LEFSCHETZ FIBRATIONS ON COMPACT STEIN MANIFOLDS

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ABSTRACT. Here we prove that up to diffeomorphism every compact Stein manifold W of dimension $2n+2>4$ admits a Lefschetz fibration over the disk D^2 with Stein regular fibers, such that the monodromy of the fibration is a symplectomorphism induced by compositions of right-handed Dehn twists along embedded Lagrangian n-spheres on the generic fiber. This generalizes the Stein surface case of $n=1$, previously proven by Loi-Piergallini and Akbulut-Ozbagci. More precisely, we show that up to Liouville isomorphism any Weinstein domain W admits a compatible compact convex Lefschetz fibration with Weinstein regular fibers and with the same monodromy description stated above. Moreover, the boundary convex open book supports the induced contact structure on ∂W.

1. INTRODUCTION

In [AO] (see also [LP]), it was proved that every compact Stein surface admits a positive allowable Lefschetz fibration over D^2 with bounded fibers, and conversely every 4-dimensional positive Lefschetz fibration over D^2 with bounded fibers is a Stein surface. Here we prove the following, which can be thought as a generalization of this result to higher dimensions:

Theorem 1.1. *For any Stein domain W of dimension $2n+2>4$, there exists a Liouville domain W', diffeomorphic to W, such that with the underlying Liouville structure W is Liouville isomorphic to W', and W' admits a Liouville Lefschetz fibration with Stein generic fibers. The monodromy of the fibration is a symplectomorphism induced by compositions of right-handed Dehn twists along embedded Lagrangian n-spheres on the generic fiber. Moreover, the induced convex open book is compatible with the induced contact structure on $\partial W'$.*

For precise definitions, we refer the reader to the next section. Theorem 1.1 and its Weinstein counterpart (Theorem 3.1) have the following immediate consequence in the smooth category:

Theorem 1.2. *Up to diffeomorphism any compact Stein (or Weinstein) manifold of dimension $2n+2>4$ admits a Lefschetz fibration over D^2 with Stein regular fibers.* □

Eliashberg showed that any compact smooth manifold W^{2n+2} of dimension $2n+2>4$ has handles of index $\leq n+1$ if and only if it has a Stein structure [E1]. So we can assume that W^{2n+2} is obtained from a subcritical Stein manifold by attaching handles with index $n+1$. Similar result also holds for compact Weinstein manifolds [W, CE]. Also a theorem of Cieliebak [C] implies that any compact subcritical Stein (or Weinstein) manifold decomposes as $X^{2n}\times D^2$, where X is a compact Stein (or Weinstein) manifold. We use this decomposition as the starting point of our proof. We should point out that our proof here does not apply to the case of Stein surfaces ($n=1$ case), where one needs a different approach of [AO]. By using [K], in dimension

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6, Theorem 1.2 comes with the following converse: *If a smooth compact 6-manifold W admits a Lefschetz fibration over D^2 with fibers having nonempty boundary, then there exists a Stein (and hence a Weinstein) structure on W.* (See Remark 3.16). Since the previous posting of our paper, we were informed that in a talk Giroux also announced a result similar to the one given in Theorem 1.1.

Before starting the proof of Theorem 1.1 in Section 3, we will first recall some definitions and basic facts in Sections 2. For more details and the proofs of the statements, we refer the reader to [K] for Lefschetz fibrations, to [E1, CE] for Stein and Weinstein manifolds, to [Ge, Gi1] for contact structures and open books, and to [MS] for symplectic geometry.

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2. Preliminaries

2.1. **Liouville, Weinstein and Stein Manifolds.** We first recall Weinstein and Stein manifolds and their relations by following [EG], [CE], [S3] and [W]: A contact manifold (M, Λ) is called *strongly symplectically filled* by a symplectic manifold (W, Ω) if there exist an *expanding Liouville vector field* Z of Ω, satisfying $\mathcal{L}_Z\Omega = \Omega$, which is defined (at least) locally near $\partial W = M$ such that Z is transverse to M and $\iota_Z\Omega = \Lambda$. Such a boundary is called *convex.* A *compact convex symplectic manifold* is a compact manifold W with boundary, together with an exact symplectic form $\Omega = d\Lambda$, such that the Ω-dual vector field Z of Λ (i.e., $\iota_Z\Omega = \Lambda$) transversally points out from ∂W. In such a case, Λ is called a *Liouville form* on W, and $\Lambda|_{\partial W}$ is a contact form which makes ∂W convex. We will interchangeably denote compact convex symplectic manifolds in the forms (W, Ω, Λ, Z), (W, Ω, Λ), (W, Ω, Z) or (W, Λ, Z), or sometimes by pairs of the form (W, Λ). (Note that Λ determines both Ω and Z, and also any two of the three determine the third.)

