBITAK grant 1109B321200181.

The following definition is due to Giroux [11]: A contact structure ξ on M is said to be *supported by* (or *carried by*, or *compatible with*) an open book $\mathcal{OB} = (B, \varphi)$ on M if there exists a contact form α for ξ such that

(i) $(B, \alpha|_{TB})$ is a contact manifold.
(ii) For any $t \in S^1$, the page $X = \varphi^{-1}(t)$ is a symplectic manifold with symplectic form $d\alpha$.
(iii) If $\bar{X}$ denotes the closure of a page X in M, then the orientation of $B \cong \partial \bar{X}$ induced by its contact form $\alpha|_{TB}$ coincides with its orientation as the boundary of $(\bar{X}, d\alpha)$.

We will say that an open book $\mathcal{OB}$ on M is called a *contact open book* if it carries a contact structure on M.

In the smooth category, given an n-dimensional submanifold L of a closed $(2n+1)$-manifold M admitting an open book $\mathcal{OB}$, it is, in general, not possible to isotope L so that it becomes disjoint from a page $\bar{X}$ of $\mathcal{OB}$. However, if the spine of $\bar{X}$ is n-dimensional, the general position argument shows that we can make them disjoint. Moreover, it is known (see below for details) that any Weinstein domain of dimension $2n$ is homotopy equivalent to its core which is an n-dimensional CW-complex. Here we first prove:

Theorem 1.1. *Let (M, ξ) be a contact manifold which admits a contact open book $\mathcal{OB}$ supporting ξ. If the pages of $\mathcal{OB}$ are Weinstein, then there exist a contact structure ξ' on M, which is isotopic to ξ and supported by $\mathcal{OB}$, and a contactomorphism $f : (M, \xi) \to (M, \xi')$ such that the image $f(L)$ of any compact embedded Legendrian submanifold L of (M, ξ) can be Legendrian isotoped until it becomes disjoint from the closure of a page of $\mathcal{OB}$.*

Legendrian contact homology was introduced by Eliashberg, Givental and Hofer in [8], and also, independently, by Chekanov in [3] for Legendrian knots in $\mathbb{R}^3$. Later, Ekholm, Etnyre and Sullivan (see [5], [7]) introduced Legendrian contact homology for Legendrian submanifolds in $\mathbb{R}^{2n+1}$, and in the contact manifolds of the form $X \times \mathbb{R}$ where X is exact symplectic. These are obtained by associating a differential graded algebra to any given Legendrian submanifold L whose homology is a Legendrian isotopy invariant, called the Legendrian contact homology of L. For instance, using this invariant, it was shown that there are non-isotopic Legendrian submanifolds of $\mathbb{R}^{2n+1}$ [6]. Theorem 1.1 has the following consequence:

Theorem 1.2. *Let $\mathcal{OB}$ be a contact open book on M supporting a contact structure $\xi_{\mathcal{OB}}$. If the pages of $\mathcal{OB}$ are Weinstein, then one can associate a Legendrian contact homology to any compact embedded Legendrian submanifold $L \subset (M, \xi_{\mathcal{OB}})$ which is invariant under Legendrian isotopies.*

In any odd dimension $2n+1$, it is known that supporting open books with Weinstein pages always exist for all closed contact manifolds, and there is a one-to-one correspondence between the set of supporting open books with Weinstein pages (upto positive stabilization) and the set of supported contact structures (upto isotopy). These results are due to Giroux (2000) for $n = 1$ and due to Giroux-Mohsen (2002) for $n > 1$ (see [9]). With the help of these we also prove:

Theorem 1.3. *Let (M, ξ) be a closed contact manifold. Then for any compact embedded Legendrian submanifold $L \subset (M, \xi)$, one can associate a Legendrian contact homology which is invariant under Legendrian isotopies.*

Evidently, there are some other definitions of Legendrian contact homologies in literature, which might be measuring something different then ours (e.g. [2], [13]), it is not clear to us how they are related to the one defined here (we thank to T. Ekholm for pointing this to us).

The proofs of the above results will be given in Section 3 and 4. One of the arguments used in proving Theorem 1.1 is making the given contact open book $\mathcal{OB}$ "β-standard" (see the next definition) by altering the compatible contact structure ξ in its isotopy class.

Definition 1.4. Let $(X^{2n}, d\beta)$ an exact symplectic manifold, and M^{2n+1} be a manifold with a open book structure $\mathcal{OB}$ with pages X, supporting a contact structure ξ. Then we say that the $\mathcal{OB}$ is *β-standard* if there exists a contact form α for ξ, such that α restricts to β on every page.

We remark that any abstract contact open book is β-standard with respect to the contact form constructed by gluing the contact form $\beta + d\theta$ on the mapping torus $X \times \mathbb{R}/\sim$ with the one on $\partial X \times D^2$. This construction (which will be given in Remark 3.1) is originally due to Thurston and Winkelnkemper [14] for dimension three, and to Giroux [12] for higher dimensions.

Acknowledgments. The authors would like to thank NSF and TUBITAK for their supports.

2. Preliminaries

A *Liouville domain* (or a *compact convex symplectic manifold*) is a pair (W, Λ) where W^{2n+2} is a compact manifold with boundary, together with a *Liouville structure* (or a *convex symplectic structure*), which meant there is a 1-form Λ on W such that $\Omega = d\Lambda$ is symplectic and the Ω-dual vector field Z of Λ defined by $\iota_Z \Omega = \Lambda$ (or, equivalently, $\mathcal{L}_Z \Omega = \Omega$), where ι denotes the interior product and $\mathcal{L}$ denotes the Lie derivative, should point strictly outwards along ∂W. Since Ω and Z (resp. Ω and Λ) together uniquely determine Λ (resp. Z), one can replace the notation with the triple (W, Ω, Z) (resp. (W, Ω, Λ)). Here Z is called *Lioville vector field*. The 1-form $\Lambda_\partial := \Lambda|_{\partial W}$ is contact (i.e., $\Lambda_\partial \wedge (d\Lambda_\partial)^n > 0$), and the contact manifold $(\partial W, \mathrm{Ker}(\Lambda_\partial))$ is called the *convex boundary* of (W, Λ).

In the non-compact case (i.e., when W is an open manifold), if we further assume that Z is complete (i.e., its flow exists for all times), and also that there exists an exhaustion $W = \bigcup_{k=1}^{\infty} W^k$ by compact domains $W^k \subset W$ such that each $(W_k, \Lambda|_{W_k})$ is a Liouville domain with the convex boundary $(\partial W_k, \Lambda|_{\partial W_k})$ for all $k \geq 1$, then (W, Λ, Z) is called a *Liouville manifold* (see [4] for details). A *Liouville cobordism* (W, Λ, Z) is a compact cobordism W with a Liouville structure (Λ, Z) such that Z points outwards along $\partial_+ W$ and inwards along $\partial_- W$. Note that any Liouville domain is a Liouville cobordism with $\partial_- W = \emptyset$.

If $Z^{-t} : W \to W$ $(t > 0)$ denotes the contracting flow of Z, then the *core* (or *skeleton*) of the Liouville manifold (W, Λ) is defined to be the set

$$\mathrm{Core}(W, \Lambda) := \bigcup_{k=1}^{\infty} \bigcap_{t>0} Z^{-t}(W^k).$$

The interior of the core of any Liouville manifold is empty (Lemma 11.1, [4]), and so the core of any Liouville domain (W, Λ, Z) is compact. If M denotes the convex boundary ∂W, then one can see that the negative half of the symplectization $(M \times \mathbb{R}, d(e^t \Lambda|_M))$ symplectically embeds into W (as a collar neighborhood of M in W) so that its complement in W is $\mathrm{Core}(W, \Lambda)$ and the embedding matches the positive t-direction of $\mathbb{R}$ with Z. The *completion* $(\widehat{W}, \widehat{\Lambda})$ of a Liouville domain (W, Λ) is obtained from W by gluing the positive part $\partial W \times [0, \infty)$ of the *symplectization* $(\partial W \times \mathbb{R}, d(e^t \Lambda_\partial))$ of its convex boundary. Two Liouville domains (W_i, Λ_i) are said to be *Liouville isomorphic* if there exists a diffeomorphism $\phi : (\widehat{W_0}, \widehat{\Lambda}_0) \to (\widehat{W_1}, \widehat{\Lambda}_1)$, called a *Liouville isomorphism*, between their completions $(\widehat{W_i}, \widehat{\Lambda}_i)$ such that $\phi^*(\widehat{\Lambda}_1) = \widehat{\Lambda}_0 + df$ where

f is some compactly supported smooth function on $\widehat{W}_0$.

To define Weinstein manifolds and domains, we need three preliminary definitions:

Definition 2.1. (i) A vector field Z on a smooth manifold W is said to *gradient-like* for a smooth function $\Psi : W \to \mathbb{R}$ if $Z \cdot \Psi = \mathcal{L}_Z \Psi > 0$ away from the critical point of Ψ.
(ii) A real-valued function is said to be *exhausting* if it is proper and bounded from below.
(iii) An exhausting function $\Psi : W \to \mathbb{R}$ on a symplectic manifold (W, Ω) is said to be Ω-*convex* if there exists a complete Liouville vector field Z which is gradient-like for Ψ.

Definition 2.2. A *Weinstein manifold* (W, Ω, Z, Ψ) is a symplectic manifold (W, Ω) which admits a Ω-convex Morse function $\Psi : W \to \mathbb{R}$ whose complete gradient-like Liouville vector field is Z. The triple (Ω, Z, Ψ) is called a *Weinstein structure* on W. A *Weinstein cobordism* (W, Ω, Z, Ψ) is a Liouville cobordism (W, Ω, Z) whose Liouville vector field Z is gradient-like for a Morse function $\Psi : W \to \mathbb{R}$ which is constant on the boundary ∂W. A Weinstein cobordism with $\partial_- W = \emptyset$ is called *Weinstein domain.*

Remark 2.3. Any Weinstein manifold (W, Ω, Z, Ψ) can be exhausted by Weinstein domains $W_k = \{\Psi^{-1}(-\infty, d_k]\} \subset W$ where $\{d_k\}$ is an increasing sequence of regular values of Ψ, and therefore, any Weinstein manifold is a Liouville manifold. In particular, any Weinstein domain is a Liouville domain. Also note that any Weinstein domain (W, Ω, Z, Ψ) has the convex boundary $(\partial W, \mathrm{Ker}(\iota_Z \Omega|_{\partial W}))$, and the completion of a Weinstein domain is a Weinstein manifold.

The following topologically characterizes Weinstein domains and will be used later.

Theorem 2.4 ([15], see also Lemma 11.13 in [4])**.** *Any Weinstein domain of dimension* $2n$ *admits a handle decomposition whose handles have indices at most* n.

3. Legendrian Embeddings into Contact Open Books

In this section we'll prove Theorem 1.1. Let us start with by recalling how an open book $\mathcal{OB} = (B, \varphi)$ on a closed manifold M determines an abtract open book (X, h): By definition B is a codimension-two subset of M with a trivial normal bundle $B \times D^2$ and $\varphi : M \setminus B \to S^1$ is a fiber bundle map agreeing with the angular coordinate on the D^2-factor. To describe an abstract open book, we first pick a page $X = \varphi^{-1}(p)$ for some $p \in S^1$. Then the monodromy $h : X \to X$ can be read from the first return map of a vector field on M transversal to the fibers of φ. Now using the resulting abstract open book we can construct a closed manifold

$$M(X, h) := \Sigma(X, h) \cup_\partial (\partial X \times D^2)$$

where $\Sigma(X, h)$ is the mapping torus determined by h. Observe that the construction defines an open book decomposition $\mathcal{OB}_{(X,h)}$ on $M(X, h)$, and also that M can be identified with $M(X, h)$ via some diffeomorphism Υ respecting the fibration maps on $M \setminus B$ and $M(X, h) \setminus (\partial X \times \{0\})$.

Next, from a collection of results from [11] and [12] (see also Section 7.3 of [10] for a detailed explanation) we recall the construction of a contact form (under suitable assumptions) on the manifold $M(X, h)$ in the following remark.

Remark 3.1. Given an abstract open book (X, h), consider the closed manifold $M(X, h) = \Sigma(X, h) \cup_\partial (\partial X \times D^2)$. Suppose that there exists a Liouville form β on X and $h \in \mathrm{Symp}(X, d\beta)$. Then one can construct a contact structure ξ_β on $M(X, h)$ so that $\mathcal{OB}_{(X,h)}$ becomes β-standard. Here we recall the explicit construction of the contact form α_β defining ξ_β: Since $h \in \mathrm{Symp}(X, d\beta)$, the form $h^*(\beta) - \beta$ is closed, and it can be made exact by deforming h through symplectomorphisms which are identity near ∂X. Such a deformation of h changes $M(X,h)$ in its diffeomorphism class, and so we may assume that $h^*(\beta) - \beta = -d\rho$ for some smooth function $\rho : X \to \mathbb{R}$. Adding a large enough constant, one can assume that ρ is strictly positive everywhere on X, and so we can use ρ to construct a smooth mapping torus

$$\Sigma(X,h)_\rho := X \times \mathbb{R}/\sim_\rho \quad \text{where} \quad (x,z) \sim_\rho (h(x), z+\rho(x)\,).$$

Consider the contact form $\alpha = \beta + dz$ on $X \times \mathbb{R}$, and let $\sigma_\rho : X \times \mathbb{R} \to X \times \mathbb{R}$ be the diffeomorphism defining $\sim_\rho$, that is, $\sigma_\rho(x,z) = (h(x), z + \rho(x)\,)$. Then we compute

$$\sigma_\rho^*(\beta + dz) = h^*(\beta) + d(z+\rho) = \beta - d\rho + dz + d\rho = \beta + dz$$

which shows that $\alpha = \beta + dz$ decends to a contact form, say $\tilde{\alpha}$, on $\Sigma(X,h)_\rho$. Then using appropriate cut-off functions, one can construct a contact form α_β on

$$\Sigma(X,h)_\rho \cup_\partial (\partial X \times D^2) \approx M(X,h)$$

by smoothly gluing $\tilde{\alpha}$ with the contact form $\beta|_{\partial X} + \frac{1}{2}r^2 d\theta$ on the normal bundle $\partial X \times D^2$.

The key observation for the proof of Theorem 1.1 is the following:

Theorem 3.2. *Let X be a Weinstein domain of dimension $2n \geq 2$ whose underlying Liouville structure is given by the Liouville form β. For $h \in \mathrm{Symp}(X, d\beta)$, consider the contact manifold $(M(X,h), \xi_\beta := \mathrm{Ker}(\alpha_\beta))$ as in Remark 3.1. Let $\phi : \Sigma^n \hookrightarrow (M(X,h), \xi_\beta)$ be a Legendrian embedding of a compact Legendrian submanifold $S := \phi(\Sigma) \subset M(X,h)$. Then S can be Legendrian isotoped (through Legendrian embeddings) to another embedded Legendrian submanifold S' which is disjoint from the closure of a page of $\mathcal{OB}_{(X,h)}$ on $M(X,h)$.*

In order to prove this theorem we will make use of contact vector fields. A vector field on a contact manifold is said to be *contact* if its flow preserves the contact distrubution. The following fundamental lemma in contact geometry characterizes contact vector fields on a given contact manifold. More details can be found, for instance, in [10].

Lemma 3.3. *Let $(M, \mathrm{Ker}(\alpha))$ be any contact manifold and R_α denote the Reeb vector field of α. Then there is a one-to-one correspondence between the set $\{Z \in \Gamma(M) \,|\, Z \text{ is contact}\}$ of all contact vector fields on M and the set $\{H : M \to \mathbb{R} \,|\, H \text{ is smooth}\}$ of all smooth functions on M. The correspondence is given by $Z \to H_Z := \alpha(Z)$ (H_Z is called the "contact Hamiltonian" of the contact vector field Z), and $H \to Z_H$ where Z_H is the contact vector field uniquely determined by the equations $\alpha(Z_H) = H$ and $\iota_{Z_H} d\alpha = dH(R_\alpha)\alpha - dH$.* □

We note that a similar statement also holds between locally defined contact vector fields and locally defined smooth functons.

Proof of Theorem 3.2. For simplicity we will write $M(X,h) = Y$ and $\alpha_\beta = \alpha$. We may assume that Legendrian submanifold $S = \phi(\Sigma^n) \subset (Y, \mathrm{Ker}(\alpha))$ is connected since all the argument used below can be adapted (or generalized) to the case where S has more than one connected component. So we have a Legendrian embedding of a compact connected manifold Σ in the open book $\mathcal{OB}_{(X,h)} = (B, \varphi)$ on Y corresponding to the abstract open book (X,h). So we have

$$\phi : \Sigma^n \hookrightarrow Y = Y_1 \cup Y_2 = \Sigma(X,h)_\rho \cup (B \times D^2)$$

where $B = B \times \{0\} \subset B \times D^2 \subset Y$ is the binding, $Y_2 := B \times D^2$, and $Y_1 := \Sigma(X,h)_\rho$ is the mapping torus as above. We may consider $(Y_2, \mathrm{Ker}(\alpha|_{Y_2}))$ as the contact manifold

$$(B \times D^2, \mathrm{Ker}(\beta + (x/2)\,dy - (y/2)\,dx))$$

where (x, y) are the coordinates on the unit disk D^2. Consider the following vector fields:

$$Z_1 = \partial x - (y/2)R_\beta, \quad Z_2 = \partial y + (x/2)R_\beta, \quad Z_3 = \chi + \theta\, \partial\theta, \quad Z_4 = R_\alpha \tag{1}$$

where χ is the $d\beta$-dual vector field of β, R_α (resp. R_β) denotes the Reeb vector field of α (resp. $\beta|_B$), and θ is the S^1-coordinate in the fibration $\varphi : Y \setminus B \to S^1$ determined by $\mathcal{OB}_{(X,h)}$. Here $\partial x = \partial/\partial x, \partial y = \partial/\partial y$, and so on.... Note that the first two are defined on $Y_2 = B \times D^2$, and the third is defined on the contact manifold $(Y_1 \setminus X, \mathrm{Ker}(\alpha|_{Y_1 \setminus X}))$ where X is any fixed page of $\mathcal{OB}_{(X,h)}$. Here we note that α restricts to β on every page of $\mathcal{OB}_{(X,h)}$ (from its construction described in Remark 3.1) and $Y_1 \setminus X = X \times \mathbb{R}$, and so, in particular, we may also write $Z_3 = \chi + z\, \partial z$ where z is the $\mathbb{R}$-coordinate on $X \times \mathbb{R}$. It is easy to check that:

$$\mathcal{L}_{Z_i}\alpha = \begin{cases} 0 & \text{if } i = 1, 2, 4 \\ \alpha & \text{if } i = 3 \end{cases}$$

So they are all contact vector fields on the regions where they are defined. In fact, if H_i denotes the contact Hamiltonian function corresponding to Z_i as in Lemma 3.3, then we have

$$H_1 = -y, \quad H_2 = x, \quad H_3 = \theta, \quad H_4 \equiv 1.$$

We first see that one can make S transverse to the binding B:

Lemma 3.4. *In any pre-given ϵ-neighbourhood of S, one can Legendrian isotope S (in a small neighbourhood of the binding B) so that it becomes everywhere transverse to B.*

Proof. Let N_ϵ be any ϵ-neighbourhood of S. We will use the fact that Legendrian (more generally, isotropic) submanifolds stays Legendrian (isotropic) under the flows of contact vector fields. Note that for any constants $a_1, a_2 \in \mathbb{R}$ the vector field $Z = a_1 Z_1 + a_2 Z_2$ is contact with the contact Hamiltonian $H_Z = a_1 H_1 + a_2 H_2$. If S intersects B transversally, then there is nothing to prove. If not, let $K \subset S \cap (B \times \{(0, 0)\})$ be the region where they don't intersect transversally. Consider $Z_1|_{B\times\{(0,0)\}} = \partial x, Z_2|_{B\times\{(0,0)\}} = \partial y$ on $B = B \times \{(0, 0)\}$. For any $p \in K$, the tangent space

$$(\{\mathbf{0}\} \times TD^2)|_p \subset (TB \times TD^2)|_p$$

does not lie in $TS|_p$ (otherwise S and B would intersect transversally at $p \in K$). Therefore, there exists a vector $v = a_1\partial x + a_2\partial y \in (\{\mathbf{0}\} \times TD^2)|_{B\times\{(0,0)\}}$ (for some constants $a_1, a_2 \in \mathbb{R}$) which is everywhere transverse to $S \cap B$. We consider $Z = a_1 Z_1 + a_2 Z_2$ as the smooth extension of v to the whole $B \times D^2$. Note that Z will stay transverse to S in a small neighbourhood $N_\delta := B \times \{(x, y) \in D^2 \,|\, x^2 + y^2 < \delta\}$ for some $\delta > 0$. Let $N_S K \subset N_\delta$ be a small neighbourhood of K in S. Now choose a regular value $q \in \{(x, y) \in D^2 \,|\, x^2 + y^2 < \delta\} \subset D^2$ of the composition

$$\Sigma^n \supset U \xrightarrow{\phi} B \times D^2 \xrightarrow{\pi_2} D^2, \text{ where } \phi(\Sigma) = S$$

where π_2 is the projection and $U = \phi^{-1}(B \times \mathrm{int}(D^2))$, such that $q \in \pi_2(N_S K)$ lies on the line segment joining $(0, 0)$ and (a_1, a_2), and the above composition has no critical value other than $(0, 0)$ on the line segment, say l_q, joining $(0, 0)$ and q. Note that, by construction, S intersects the identical copy $B_q = \pi_2^{-1}(q)$ of B transversally, and Z is everywhere transverse to $\pi_2^{-1}(l_q)$. Let $\mu : B \times D^2 \to \mathbb{R}$ be a cut-off function such that $\mu \equiv 1$ on a neigbourhood $N \subset N_\epsilon$ of $\pi_2^{-1}(l_q)$ in N_δ, and $\mu \equiv 0$ on outside of a slightly larger neigbourhood $N_\epsilon \subset N'$ (see Figure 1).

Next consider the contact vector field Z_μ corresponding to the contact Hamiltonian $H_\mu := \mu H_Z$. By the choice of μ, Z_μ agrees with Z on N and it is identically zero outside N'. Using the backward flow Z_μ^{-t} of Z_μ we define the following 1-parameter smooth family:

$$\Phi_t : \Sigma^n \longrightarrow Y, \quad \Phi_t(p) = Z_\mu^{-t}(\phi(p)), \quad t \in [0, T]$$

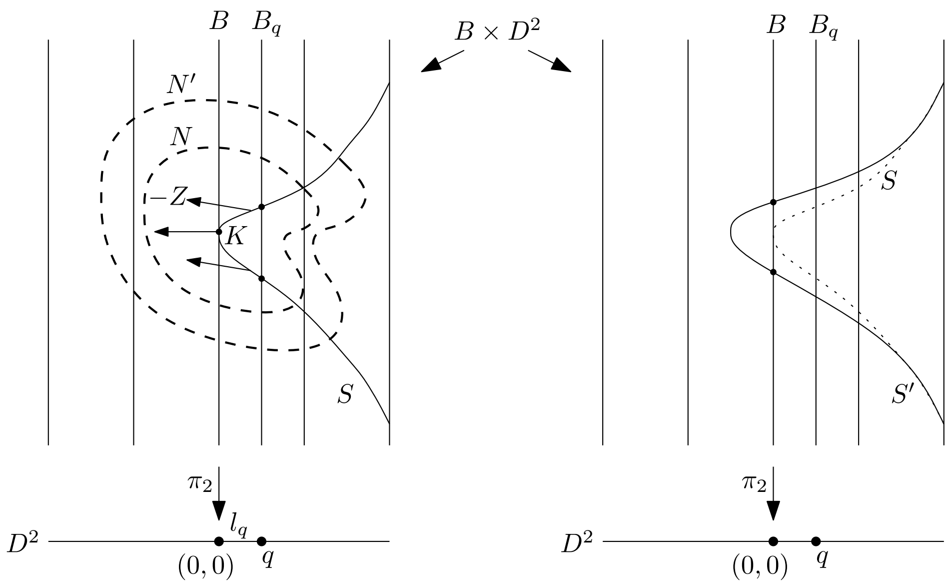

FIGURE 1. Isotoping S to another Legendrian submanifold S' which is transverse to the binding $B = B \times \{(0,0)\}$.

where T is the time elapsed during the points of $S \cap B_q$ are moved to their final images in $B = B \times \{(0,0)\}$ under the backward flow of Z_μ. (Note that all the points of $S \cap B_q \subset N$ reach B at the same time because the "horizontal" components of $Z_\mu|_N = Z|_N$ is defined by the constants a_1, a_2.) Observe that $\Phi_0 = \phi$, for each $t \in [0,T]$ we have $\Phi_t = \phi$ outside N' and $\Phi_t(\Sigma^n)$ is Legendrian, and $S' := \Phi_1(\Sigma^n) \subset Y$ is everywhere transverse to the binding B as depicted in Figure 1. Finally, by choosing δ small enough, one can guarantee that the isotopy Φ_t stays in the pre-given ϵ-neighbourhood N_ϵ of S.