Definition 2.1. A *complete convex symplectic manifold* (or simply a *Liouville manifold*) is a noncompact manifold W together with an exact symplectic form $\Omega = d\Lambda$ such that

(i) the (expanding) Ω-dual vector field Z of Λ is *complete* (i.e., its flow exists for all times), and
(ii) (W, Λ, Z) is *(symplectically) convex* which means that there exists an exhaustion $W = \bigcup_{k=1}^{\infty} W^k$ by compact domains $W^k \subset W$ with smooth boundaries along which Z points outward. In other words, $(W_k, \Lambda|_{W_k})$ is a compact convex symplectic manifold with convex boundary $(\partial W_k, \Lambda|_{\partial W_k})$ for all $k \geq 1$.

The notations introduced above for compact convex symplectic manifolds will be also used for Liouville manifolds (as well as for Liouville cobordisms and domains, see below). If $Z^{-t} : W \to W$ $(t > 0)$ denotes the contracting flow of Z, then the *core* (or *skeleton*) of the Liouville manifold (W, Λ) is defined to be the set

$$\mathrm{Core}(W, \Lambda) := \bigcup_{k=1}^{\infty} \bigcap_{t>0} Z^{-t}(W^k).$$

A *Liouville cobordism* (W, Λ, Z) is a compact cobordism W with a convex symplectic structure (Λ, Z) such that Z points outwards along $\partial_+ W$ and inwards along $\partial_- W$. A Liouville cobordism with $\partial_- W = \emptyset$ is called a *compact complete convex symplectic manifold* (or simply a *Liouville domain*).

Remark 2.2. As the interior of the core of any Liouville manifold is empty (Lemma 11.1, [CE]), the core of any Liouville domain (W, Λ, Z) is compact. If M denotes the convex boundary

$\partial_+ W(=\partial W)$, then one can see that the negative half of the symplectization $(M \times \mathbb{R}, d(e^t \Lambda|_M))$ symplectically embeds into W (as a collar neighborhood of M in W) so that its complement in W is $\mathrm{Core}(W, \Lambda)$ and the embedding matches the positive t-direction of $\mathbb{R}$ with Z. The *completion* of a Liouville domain W is a Liouville manifold $(\hat{W}, \hat{\Lambda}, \hat{Z})$ obtained from W by gluing the positive part of the symplectization of its contact boundary. Two Liouville domains (W_i, Λ_i) are said to be *Liouville isomorphic* if there exists a diffeomorphism $\phi : (\hat{W}_0, \hat{\Lambda}_0) \to (\hat{W}_1, \hat{\Lambda}_1)$, called a *Liouville isomorphism*, between their completions $(\hat{W}_i, \hat{\Lambda}_i)$ such that $\phi^*(\hat{\Lambda}_1) = \hat{\Lambda}_0 + df$ where f is some compactly supported smooth function on $\hat{W}_1$.

We will need to deform Liouville structures in their homotopy classes. For completeness we now give related definitions and facts.

Definition 2.3. ([CE]) A smooth family (W, Ω_t, Z_t), $t \in [0, 1]$, of Liouville manifolds is called a *simple Liouville homotopy* if there exists a smooth family of exhaustions $W = \cup_{k=1}^{\infty} W_t^k$ by compact domains $W_t^k \subset W$ with smooth boundaries along which Z_t is outward pointing. A smooth family (W, Ω_t, Z_t), $t \in [0, 1]$, of Liouville manifolds is called *Liouville homotopy* if it is a composition of finitely many simple homotopies. Also a Liouville manifold is of *finite type* if its core is compact.