□

By the above lemma we may assume that the Legendrian submanifold $S = \phi(\Sigma^n)$ is transverse to the binding B. Next, by picking a regular value $p \in S^1$ for the projection $\pi_2 : \phi(\Sigma^n) \cap \Sigma(X,h)_\rho \to S^1$ we can assume that for the page $\varphi^{-1}(p) \approx X$ of $\mathcal{OB}_{(X,h)}$, the intersection

$$L := \phi(\Sigma^n) \cap \varphi^{-1}(p) = S \cap X$$

is a properly imbedded $n-1$ dimensional submanifold $(L, \partial L) \subset (X, B = \partial X)$ meeting the binding along an $n-2$ dimensional submanifold ∂L (for simplicity we will write X for $\varphi^{-1}(p)$).

Lemma 3.5. *In any pre-given ϵ-neighbourhood of S, one can Legendrian isotope S (in a small neighbourhood of the page X) so that $L = S \cap X$ becomes disjoint from $\Delta = \mathrm{Core}(X, \beta)$.*

Proof. Since X^{2n} is Weinstein (by assumption), $\dim(\Delta) = n$ by Theorem 2.4. Also we have $\dim(\Delta) + \dim(L^{n-1}) = 2n-1 < 2n = \dim(X)$. Hence, by the general position in X, we can (topologically) isotope L^{n-1} to a nearby copy which is disjoint from Δ. This means that there exists a vector field Z on X which is transverse to both L and Δ along their intersection $L \cap \Delta$. In what follows, using contact vector fields which are compactly supported near $L \cap \Delta$ (and which are generated from Z), we will construct an isotopy which transforms L to some nearby copy L' (disjoint from Δ), and recognize this isotopy (in X) as the restriction of a local Legendrian isotopy (in Y) moving S to another Legendrian submanifold, say S'.

Recall that there is a canonical contact model (Legendrian Neighbourhood Theorem) for the tubular neighbourhood $N_Y(S)$ of S in Y. That is, there exists a contactomorphism

$$\Upsilon : (T^*\Sigma^n \times \mathbb{R}, \mathrm{Ker}(\mathbf{q}d\mathbf{p} + dz)) \longrightarrow (N_Y(S), \mathrm{Ker}(\bar{\beta}^{st}|_{N_Y(S)}))$$

from the 1-jet bundle $(T^*\Sigma^n \times \mathbb{R}, \mathrm{Ker}(\mathbf{q}d\mathbf{p} + dz))$ where $\mathbf{p} = (p_1, ..., p_n), \mathbf{q} = (q_1, ..., q_n)$ are the standard coordinates on $T^*\Sigma^n$ and z is the real coordinate. (Here Υ maps the zero section $\Sigma_0 = \{\mathbf{q} = \mathbf{0}\} \times \{0\} \subset T^*\Sigma^n \times \mathbb{R}$ onto S.) Observe that, on $T^*\Sigma^n \times \mathbb{R}$, there are $2n+1$ linearly independent contact vector fields:

$$Z'_1 = \partial p_1, ..., Z'_n = \partial p_n, \quad Z'_{n+1} = \partial q_1 - p_1 \partial z, ..., Z'_{2n} = \partial q_n - p_n \partial z, \quad Z'_{2n+1} = \partial z \tag{2}$$

The corresponding contact Hamiltonian functions (as in Lemma 3.3), respectively, are

$$H'_1 = q_1, ..., H'_n = q_n, \quad H'_{n+1} = -p_1, ..., H'_{2n} = -p_n, \quad H'_{2n+1} = 1. \tag{3}$$

We will use these contact vector fields for local Legendrian isotopies in $N_Y(S) \cong T^*\Sigma^n \times \mathbb{R}$ that we need for our purpose.

Let $L \cap \Delta = \sqcup_{i=1}^s L_i$ where L_i's are (disjoint) connected components. Note that L is compact as both $S = \phi(\Sigma^n)$ and X are compact. Moreover, the core Δ is compact, and so $L \cap \Delta$ is also compact from which we conclude that each L_i is a compact CW-complex of finite type. Denote by L_i^K the K-skeleton of L_i for $K = 0, 1, ..., n-1$. In particular, we have $L_i = L_i^{n-1}$ (Figure 2-a). Using the isotopies mentioned above, we will first make the closure of every $(n-1)$-cell in each L_i disjoint from Δ, and then do the same for $(n-2)$-cells, and so on...

Let $\{E_{i,1}^K, ..., E_{i,l_i^K}^K\}$ be the set of all K-cells in L_i^K. For any $1 \leq j \leq l_i^K$, consider the following open cover for the closure $\overline{E_{i,j}^K}$ (recall the vector field Z in the proof of Lemma 3.5):

$$\mathcal{U}_{i,j}^K := \{U_x \mid x \in E_{i,j}^K, U_x \text{ is a nbhd of } x \text{ in } \overline{E_{i,j}^K} \text{ s.t. } Z(x) \pitchfork \overline{U_x}\}.$$

Clearly, $\mathcal{U}_{i,j}^K$ covers $\overline{E_{i,j}^K}$ which is compact. So, $\mathcal{U}_{i,j}^K$ has a finite subcover

$$\{U_{x_{i,j,1}^K}, U_{x_{i,j,2}^K}, ..., U_{x_{i,j,m_{i,j}^K}^K}\}$$

for some finite number of points $\{x_{i,j,1}^K, x_{i,j,2}^K, ..., x_{i,j,m_{i,j}^K}^K\}$ in $E_{i,j}^K$. We label these points in such a way that the neighbourhood of any point has nonempty intersection with the union of the neighbourhoods of the preceding points in the list as depicted in Figure 2-b (for the case $K = n-1$).

We will first make $\overline{U_{x_{i,j,1}^{n-1}}}$ disjoint from Δ. From the definition of $\mathcal{U}_{i,j}^{n-1}$, we have a constant vector field $Z(x_{i,j,1}^{n-1}) \in TX|_{\overline{U_{x_{i,j,1}^{n-1}}}}$ which is everywhere transverse to $\overline{U_{x_{i,j,1}^{n-1}}}$. (Here at every point of $\overline{U_{x_{i,j,1}^{n-1}}}$, we have the same vector $Z(x_{i,j,1}^{n-1})$.) Observe that, since $\{Z'_i\}_{i=1}^{2n+1}$ is a linearly independent set, the pull-back vector field $\Upsilon^*(Z(x_{i,j,1}^{n-1}))$ can be written as the unique linear combination

$$\Upsilon^*(Z(x_{i,j,1}^{n-1})) = c_1 Z'_1 + \cdots c_{2n+1} Z'_{2n+1}$$

for some unique constants $c_1, ..., c_{2n+1} \in \mathbb{R}$. Since these constants depend on the point $x_{i,j,1}^{n-1}$, we set the notation

$$\mathbf{Z}_{x_{i,j,1}^{n-1}} := c_1 Z'_1 + \cdots c_{2n+1} Z'_{2n+1}.$$

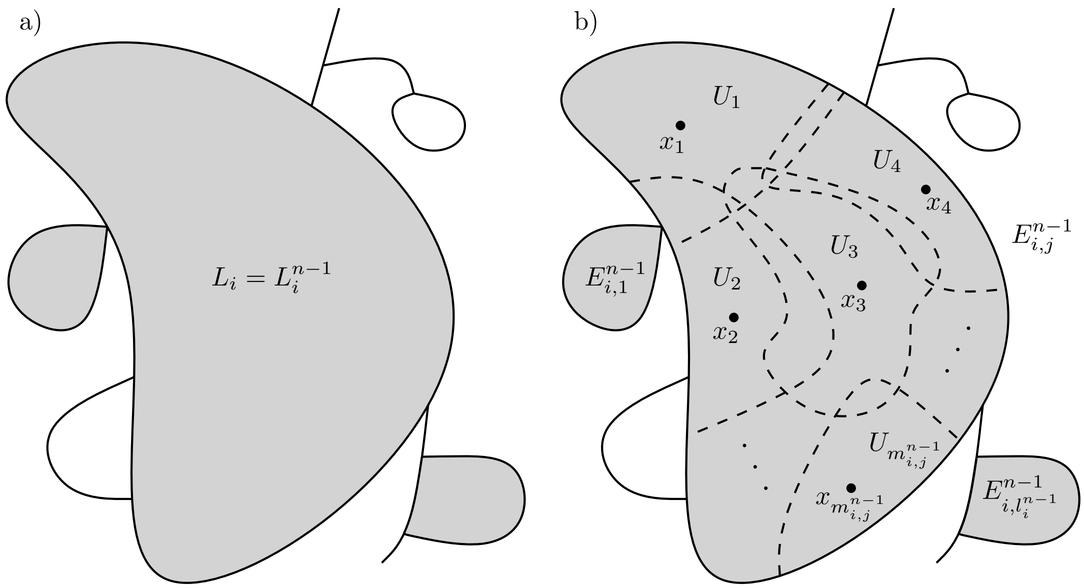

FIGURE 2. a) A typical connected component $L_i = L_i^{n-1}$ of $L \cap \Delta$, b) Finite subcover of $\mathcal{U}_{i,j}^{n-1}$ (where we write $x_{i,j,1}^{n-1} = x_1, ..., x_{i,j,m_{i,j}^{n-1}}^{n-1} = x_{m_{i,j}^{n-1}}$, and similarly $U_{x_{i,j,1}^{n-1}} = U_1, ..., U_{x^{n-1}_{i,j,m^{n-1}_{i,j}}} = U_{m^{n-1}_{i,j}}$ for simplicity).

Let us write $V_{x^{n-1}_{i,j,1}}$ for the pre-image $\Upsilon^{-1}(U_{x^{n-1}_{i,j,1}})$. Observe that $\mathbf{Z}_{x^{n-1}_{i,j,1}}$ is a contact vector field on $T^*\Sigma^n \times \mathbb{R}$ with the contact Hamiltonian

$$\mathbf{H}_{x^{n-1}_{i,j,1}} := c_1 H'_1 + \cdots c_{2n+1} H'_{2n+1},$$

and also that it is everywhere transverse to $\overline{V_{x^{n-1}_{i,j,1}}} = \Upsilon^{-1}(\overline{U_{x^{n-1}_{i,j,1}}}) \subset \Upsilon^{-1}(L) \subset \Sigma_0$.

Denote by $\widetilde{V_{x^{n-1}_{i,j,1}}}$ a small neighbourhood of $\overline{V_{x^{n-1}_{i,j,1}}}$ in $\Upsilon^{-1}(N_Y(S) \cap X) \subset T^*\Sigma^n \times \mathbb{R}$. Let $\mu_{x^{n-1}_{i,j,1}} : \Upsilon^{-1}(N_Y(S) \cap X) \to \mathbb{R}$ be a smooth cut-off function such that $\mu_{x^{n-1}_{i,j,1}} \equiv 1$ near $\overline{V_{x^{n-1}_{i,j,1}}}$, and $\mu_{x^{n-1}_{i,j,1}} \equiv 0$ on the complement $\Upsilon^{-1}(N_Y(S) \cap X) \setminus \widetilde{V_{x^{n-1}_{i,j,1}}}$. Now consider the contact vector field $\mathbf{Z}_{\mu_{x^{n-1}_{i,j,1}}}$ whose corresponding contact Hamiltonian is equal to $\mu_{x^{n-1}_{i,j,1}} \mathbf{H}_{x^{n-1}_{i,j,1}}$. By the choice of the cut-off function $\mathbf{Z}_{\mu_{x^{n-1}_{i,j,1}}}|_{\overline{V_{x^{n-1}_{i,j,1}}}} = \mathbf{Z}_{x^{n-1}_{i,j,1}}$, and so $\mathbf{Z}_{\mu_{x^{n-1}_{i,j,1}}}$ is also transverse to $\overline{V_{x^{n-1}_{i,j,1}}}$. Using the flow $\mathbf{Z}^t_{\mu_{x^{n-1}_{i,j,1}}}$ we isotope $\overline{V_{x^{n-1}_{i,j,1}}}$ to its nearby copy $\mathbf{Z}^t_{\mu_{x^{n-1}_{i,j,1}}}(\overline{V_{x^{n-1}_{i,j,1}}})$ for some fixed time t. Note that pushing along a transverse contact vector field implies that $\mathbf{Z}^t_{\mu_{x^{n-1}_{i,j,1}}}(\overline{V_{x^{n-1}_{i,j,1}}})$ is disjoint from Δ, and is still isotropic in $(T^*\Sigma^n \times \mathbb{R}, \mathrm{Ker}(\mathbf{q}d\mathbf{p} + dz))$. Indeed, since the transversality is an open condition, we know that the isotropic image $\mathbf{Z}^t_{\mu_{x^{n-1}_{i,j,1}}}(\widehat{V_{x^{n-1}_{i,j,1}}})$ is disjoint from Δ where $\widehat{V_{x^{n-1}_{i,j,1}}} \supset \overline{V_{x^{n-1}_{i,j,1}}}$ is a neighbourhood of $\overline{V_{x^{n-1}_{i,j,1}}}$ in L_i such that

$$\overline{V_{x^{n-1}_{i,j,1}}} \subset \widehat{V_{x^{n-1}_{i,j,1}}} \subset \widetilde{V_{x^{n-1}_{i,j,1}}} \cap L_i.$$

Similarly, we can make the closure of all the other open sets in the above finite subcover of $\mathcal{U}_{i,j}^{n-1}$ disjoint from Δ (this will isotope the whole closed $(n-1)$-cell $\overline{E_{i,j}^{n-1}}$ to some isotropic copy which is disjoint from Δ). However, for each such closure, the choice of how much we push it (using the flow of the corresponding contact vector field) needs a litle bit of more care: Let us discuss this in an inductive way: Suppose that we have already isotoped the union

$$\overline{V_{x_{i,j,1}^{n-1}}} \cup \overline{V_{x_{i,j,2}^{n-1}}} \cup \cdots \cup \overline{V_{x_{i,j,k}^{n-1}}}, \quad \text{for some } k \in \{1, ..., m_{i,j}^{n-1} - 1\}$$

along the contact vector fields $\{\mathbf{Z}_{\mu_{x_{i,j,m}^{n-1}}}\}_{m=1}^{k}$ (where the smooth cut-off functions $\mu_{x_{i,j,m}^{n-1}} : \Upsilon^{-1}(N_Y(S) \cap X) \to \mathbb{R}$ are constructed in the same way as above) so that the image of the union

$$\bigcup_{m=1}^{k} \widehat{V_{x_{i,j,m}^{n-1}}}$$

is isotropic in $(T^*\Sigma^n \times \mathbb{R}, \mathrm{Ker}(\mathbf{q}d\mathbf{p} + dz))$ and is disjoint from Δ where $\widehat{V_{x_{i,j,m}^{n-1}}} \subset \widetilde{V_{x_{i,j,m}^{n-1}}} \cap L_i$ is a small neighbourhood of $\overline{V_{x_{i,j,m}^{n-1}}}$ in L_i. Now we would like to push (i.e., isotope) $\overline{V_{x_{i,j,k+1}^{n-1}}}$ using $\mathbf{Z}_{\mu_{x_{i,j,k+1}^{n-1}}}$. Observe that the region

$$\overline{V_{x_{i,j,k+1}^{n-1}}} \cap (\widehat{V_{x_{i,j,1}^{n-1}}} \cup \cdots \cup \widehat{V_{x_{i,j,k}^{n-1}}})$$

has been already made disjoint from Δ, and also that $\mathbf{Z}_{\mu_{x_{i,j,k+1}^{n-1}}}$ might be tangent to the image of this region at some points, or even its flow might transform some points in the region back to Δ (if we let them flow too much). On the other hand, since transversality is an open condition there exists a codim-0 subset $\overline{\overline{V_{x_{i,j,k+1}^{n-1}}}} \subset \overline{V_{x_{i,j,k+1}^{n-1}}}$ with a codim-0 nonempty intersection

$$\overline{\overline{V_{x_{i,j,k+1}^{n-1}}}} \cap (\widehat{V_{x_{i,j,1}^{n-1}}} \cup \cdots \cup \widehat{V_{x_{i,j,k}^{n-1}}})$$

to which $\mathbf{Z}_{\mu_{x_{i,j,k+1}^{n-1}}}$ is everywhere transverse. Therefore, we can make the union

$$\overline{V_{x_{i,j,1}^{n-1}}} \cup \cdots \cup \overline{V_{x_{i,j,k+1}^{n-1}}}$$

disjoint from Δ by pushing (in an appropriate amount) along $\mathbf{Z}_{\mu_{x_{i,j,k+1}^{n-1}}}$ as shown in Figure 3.

Repeating the above process we can make the union $\overline{E_{i,1}^{n-1}} \cup \cdots \cup \overline{E_{i,l_i^{n-1}}^{n-1}}$ of the closures of all $(n-1)$-cells disjoint from Δ by pushing to a nearby isotropic copy in $\Upsilon^{-1}(N_Y(S) \cap X)$. Note that for any particular closure in the union if some part of it has been already pushed (this might happen if it has a common boundary part with another $(n-1)$-cell which has been pushed earlier), then we isotope it in an appropriate amount (as in the above discussion) so that previously pushed regions in the cell would not be moved back to Δ.

Similarly, we can deal with the union $\{\overline{E_{i,1}^{K}} \cup \cdots \cup \overline{E_{i,l_i^K}^{K}}\}$ of all closed K-cells in L_i (for $0 \leq K \leq n-2$) in the same way (under the assumption that all closed $(K+1)$-cells have been already made disjoint from Δ. Note that we do not need to push any K-cell which appears as a part of the boundary of some $(K+1)$-cell(s) because such K-cells have been already made disjoint from Δ in the previous step. This process deforms the connected component L_i to its image L_i' which is isotropic and disjoint from Δ. Repeating this for each connected component, we conclude that there exists an isotopy of the embeddings in X from L to its nearby isotropic

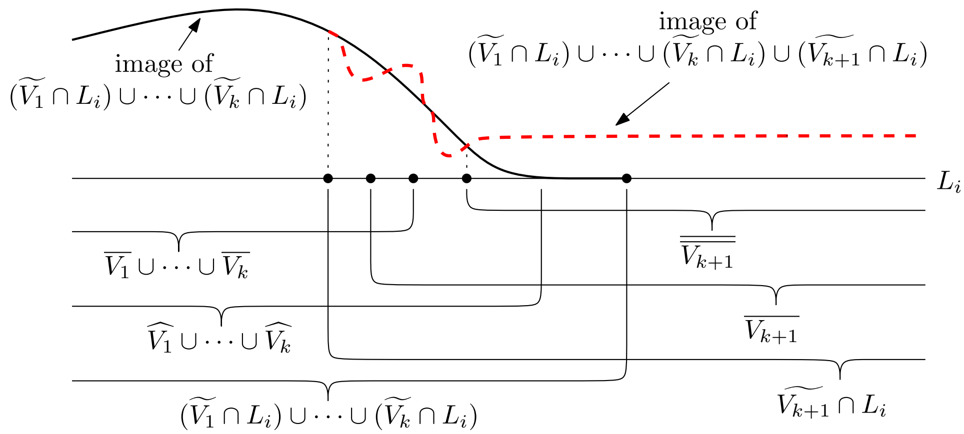

FIGURE 3. Making $\overline{V_{x^{n-1}_{i,j,k+1}}}$ disjoint from Δ (where for each $m = 1, ..., k+1$ we write $\overline{V_{x^{n-1}_{i,j,m}}} = \overline{V_m}$, $\widehat{V_{x^{n-1}_{i,j,m}}} = \widehat{V_m}$, $\widetilde{V_{x^{n-1}_{i,j,m}}} = \widetilde{V_m}$, and also $\overline{\overline{V_{x^{n-1}_{i,j,k+1}}}} = \overline{\overline{V_{k+1}}}$ for simplicity).

copy L' which is disjoint from Δ. This isotopy is generated by contact vector fields $\mathbf{Z}_{\mu_{x^K_{i,j,k}}}$ and compactly supported in the neighbourhood

$$\widetilde{V} := \bigcup_{0 \le i \le s} \bigcup_{0 \le K \le n-1} \bigcup_{0 \le j \le l^K_i} \bigcup_{0 \le k \le m^K_{i,j}} \widetilde{V_{x^K_{i,j,k}}} \subset \Upsilon^{-1}(N_Y(S) \cap X)$$

Next, we want to extend this local isotopy of L in $\Upsilon^{-1}(N_Y(S) \cap X) \times \{0\}$ to a local Legendrian isotopy of S compactly supported in $\widetilde{V} \times [-1,1] \subset \Upsilon^{-1}(N_Y(S) \cap X) \times [-1,1] \subset T^*\Sigma^n \times \mathbb{R}$. (Here since L is codimension 1 submanifold of S, we can consider the tubular neighbourhood of L in S as the product $L \times [-1,1]$ such that L corresponds to $L \times \{0\}$.) Consider the smooth cut-off function

$$f : \Upsilon^{-1}(N_Y(S) \cap X) \times [-1,1] \to \mathbb{R}, \quad f(x,t) = \mu(t)$$

where $\mu : [-1,1] \to \mathbb{R}$ is a smooth cut-off function which is equal to 1 near $t = 0$, and 0 near $t = \pm 1$. Also let $\widetilde{\mu}_{x^K_{i,j,k}} : \Upsilon^{-1}(N_Y(S) \cap X) \times [-1,1] \to \mathbb{R}$ be the extension of $\mu_{x^K_{i,j,k}}$ given by

$$\widetilde{\mu}_{x^K_{i,j,k}}(x,t) = \mu_{x^K_{i,j,k}}(x).$$

Denote by $\widetilde{\mathbf{Z}}_{\mu_{x^K_{i,j,k}}}$ the contact vector field on $\Upsilon^{-1}(N_Y(S) \cap X) \times [-1,1]$ whose coresponding contact Hamiltonian (as in Lemma 3.3) is equal to $f\widetilde{\mu}_{x^K_{i,j,k}} \mathbf{H}_{x^K_{i,j,k}}$. Now we isotope S to its nearby Legendrian copy S' by applying the flow maps of $\widetilde{\mathbf{Z}}_{\mu_{x^K_{i,j,k}}}$ in the same order and amount that we apply the flow maps of $\mathbf{Z}_{\mu_{x^K_{i,j,k}}}$ to isotope L to L'. By construction, this isotopy is compactly supported in $\widetilde{V} \times [-1,1]$, and its restriction to $\Upsilon^{-1}(N_Y(S) \cap X) \times \{0\}$ is the isotopy taking L to L' (constructed above) as $\widetilde{\mathbf{Z}}_{\mu_{x^K_{i,j,k}}}|_{\Upsilon^{-1}(N_Y(S)\cap X)\times\{0\}} = \mathbf{Z}_{\mu_{x^K_{i,j,k}}}$. In particular, for the new Legendrian submanifold S', its intersection $L' = S' \cap X$ is disjoint from the core Δ of X.

Finally, we note that by working on a small enough neighbourhood $N_Y(S)$ one can guarantee that S' lies in any pre-given ϵ-neighbourhood of S in Y. $\square$

By the last lemma we may assume that the transverse intersection $L = S \cap X$ of the Legendrian submanifold $S = \phi(\Sigma^n)$ is disjoint from the core Δ of $X = \varphi^{-1}(p)$. To finish the proof of Theorem 3.2, we will construct an isotopy of Legendrian embeddings of S which will be compactly supported in a neighbourhood of X in Y and will push L completely outside X. To this end, we will first isotope S along a contact vector field generated from Z_3 given in the list (1) above so that the part $L \cap Y_1$ of L (recall $Y_1 = \Sigma(X,h)_\rho$ is the mapping torus of the open book $\mathcal{OB}_{(X,h)}$) is completely pushed into the interior of $Y_2 = B \times D^2$ (tubular neighbourhood of the binding $B = \partial X$). Then using Z_1, Z_2 of the list (1) we will isotope S until $L \cap Y_2$ completely crosses the binding $B = B \times \{0\}$.

Remark 3.6. So far, when we write X we meant the whole page (in particular, ∂X was the binding B). However, for what follows it is better to use the abstract open book desription as in the previous paragraph. Therefore, from now on X will denote the complement of the collar neighbourhood of the binding B in the corresponding page.

From its construction the contact manifold $(Y_1 = \Sigma(X,h)_\rho, \alpha|_{Y_1})$ is obtained as the quotient space of $(X \times \mathbb{R}, \beta + dz)$ using the equivalence relation $\sim_\rho$. Recall that we have

$$Y_1 = X \times \mathbb{R}/\sim_\rho \quad \text{where} \quad (x,z) \sim_\rho (h(x), z + \rho(x)\,).$$

By translating (i.e., isotoping) S along the Reeb direction $Z_4 = R_\alpha$ (which corresponds to ∂z on $X \times \mathbb{R}$), we may assume that $X = \varphi^{-1}(p)$ corresponds to $X \times \{0\}$ under this identification. Therefore, we can identify a neighbourhood of X in Y_1 with $X \times [-a,a] \subset X \times \mathbb{R}$ for some real number $0 < a < K$ where $K > 0$ is a constant satisfying

$$K < \rho(x), \quad \forall x \in X.$$

(Note that such K exists since X is compact and ρ is a strictly positive continuous function). The contact form α on Y is equal to $\beta + dz$ on $X \times [-a,a]$ where Z_3 takes the form $Z := \chi + z\partial z$ as mentioned earlier. By choosing a small enough, we may guarantee that the intersection $S \cap (X \times [-a,a])$ is equal to $L \times [-a,a]$ (S and $X = X \times \{0\}$ intersect transversally), and also that $L \times [-a,a]$ is disjoint from $\Delta \times [-a,a]$ (this is because all cut-off functions which we used to isotope S to S' in the proof of Lemma 3.5 are all equal to 1 near $(L \times \{0\}) \cap (\Delta \times \{0\})$.)

One can think of Y_1 slightly larger by considering $Y_2 = B \times D^2$ slightly smaller. More precisely, let $N = B \times D \subset Y_2$ be a smaller neighbourhood of B where $D \subset D^2$ is a smaller disk in $\mathbb{R}^2$ around the origin. By expanding each (X, β) in Y_1 to a larger domain $(\tilde{X}, \tilde{\beta})$, we get another decomposition $Y = Y_1' \cup N$ where $Y_1' = Y \setminus N$. Note that extending the above identification, a neighbourhood of $\tilde{X}$ in Y_1' can be identified with $\tilde{X} \times [-a,a]$ ($\tilde{X} = \tilde{X} \times \{0\}$) on which the contact form α is given as $\alpha = \tilde{\beta} + dz$ and we have the extension

$$\tilde{Z} = \tilde{\chi} + z\partial z$$

of Z where $\tilde{\chi}$ is the $d\tilde{\beta}$-dual of $\tilde{\beta}$. We remark that $\tilde{Z}$ is contact with the contact Hamiltonian $H = z$. Let $\mu_1 : Y \to \mathbb{R}$ be a smooth cut-off function such that $\mu_1 \equiv 1$ near $\tilde{X} \times \{0\}$ and $\mu_1 \equiv 0$ in the complement $Y \setminus \tilde{X} \times (-\epsilon, \epsilon)$ for some $0 < \epsilon < a$ which will be determined later. Denote by Z_{μ_1} the contact vector field on Y which corresponds to the contact Hamiltonian $\mu_1 H_3$.

Now we first push S using the flow $Z^t_{\mu_1}$ until the image of L lies completely in the interior $\text{int}(Y_2)$ of Y_2 as follows: First note that $Z_{\mu_1} = \tilde{\chi}$ on $\tilde{X} = \tilde{X} \times \{0\}$ (as $z = 0$ there). Since L is disjoint from the core of $\tilde{X}$ (which is the same as that of X), for every point $x \in L \cap \tilde{X}$ there exists a unique flow line of $\tilde{\chi}$ (i.e., of $Z_{\mu_1}|_{\tilde{X}}$) passing through x. All such flow lines reach the region $\tilde{X} \cap \mathrm{int}(Y_2 \setminus N)$. Consider the set

$$A = \{t \in (0, \infty) \mid Z^t_{\mu_1}(x) \in \tilde{X} \cap \mathrm{int}(Y_2 \setminus N) \quad \text{for all } x \in L \cap \tilde{X}\}.$$

Since $L \cap \tilde{X}$ is a compact, the set A is non-empty. Choose a finite number $T > 0$ from A. We isotope the Legendrian sphere $S \subset Y$ using the flow maps $\{Z^t_{\mu_1} \mid 0 \leq t \leq T\}$. Indeed, by the construction of Z_{μ_1}, we only push the region $(L \cap \tilde{X}) \times (-\epsilon, \epsilon) \subset S$ during the isotopy. Observe that by choosing $\epsilon > 0$ small enough, we can guarantee that the image $Z^T_{\mu_1}(L \times (-\epsilon, \epsilon))$ completely lies in $\tilde{X} \times [-a, a]$. Therefore, $Z^T_{\mu_1}(S)$ is an embedded Legendrian which is Legendrian isotopic to S (see Figure 4).

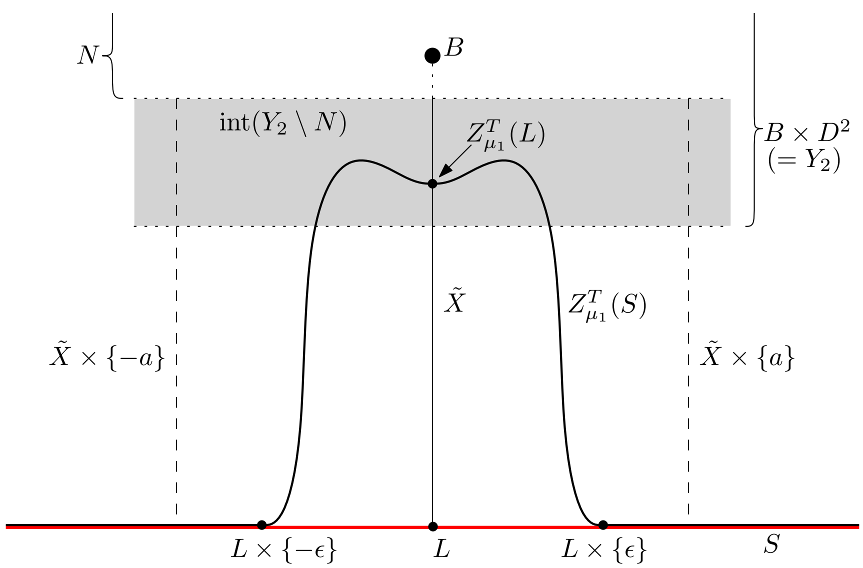

FIGURE 4. Making L closer to the binding via a Legendrian isotopy of S.

By the choice of the flow parameter T above, we know that $Z^T_{\mu_1}(L)$ is completely lie in the region $\mathrm{int}(Y_2)$. For $0 < r < 1$ consider the disk

$$D_r = \{(x, y) \in D^2 \mid x^2 + y^2 \leq r^2\} \subset D^2$$

and the neighborhood $N_r = B \times D_r \subset \mathrm{int}(B \times D^2) = \mathrm{int}(Y_2)$ of the binding. Then after the above isotopy we know that $Z^T_{\mu_1}(L) \subset N_r$ for some r. Let $(a_1, a_2) \in D^2$ be a point which corresponds to the angular coordinate $p \in S^1$ (recall that $\tilde{X} = \tilde{X} \times \{p\}$). Recall the contact vector fields Z_1, Z_2 from the list (1) and their contact Hamiltonians H_1, H_2. Then the vector field $Z' = a_1 Z_1 + a_2 Z_2$ is also contact whose contact Hamiltonian is given by $H' = a_1 H_1 + a_2 H_2$. Let $\mu_2 : Y \to \mathbb{R}$ be a smooth cut-off function such that $\mu_2 \equiv 1$ near N_r and $\mu_2 \equiv 0$ in the complement $Y \setminus Y_2$. Denote by Z_{μ_2} the contact vector field on Y which corresponds to the contact Hamiltonian $\mu_2 H'$. Now we isotope $Z^T_{\mu_1}(S)$ using the flow maps of the contact vector field Z_{μ_2} until $Z^T_{\mu_1}(L)$ completely crosses the binding. Say for $T' > 0$ the image of $Z^T_{\mu_1}(L)$ under the flow map $Z^{T'}_{\mu_2}$ completely crosses the binding. As a result, the closure of the page $\varphi^{-1}(p)$ of the open book $\mathcal{OB}_{(X,h)}$ is disjoint from the final image $S' := Z^{T'}_{\mu_2}(Z^T_{\mu_1}(S))$ which is Legendrian isotopic to $S = \phi(\Sigma^n)$ as claimed. This finishes the proof of Theorem 3.2. □

Proof of Theorem 1.1. Let $\mathcal{OB}$ be an open book decomposition carrying a contact structure ξ on a (closed) manifold M of dimension $2n+1 \geq 3$. In particular, there exists a contact form α for ξ such that α restricts to a Liouville form on every page of $\mathcal{OB}$. By assumption, for any page X of $\mathcal{OB}$, the restriction $\alpha|_X$ is, indeed, the underlying Liouville form of a Weinstein structure on X.

Pick a page X equipped with the Liouville form $\beta := \alpha|_X$ which is, by assumption, the underlying Liouville form of a Weinstein structure on X. Denote by $h \in \mathrm{Symp}(X, d\beta)$ the monodromy of $\mathcal{OB}$ and consider the manifold $M(X,h)$ equipped with the contact structure $\xi_\beta = \mathrm{Ker}(\alpha_\beta)$ where α_β is the contact form on $M(X,h)$ constructed as in Remark 3.1. Let

$$\xi_X := (\Upsilon_X^{-1})_*(\xi_\beta)$$

be the contact structure on M obtained by pushing forward ξ_β using the inverse of the identification map $\Upsilon_X : M \to M(X,h)$. As explained at the beginning of this section, Υ_X respects the fibration maps on $M \setminus B$ and $M(X,h) \setminus (\partial X \times \{0\})$ associated to the open books $\mathcal{OB}$ and $\mathcal{OB}_{(X,h)}$, respectively. Therefore, we have a contactomorphism

$$\Upsilon_X : (M, \xi_X) \to (M(X,h), \xi_\beta)$$

mapping pages of $\mathcal{OB}$ to those of $\mathcal{OB}_{(X,h)}$.

Next, observe that ξ and ξ_X are supported by the same open book $\mathcal{OB}$ (by construction of ξ_X), so we know, by Giroux's work, that there exists an isotopy ξ_t $(t \in [0,1])$ of contact structures on M connecting $\xi_0 = \xi$ and $\xi_1 = \xi_X$. Then Gray's Stability (see, for instance, Theorem 2.2.2 of [10]) implies that there is a diffeotopy

$$\Xi_t : M \to M, \quad t \in [0,1],$$

such that $(\Xi_t)_*(\xi_0) = \xi_t$ for each t. In particular, $(\Xi_1)_*(\xi) = \xi_X$, and hence we obtain two contactomorphisms

$$f_X := \Xi_1 : (M, \xi) \to (M, \xi_X), \quad \Upsilon_X \circ f_X : (M, \xi) \to (M(X,h), \xi_\beta).$$

Suppose now we are given a Legendrian embedding $\psi : \Sigma^n \hookrightarrow (M, \xi)$ of a compact Legendrian submanifold $L = \psi(\Sigma^n)$. By pushing forward ψ using the above contactomorphisms, we obtain two Legendrian embeddings

$$f_X \circ \psi : \Sigma^n \hookrightarrow (M, \xi_X), \quad \phi := \Upsilon_X \circ f_X \circ \psi : \Sigma^n \hookrightarrow (M(X,h), \xi_\beta).$$

We set $S := \phi(\Sigma^n)$. By Theorem 3.2 we have a smooth 1-parameter family

$$\Phi_t : \Sigma^n \hookrightarrow (M(X,h), \xi_\beta), \quad t \in [0,1]$$

of Legendrian embeddings such that $\Phi_0(\Sigma^n) = S$ and the Legendrian submanifold $S' := \Phi_1(\Sigma^n)$ is disjoint from the closure of a page of the open book $\mathcal{OB}_{(X,h)}$ on $M(X,h)$ associated to (X,h). By composing Φ_t with the contactomorphism Υ_X^{-1}, we obtain a smooth 1-parameter family

$$\Psi_t := \Upsilon_X^{-1} \circ \Phi_t : \Sigma^n \hookrightarrow (M, \xi_X), \quad t \in [0,1]$$

of Legendrian embeddings such that $\Psi_0(\Sigma^n) = f_X(L)$. Finally, using this isotopy and the fact that Υ_X^{-1} is mapping pages of $\mathcal{OB}_{(X,h)}$ to the corresponding pages of $\mathcal{OB}$, we conclude that the Legendrian submanifold

$$\Psi_1(\Sigma^n) = \Upsilon_X^{-1}(S')$$

is Legendrian isotopic to $f_X(L)$ and disjoint from a page of the open book $\mathcal{OB}$. Thus, setting $f := f_X$ and $\xi' := \xi_X$ finishes the proof of Theorem 1.1. □

4. Legendrian Contact Homology

As mentioned at the beginning, Ekholm, Etnyre and Sullivan (see [5]) introduced *Legendrian contact homology* $\mathcal{LCH}(L)$ for a Legendrian submanifold L embedded in $\mathbb{R}^{2n+1}$ equipped with the standard contact structure, and then later in the contact manifolds of the form $X \times \mathbb{R}$ where X is exact symplectic ([7]). This is done by associating a differential graded algebra (DGA) to a given Legendrian submanifold L whose homology is Legendrian isotopy invariant. In [7] the contact structure on $X \times \mathbb{R}$ is of the form $\beta + dz$ where z is the $\mathbb{R}$-coordinate and β is a fixed primitive for the exact symplectic structure on X. We will write

$$\mathcal{LCH}_{(X\times\mathbb{R},\beta)}(L)$$

for the Legendrian contact homology of a Legendrian submanifold L in $(X \times \mathbb{R}, \mathrm{Ker}(\beta + dz))$. The generators of DGA for $\mathcal{LCH}_{(X\times\mathbb{R},\beta)}(L)$ are the self-intersections of the projection of L onto X under the Lagrangian projection $X \times \mathbb{R} \to X \times \{0\}$. By an ϵ-small Legendrian isotopy near L, one may assume that all such self-intersections are transverse double points. Then one can define a boundary map for the Legendrian contact homology using the reeb chords in $X \times \mathbb{R}$ joining the points on L whose projections are these transverse double points (see [7] for details).

The goal of the present section is to prove Theorem 1.2 and Theorem 1.3. We will use the notations introduced for the proofs of Theorem 1.1 and Theorem 3.2. Let us start with a compact embedded Legendrian submanifold

$$L \subset (M(X,h), \xi_\beta)$$

of the abstract contact manifold $(M(X,h), \xi_\beta)$ constructed from an abstract contact open book (X,h) with Weinstein pages (See Remark 3.1). By Theorem 3.2, L can be Legendrian isotoped to another embedded Legendrian submanifold L' which is disjoint from the closure of a page, say X_0, of the open book $\mathcal{OB}_{(X,h)}$ on $M(X,h)$ associated to (X,h). By cutting $M(X,h)$ along the closure $\bar{X}_0$, we obtain a contactomorphism (respecting fibration maps and Reeb directions)

$$\Theta_{X_0} : (M(X,h) \setminus \bar{X}_0, \xi_\beta) \longrightarrow (X \times \mathbb{R}, \mathrm{Ker}(\beta + dz)).$$

Then $\Theta_{X_0}(L')$ is a compact embedded Legendrian submanifold in $(X \times \mathbb{R}, \mathrm{Ker}(\beta + dz))$, and therefore, one can make the following definition:

Definition 4.1. The *Legendrian contact homology*

$$\mathcal{LCH}_{(M(X,h),\xi_\beta)}(L)$$

of L in $(M(X,h), \xi_\beta)$ is defined to be the Legendrian contact homology

$$\mathcal{LCH}_{(X\times\mathbb{R},\beta)}(\Theta_{X_0}(L'))$$

of $\Theta_{X_0}(L')$ in $(X \times \mathbb{R}, \mathrm{Ker}(\beta + dz))$.

Remark 4.2. Note that if a given L already misses the closure of a page X_0, then L' can be taken as L. That is, we simply define

$$\mathcal{LCH}_{(M(X,h),\xi_\beta)}(L) \doteq \mathcal{LCH}_{(X\times\mathbb{R},\beta)}(\Theta_{X_0}(L)).$$

Proposition 4.3. *$\mathcal{LCH}_{(M(X,h),\xi_\beta)}(L)$ is well-defined and invariant under Legendrian isotopies.*

Proof. The only choices made in the definition of $\mathcal{LCH}_{(M(X,h),\xi_\beta)}(L)$ are the choice of a page X_0 along which we cut $M(X,h)$ (after making L and $\bar{X}_0$ disjoint), the Legendrian isotopy of L which results in L', and the identification map Θ_{X_0} (which is a contactomorphism).

Suppose Θ'_{X_0} is another contactomorphism identifying $(M(X,h)\setminus\bar{X}_0,\xi_\beta)$ with $(X\times\mathbb{R},\mathrm{Ker}(\beta+dz))$. Then the map $\Theta'_{X_0}\circ\Theta^{-1}_{X_0}$ is a self-contactomorphism of $(X\times\mathbb{R},\mathrm{Ker}(\beta+dz))$ which maps $\Theta_{X_0}(L')$ to $\Theta'_{X_0}(L')$, and therefore, by functoriality, there is an DGA-isomorphism

$$\mathcal{LCH}_{(X\times\mathbb{R},\beta)}(\Theta_{X_0}(L'))\cong\mathcal{LCH}_{(X\times\mathbb{R},\beta)}(\Theta'_{X_0}(L'))$$

induced by $\Theta'_{X_0}\circ\Theta^{-1}_{X_0}$. Hence, the definition of $\mathcal{LCH}_{(M(X,h),\xi_\beta)}(L)$ is independent of the chosen identification Θ_{X_0} upto isomorphism of DGA's.

In order to see that the definition is independent of the page X_0, suppose $X_1\subset M(X,h)$ is another page of $\mathcal{OB}_{(X,h)}$ and assume L is Legendrian isotoped to L' so that $L'\cap\bar{X}_0=\emptyset$ (as in Theorem 3.2). In particular, L' does not meet ∂X_1 as well. Consider the image $\Theta_{X_0}(L')$ under some fixed identification (contactomorphism)

$$\Theta_{X_0}:(M(X,h)\setminus\bar{X}_0,\xi_\beta=\mathrm{Ker}(\alpha_\beta))\longrightarrow(X\times\mathbb{R},\mathrm{Ker}(\beta+dz)).$$

Note that the Reeb vector field R_{α_β} of the contact form α_β (restricted to $M(X,h)\setminus\bar{X}_0$) is mapped to the Reeb vector field $\partial/\partial z$ under the push forward map of Θ_{X_0}. By shifting all the points (in a finite time) of the compact set $\Theta_{X_0}(L')$ along the contact vector field $\partial/\partial z$, one can obtain a Legendrian submanifold $K\subset X\times\mathbb{R}\setminus\Theta_{X_0}(X_1)$ which is Legendrian isotopic to $\Theta_{X_0}(L')$ in $X\times\mathbb{R}$ as depicted in Figure 5. By pulling back this isotopy using $\Theta^*_{X_0}$ we obtain a Legendrian submanifold $L''\doteq\Theta^{-1}_{X_0}(K)\subset M(X,h)\setminus(\bar{X}_0\cup\bar{X}_1)$ which is Legendrian isotopic to L' in $M(X,h)\setminus\bar{X}_0$.

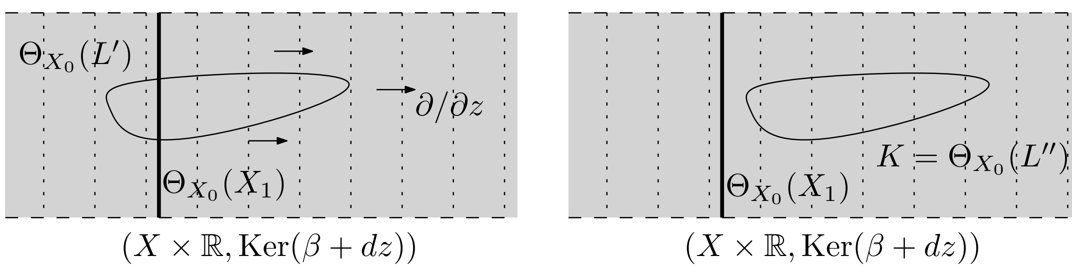

FIGURE 5. Pushing along the Reeb direction $\partial/\partial z$.

Now since $\mathcal{LCH}_{(X\times\mathbb{R},\beta)}$ is invariant under Legendrian isotopies (see [7]), we have

$$\mathcal{LCH}_{(X\times\mathbb{R},\beta)}(\Theta_{X_0}(L'))\cong\mathcal{LCH}_{(X\times\mathbb{R},\beta)}(K)=\mathcal{LCH}_{(X\times\mathbb{R},\beta)}(\Theta_{X_0}(L'')).$$

Note that L'' can be used to define $\mathcal{LCH}_{(M(X,h),\xi_\beta)}(L)$ after cutting $M(X,h)$ along $\bar{X}_1$. Moreover, observe Θ_{X_0} can be considered as an identification map which can be used to replace the one

$$\Theta_{X_1}:(M(X,h)\setminus\bar{X}_1,\xi_\beta)\longrightarrow(X\times\mathbb{R},\mathrm{Ker}(\beta+dz))$$

obtained by cutting along $\bar{X}_1$. Such a replacement corresponds to a self-contactomorphism $\Theta_{X_0}\circ\Theta^{-1}_{X_1}$ of $(X\times\mathbb{R},\mathrm{Ker}(\beta+dz))$ (shifting along the z-direction), and therefore, will produce an DGA-isomorphism. Hence, we have

$$\mathcal{LCH}_{(X\times\mathbb{R},\beta)}(\Theta_{X_0}(L'))\cong\mathcal{LCH}_{(X\times\mathbb{R},\beta)}(\Theta_{X_0}(L''))\cong\mathcal{LCH}_{(X\times\mathbb{R},\beta)}(\Theta_{X_1}(L''))$$

which proves that the definition $\mathcal{LCH}_{(M(X,h),\xi_\beta)}(L)$ is independent of the choice of X_0.

Before checking $\mathcal{LCH}_{(M(X,h),\xi_\beta)}(L)$ is independent of the choice L' in its definition, let's first check a more general statement which has been also claimed in Proposition 4.3.

Lemma 4.4. *$\mathcal{LCH}_{(M(X,h),\xi_\beta)}(L)$ is invariant under Legendrian isotopies of L.*

Proof. Suppose that $L_0 = L, L_1$ are Legendrian isotopic in $(M(X,h), \xi_\beta)$, and let

$$\Phi : \Sigma^n \times [0,1] \longrightarrow (M(X,h), \xi_\beta)$$

be an isotopy of Legendrian embeddings such that the smooth one-parameter family

$$\Phi_t : \Sigma^n \hookrightarrow (M(X,h), \xi_\beta)$$

given by $\Phi_t(p) = \Phi(p,t)$ defines a Legendrian isotopy between $L = L_0 = \Phi_0(\Sigma)$ and $L_1 = \Phi_1(\Sigma)$. First, by following similar steps as in the proof of Lemma 3.4, we may assume that the image $\Phi(\Sigma^n \times [0,1])$, i.e., the whole isotopy connecting L_0 to L_1, is everywhere transverse to the binding of the open book $\mathcal{OB}_{(X,h)}$. Secondly, modifying the proof of Lemma 3.5 (more precisely, by pushing along contact vector fields in a small ϵ-neighbourhood of $\Phi(\Sigma^n \times [0,1])$), one can make $\Phi(\Sigma^n \times [0,1])$ disjoint from the core $\Delta = \mathrm{Core}(X, \beta)$ of a generic page X of $\mathcal{OB}_{(X,h)}$. Here the key point is that any connected component of the intersection of $\Phi(\Sigma^n \times [0,1])$ with Δ deformation retracts onto the corresponding component of the intersection of Δ with $\Phi(\Sigma^n \times \{t\})$ for any $t \in [0,1]$. Therefore, we may assume that the image $\Phi(\Sigma^n \times [0,1])$ is disjoint from the core Δ.

Finally, as in the last part (starting after Remark 3.6) of the proof of Theorem 3.2, we first isotope $\Phi(\Sigma^n \times [0,1])$ along a contact vector field generated from Z_3 given in the list (1) so that the intersection $\Phi(\Sigma^n \times [0,1]) \cap X$ is completely pushed into the interior of $B \times D^2$ (tubular neighbourhood of the binding $B = \partial X$ of $\mathcal{OB}_{(X,h)}$). Then using Z_1, Z_2 in (1) we isotope $\Phi(\Sigma^n \times [0,1])$ until $\Phi(\Sigma^n \times [0,1]) \cap (B \times D^2)$ completely crosses the binding $B = B \times \{0\}$.

The above argument shows that a given isotopy $\Phi : \Sigma^n \times [0,1] \longrightarrow (M(X,h), \xi_\beta)$ can be isotoped (through contact vector fields) so that it becomes disjoint from the closure of a page, say X_0, and therefore, it gives rise to an isotopy $\Phi' : \Sigma^n \times [0,1] \longrightarrow (M(X,h) \setminus \bar{X}_0, \xi_\beta)$ of Legendrian embeddings which defines a Legendrian isotopy $\Phi'_t : \Sigma^n \hookrightarrow (M(X,h) \setminus \bar{X}_0, \xi_\beta)$ given by $\Phi'_t(p) = \Phi'(p,t)$ between $L' := \Phi'_0(\Sigma)$ (which is Legendrian isotopic to L_0) and $L'' := \Phi'_1(\Sigma)$ (which is Legendrian isotopic to L_1). Thus, after fixing an identification Θ_{X_0}, we obtain an isotopy

$$\Psi := \Theta_{X_0} \circ \Phi' : \Sigma^n \times [0,1] \longrightarrow (X \times \mathbb{R}, \mathrm{Ker}(\beta + dz))$$

of Legendrian embeddings which defines a Legendrian isotopy

$$\Psi'_t : \Sigma^n \hookrightarrow (X \times \mathbb{R}, \mathrm{Ker}(\beta + dz))$$

given by $\Psi'_t(p) = \Psi'(p,t)$ between $\Psi'_0(\Sigma) = \Theta_{X_0}(L')$ and $\Psi'_1(\Sigma) = \Theta_{X_0}(L'')$. Then since $\mathcal{LCH}_{(X\times\mathbb{R},\beta)}(\cdot)$ is invariant under Legendrian isotopies (see [7]), there exists an DGA-isomorphism

$$[\mathcal{LCH}_{(M(X,h),\xi_\beta)}(L_0) \doteq]\ \mathcal{LCH}_{(X\times\mathbb{R},\beta)}(\Theta_{X_0}(L')) \cong \mathcal{LCH}_{(X\times\mathbb{R},\beta)}(\Theta_{X_0}(L''))\ [\doteq \mathcal{LCH}_{(M(X,h),\xi_\beta)}(L_1)].$$

Hence, we conclude that $\mathcal{LCH}_{(M(X,h),\xi_\beta)}(L)$ is invariant under Legendrian isotopies of L. □

Finally, it remains to show that $\mathcal{LCH}_{(M(X,h),\xi_\beta)}(L)$ is independent of the choice L' in its definition. Suppose $L', L'' \subset (M(X,h) \setminus \bar{X}_0, \xi_\beta)$ are both obtained from L via Legendrian isotopies, say Φ'_t, Φ''_t, respectively, constructed as in the proof of Theorem 3.2. Here we consider $\Phi'_t, \Phi''_t : \Sigma \hookrightarrow M(X,h)$ as the compositions of all isotopies (constructed in the proof) taking $L = \Phi'_0(\Sigma) = \Phi''_0(\Sigma)$ to $L' = \Phi'_1(\Sigma)$, $L'' = \Phi''_1(\Sigma)$, respectively. Then the map Φ'_{1-2t} (where $t \in [0,1/2]$) followed by Φ''_{2t-1} (where $t \in [1/2,1]$) defines a Legendrian isotopy between L' and L'' in $(M(X,h), \xi_\beta)$, and hence the result follows from Lemma 4.4.

This finishes the proof of Proposition 4.3. □

Proof of Theorem 1.2. Assume that L is a compact embedded Legendrian submanifold of $(M, \xi_{\mathcal{OB}})$ and $\mathcal{OB}$ has Weinstein pages. By Theorem 1.1 (first apply the theorem by taking $\xi = \xi_{\mathcal{OB}}$, and then replace $\xi_{\mathcal{OB}}$ with ξ' in the statement), we may assume that L can be Legendrian isotoped in $(M, \xi_{\mathcal{OB}})$ to an another Legendrian submanifold which is disjoint from the closure of a page of $\mathcal{OB}$. Moreover, by fixing a page $(X, d\beta)$ of $\mathcal{OB}$, we have a contactomorphism (as in the proof of Theorem 1.1)

$$\Upsilon_X : (M, \xi_{\mathcal{OB}}) \to (M(X,h), \xi_\beta)$$

mapping pages of $\mathcal{OB}$ to those of $\mathcal{OB}_{(X,h)}$. Then one can make the following definition:

Definition 4.5. The *Legendrian contact homology*

$$\mathcal{LCH}_{(M,\xi_{\mathcal{OB}})}(L)$$

of L in $(M, \xi_{\mathcal{OB}})$ is defined to be the Legendrian contact homology

$$\mathcal{LCH}_{(M(X,h),\xi_\beta)}(\Upsilon_X(L))$$

of $\Upsilon_X(L)$ in $(M(X,h), \xi_\beta)$.

Here one needs to check:

Lemma 4.6. *$\mathcal{LCH}_{(M,\xi_{\mathcal{OB}})}(L)$ is well-defined and invariant under Legendrian isotopies.*

Proof. The only choice made in the definition is the choice of a page X of $\mathcal{OB}$ that we fixed. If X' is any other page of $\mathcal{OB}$ with the Liouville form β', then the corresponding first return flow determines $h' \in \mathrm{Symp}(X', d\beta')$ and we obtain another abstact open book $\mathcal{OB}_{(X',h')}$ supporting the contact structure $\xi_{\beta'}$ on $M(X', h')$ (see Remark 3.1). Then we have a contactomorphism

$$\Upsilon_{X'} : (M, \xi_{\mathcal{OB}}) \to (M(X', h'), \xi_{\beta'})$$

mapping pages of $\mathcal{OB}$ to those of $\mathcal{OB}_{(X',h')}$. Then the map

$$\Upsilon_{X'} \circ \Upsilon_X^{-1} : (M(X,h), \xi_\beta) \to (M(X', h'), \xi_{\beta'})$$

is a contactomorphism mapping pages of $\mathcal{OB}_{(X,h)}$ to those of $\mathcal{OB}_{(X',h')}$. Also it maps $\Upsilon_X(L)$ to $\Upsilon_{X'}(L)$, and so, by functoriality, there is an DGA-isomorphism

$$\mathcal{LCH}_{(M(X,h),\xi_\beta)}(\Upsilon_X(L)) \cong \mathcal{LCH}_{(M(X',h'),\xi_{\beta'})}(\Upsilon_{X'}(L))$$

induced by $\Upsilon_{X'} \circ \Upsilon_X^{-1}$. Hence, the definition of $\mathcal{LCH}_{(M,\xi_{\mathcal{OB}})}(L)$ is independent of the page X of $\mathcal{OB}$ that we fix upto isomorphism of DGA's.

In order to check the invariance of $\mathcal{LCH}_{(M,\xi_{\mathcal{OB}})}(L)$ under Legendrian isotopies of L, suppose we are given a smooth one-parameter family

$$\Psi_t : \Sigma^n \hookrightarrow (M, \xi_{\mathcal{OB}}), \quad t \in [0,1]$$

which gives a Legendrian isotopy between $L = L_0 = \Psi_0(\Sigma^n)$ and $L_1 := \Psi_1(\Sigma^n)$. Composing with the contactomorphism Υ_X, we obtain a smooth one-parameter family

$$\Phi_t := \Upsilon_X \circ \Psi_t : \Sigma^n \hookrightarrow (M(X,h), \xi_\beta), \quad t \in [0,1].$$

In other words, Φ_t is a Legendrian isotopy between $\Upsilon_X(L_0) = \Phi_0(\Sigma^n)$ and $\Upsilon_X(L_1) = \Phi_1(\Sigma^n)$. Then since $\mathcal{LCH}_{(M(X,h),\xi_\beta)}(\cdot)$ is invariant under Legendrian isotopies (Lemma 4.4), we conclude

$$\mathcal{LCH}_{(M,\xi_{\mathcal{OB}})}(L_0) \doteq \mathcal{LCH}_{(M(X,h),\xi_\beta)}(\Upsilon_X(L_0)) \cong \mathcal{LCH}_{(M(X,h),\xi_\beta)}(\Upsilon_X(L_1)) \doteq \mathcal{LCH}_{(M,\xi_{\mathcal{OB}})}(L_1)$$

as desired. □

This finishes the proof of Theorem 1.2. □

Proof of Theorem 1.3. Suppose L is a compact embedded Legendrian submanifold of a closed contact manifold (M, ξ). Say M is of dimension $2n+1$. There exists a contact open book $\mathcal{OB}$ with Weinstein pages (of dimension $2n$) supporting ξ as mentioned in the introduction. As before let us use $\xi_{\mathcal{OB}}$ to denote a contact structure supported by $\mathcal{OB}$. (Note any $\xi_{\mathcal{OB}}$ must be necessarily isotopic to ξ.) Then by simply considering ξ as such $\xi_{\mathcal{OB}}$ we define:

Definition 4.7. The *Legendrian contact homology*

$$\mathcal{LCH}_{(M,\xi)}(L)$$

of L in (M, ξ) is defined to be the Legendrian contact homology

$$\mathcal{LCH}_{(M,\xi_{\mathcal{OB}})}(L)$$

of L in $(M, \xi_{\mathcal{OB}})$.

As before we need to check:

Lemma 4.8. *$\mathcal{LCH}_{(M,\xi)}(L)$ is well-defined and invariant under Legendrian isotopies.*

Proof. The only choice made in the definition is the choice of an open book supporting ξ. Suppose that $\mathcal{OB}1$, $\mathcal{OB}2$ are open books on M both supporting ξ and having Weinstein pages. We need to show that there is an DGA-isomorphism

$$\mathcal{LCH}_{(M,\xi_{\mathcal{OB}1})}(L) \cong \mathcal{LCH}_{(M,\xi_{\mathcal{OB}2})}(L)$$

for any compact embedded Legendrian submanifold $L \subset (M, \xi)$.

By the argument used at the beginning of the proof of Theorem 1.2 and the invariance of $\mathcal{LCH}_{(M,\xi_{\mathcal{OB}})}(\cdot)$ under Legendrian isotopies, we may assume that L is already disjoint from the closures of a page X_1 of $\mathcal{OB}1$ and a page X_2 of $\mathcal{OB}2$. Let $\mathcal{OB}1_{(X_1,h_1)}$ (resp. $\mathcal{OB}2_{(X_2,h_2)}$) be an abstract contact open book for $\mathcal{OB}1$ (resp. $\mathcal{OB}2$) obtained by fixing the exact symplectic page $(X_1, d\beta_1)$ of $\mathcal{OB}1$ (resp. $(X_2, d\beta_2)$ of $\mathcal{OB}2$) that L does not meet. As before denote by $\Upsilon_{X_1} : (M, \xi_{\mathcal{OB}1}) \to (M(X_1, h_1), \xi_{\beta_1})$ (resp. $\Upsilon_{X_2} : (M, \xi_{\mathcal{OB}2}) \to (M(X_2, h_2), \xi_{\beta_2})$) a (page preserving) contactomorphism identifying $\mathcal{OB}1$ (resp. $\mathcal{OB}2$) on M with its abstract copy $\mathcal{OB}1_{(X_1,h_1)}$ on $M(X_1, h_1)$ (resp. $\mathcal{OB}2_{(X_2,h_2)}$ on $M(X_2, h_2)$).

As mentioned in the introduction, $\mathcal{OB}1$, $\mathcal{OB}2$ have a common stabilization, say $\mathcal{OB}3$, also supporting ξ and having Weinstein pages. One can describe $\mathcal{OB}3$ as an abstract open book $\mathcal{OB}3_{(X_3,h_3)}$ in the following way: Say $\mathcal{OB}3$ is obtained from $\mathcal{OB}1$ (resp. $\mathcal{OB}2$) by performing m_1 (resp. m_2) succesive positive stabilizations. A positive stabilization of an abstract open book involves two steps in first of which every (Weinstein) page gains a Weinstein handle of the critical index n (the half dimension of a page). The second step is adjoining a new positive Dehn twist to the monodromy (see, for instance, [1] for details). In our case, suppose X_3 is the Weinstein manifold obtained from X_1 by attaching m_1 (resp. m_2) $2n$-dimensional Weinstein handles H_1^j (resp. H_2^j) along Legendrian $(n-1)$-spheres L_1^j in the contact boundary $(\partial X_1, \beta_1|_{\partial X_1})$ (resp. L_2^j in the contact boundary $(\partial X_2, \beta_2|_{\partial X_2})$) for $j = 1, ..., m_i$ and $i = 1, 2$. The second step of each stabilization adjoins a positive Dehn twist τ_i^j to h_i along an (honest) Lagrangian n-sphere S_i^j obtained by gluing the (Lagrangian) core disk of H_i^j and some properly embedded Lagrangian disk in $(X_i, d\beta_i)$ together along their common boundary L_i^j in the standard way, and so we have

$$h_1 \tau_1^1 \tau_1^2 \cdots \tau_1^{m_1} = h_3 = h_2 \tau_2^1 \tau_2^2 \cdots \tau_2^{m_2}.$$

Now, since L is disjoint from the closure of X_i for each i, we have $L \cap L_i^j = \emptyset$ for each $j = 1, ..., m_i$, and so all Weinstein handles H_i^j are attached away from L which implies that L also misses a page X_3 of $\mathcal{OB}3$. So after fixing identifications $\Upsilon_{X_3} : (M, \xi_{\mathcal{OB}3}) \to (M(X_3, h_3), \xi_{\beta_3})$ and

$$\Theta_{X_3} : (M(X_3, h_3) \setminus \bar{X}_3, \xi_{\beta_3}) \longrightarrow (X_3 \times \mathbb{R}, \mathrm{Ker}(\beta_3 + dz)),$$

we obtain a compact Legendrian submanifold $\Theta_{X_3}(\Upsilon_{X_3}(L))$ in $(X_3 \times \mathbb{R}, \mathrm{Ker}(\beta_3 + dz))$ once we cut $\mathcal{OB}3_{(X_3,h_3)}$ along some $\bar{X}_3$ that $\Upsilon_{X_3}(L)$ does not meet. Also the construction of X_3 implies that the interior of $(X_i, d\beta_i)$ is a Weinstein subdomain of $(X_3, d\beta_3)$ for each i, and hence, we have two (codimension zero) contact embeddings

$$f_i : (X_i \times \mathbb{R}, \mathrm{Ker}(\beta_i + dz)) \hookrightarrow (X_3 \times \mathbb{R}, \mathrm{Ker}(\beta_3 + dz)) \quad (i = 1, 2).$$

Moreover, by choosing an identification map

$$\Theta_{X_i} : (M(X_i, h_i) \setminus \bar{X}_i, \xi_{\beta_i}) \longrightarrow (X_i \times \mathbb{R}, \mathrm{Ker}(\beta_i + dz))$$

as the restriction of Θ_{X_3} (note $(M(X_i, h_i) \setminus \bar{X}_i, \xi_{\beta_i}) \subset (M(X_3, h_3) \setminus \bar{X}_3, \xi_{\beta_3})$ by construction) we guarantee that f_i respects the obvious fibration maps onto the second factors and the Reeb directions (i.e., $\partial/\partial z$), and maps $\Theta_{X_i}(\Upsilon_{X_i}(L))$ to $\Theta_{X_3}(\Upsilon_{X_3}(L))$ for each $i = 1, 2$. The key observation here is that the Reeb coords and the projections of $\Theta_{X_i}(\Upsilon_{X_i}(L))$ and $\Theta_{X_3}(\Upsilon_{X_3}(L))$ onto X_i and X_3 (respectively) are the same. Therefore, f_i induces an DGA-isomorphism

$$F_i : \mathcal{LCH}_{(X_i \times \mathbb{R}, \beta_i)}(\Theta_{X_i}(\Upsilon_{X_i}(L))) \longrightarrow \mathcal{LCH}_{(X_3 \times \mathbb{R}, \beta_3)}(\Theta_{X_3}(\Upsilon_{X_i}(L)))$$

which implies (recall Definition 4.1 and Remark 4.2) that

$$\mathcal{LCH}_{(M(X_1,h_1),\xi_{\beta_1})}(\Upsilon_{X_1}(L)) \cong \mathcal{LCH}_{(M(X_3,h_3),\xi_{\beta_3})}(\Upsilon_{X_3}(L)) \cong \mathcal{LCH}_{(M(X_2,h_2),\xi_{\beta_2})}(\Upsilon_{X_2}(L)).$$

Hence, we conclude from Definition 4.5 that

$$\mathcal{LCH}_{(M,\xi_{\mathcal{OB}1})}(L) \cong \mathcal{LCH}_{(M,\xi_{\mathcal{OB}3})}(L) \cong \mathcal{LCH}_{(M,\xi_{\mathcal{OB}2})}(L)$$

as desired.

The invariance of $\mathcal{LCH}_{(M,\xi)}(L)$ under Legendrian isotopies of L can be easily seen as follows: Observe that any Legendrian isotopy $\Phi_t : \Sigma^n \hookrightarrow (M, \xi)$ between $L = L_0 = \Phi_0(\Sigma^n)$ and $L_1 := \Phi_1(\Sigma^n)$ can be also thought as a Legendrian isotopy between L_0 and L_1 in $(M, \xi_{\mathcal{OB}})$ by considering supported contact structure $\xi_{\mathcal{OB}}$ as ξ. Therefore, one gets by Lemma 4.6 that

$$\mathcal{LCH}_{(M,\xi)}(L_0) \doteq \mathcal{LCH}_{(M,\xi_{\mathcal{OB}})}(L_0) \cong \mathcal{LCH}_{(M,\xi_{\mathcal{OB}})}(L_1) \doteq \mathcal{LCH}_{(M,\xi)}(L_1).$$

□

This finishes the proof of Theorem 1.3. □

References

[1] S. Akbulut and M. F. Arikan, *Stabilizations via Lefschetz Fibrations and Exact Open Books*, Preprint (arXiv:1112.0519).

[2] F. Bourgeois, T. Ekholm and Y. Eliashberg, *Effect of Legendrian surgery*, Geom.Topology 16(2012) 301-389.

[3] Y. Chekanov, *Differential algebras of Legendrian links*, Invent. Math. 150 (2002), no. 3, 441483.

[4] K. Cieliebak and Y. Eliashberg, *From Stein to Weinstein and back. Symplectic geometry of affine complex manifolds*, American Mathematical Society Colloquium Publications, 59. American Mathematical Society, Providence, RI, 2012.

[5] T. Ekholm, J. Etnyre and M. Sullivan, *The Contact Homology of Legendrian Submanifolds in* $\mathbb{R}^{2n+1}$, J. Differential Geom. 71 (2005), no. 2, 177-305.

[6] T. Ekholm, J. Etnyre and M. Sullivan, *Non-isotopic Legendrian submanifolds in* $\mathbb{R}^{2n+1}$, J. Differential Geom. 71 (2005), no. 1, 85-128.

[7] T. Ekholm, J. Etnyre and M. Sullivan, *Legendrian Contact Homology in $P \times \mathbb{R}$*, Trans. Amer. Math. Soc. 359 (2007), no. 7, 3301-3335 (electronic).
[8] Y. Eliashberg, A. Givental, and H. Hofer, *Introduction to symplectic field theory*, GAFA 2000 (Tel Aviv, 1999). Geom. Funct. Anal. 2000, Special Volume, Part II, 560673.
[9] J. B. Etnyre, *Open book decompositions and the Giroux correspondence*, Lecture notes from Introductory Workshop: Symplectic and Contact Geometry and Topology, MSRI, Berkeley CA, August 17-21, 2009. (http://people.math.gatech.edu/∼etnyre/talks/MSRI-OBDSurvey.pdf)
[10] H. Geiges, *An Introduction to Contact Topology, Cambridge University Press*, (2008).
[11] E. Giroux, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing), Higher Ed. Press, (2002), pp. 405-414. MR 2004c:53144
[12] E. Giroux, *Contact structures and symplectic fibrations over the circle*, trans- parencies of a seminar talk.
[13] Z. Sylvan, *On partially wrapped Fukaya categories*, arXiv:1604.02540v2 [math.SG]
[14] W. P. Thurston, H. E. Winkelnkemper, *On the existence of contact forms*, Proc. Amer. Math. Soc. **52** (1975), 345–347.
[15] A. Weinstein, *Contact surgery and symplectic handlebodies*, Hokkaido Math. J. 20 (1991), no. 2, 241-251.

Department of Mathematics, Michigan State University, Lansing MI, USA
E-mail address: akbulut@math.msu.edu

Dept. of Mathematics, Middle East Technical University, Ankara, TURKEY
E-mail address: farikan@metu.edu.tr