Lemma 2.4 ([CE], Lemma 11.6 and Proposition 11.8)**.** *Let (W, Ω_t, Z_t), $t \in [0, 1]$, be a smooth family of Liouville manifolds with the Liouville forms Λ_t. If each (W, Ω_t, Z_t) is of finite type and the closure $\overline{\cup_{t\in[0,1]} \mathrm{Core}(W, \Omega_t, Z_t)}$ of the union of their cores is compact, then (W, Ω_t, Z_t) is a Liouville homotopy and there exists a diffeotopy $\phi_t : W \to W$ with $\phi_0 = id$ such that $\phi^*(\Lambda_t) - \Lambda_0 = df_t$ for some smooth family $f_t : W \to \mathbb{R}$. In particular, any smooth family of Liouville domains defines a Liouville homotopy for the completions and the corresponding f_t's have compact supports, and hence two Liouville domains connected by a smooth family of Liouville domains are Liouville isomorphic.*

To define Weinstein manifolds, we need three preliminary definitions:

Definition 2.5. (i) A vector field Z on a smooth manifold W is said to *gradient-like* for a smooth function $\Psi : W \to \mathbb{R}$ if $Z \cdot \Psi = \mathcal{L}_Z \Psi > 0$ away from the critical point of Ψ.
(ii) A real-valued function is said to be *exhausting* if it is proper and bounded from below.
(iii) An exhausting function $\Psi : W \to \mathbb{R}$ on a symplectic manifold (W, Ω) is said to be Ω-*convex* if there exists a complete Liouville vector field Z which is gradient-like for Ψ.

Definition 2.6. A *Weinstein manifold* (W, Ω, Z, Ψ) is a symplectic manifold (W, Ω) which admits a Ω-convex Morse function $\Psi : W \to \mathbb{R}$ whose complete gradient-like Liouville vector field is Z. The triple (Ω, Z, Ψ) is called a *Weinstein structure* on W. A *Weinstein cobordism* (W, Ω, Z, Ψ) is a Liouville cobordism (W, Ω, Z) whose Liouville vector field Z is gradient-like for a Morse function $\Psi : W \to \mathbb{R}$ which is constant on the boundary ∂W. A Weinstein cobordism with $\partial_- W = \emptyset$ is called *Weinstein domain.*

Remark 2.7. Any Weinstein manifold (W, Ω, Z, Ψ) can be exhausted by Weinstein domains $W_k = \{\Psi^{-1}(-\infty, d_k]\} \subset W$ where $\{d_k\}$ is an increasing sequence of regular values of Ψ, and therefore, any Weinstein manifold is a Liouville manifold. In particular, any Weinstein domain is a Liouville domain. Also we note that any Weinstein domain (W, Ω, Z, Ψ) has the convex boundary $(\partial W, \mathrm{Ker}(\iota_Z \Omega|_{\partial W}))$, and the completion of a Weinstein domain is a Weinstein manifold.

The following topologically characterizes Weinstein manifolds and will be used later.

Theorem 2.8 ([W], see also Lemma 11.13 in [CE])**.** *Any Weinstein manifold of dimension $2n$ admits a handle decomposition whose handles have indices at most n.*

Let us now recall Stein manifolds. A *Stein manifold* is a complex manifold (necessarily noncompact) which admits proper holomorphic embeddings in $\mathbb{C}^N$ for sufficiently large N. A complex manifold W is Stein if and only if it admits an *exhausting strictly plurisubharmonic* function, which is essentially characterized as being a proper function $\Psi : W \to \mathbb{R}$ that is bounded below and can be assumed to be a Morse function, whose regular level sets $\Psi^{-1}(c)$ are *strictly pseudoconvex*, where $\Psi^{-1}(c)$ is oriented as the boundary of the complex manifold $\Psi^{-1}(-\infty, c]$. If J is the underlying integrable almost complex structure on W, then $\Omega_\Psi := -d(d\Psi \circ J)$ defines an exact symplectic (indeed Kähler) form on W. Indeed, strictly plurisubharmonicity condition of Ψ is equivalent to the non-degeneracy condition of Ω_Ψ. We will consider Stein manifolds as triples of the form (W, J, Ψ) and denote Stein structures by pairs (J, Ψ). Also note that strictly plurisubharmonic functions are most commonly called *J-convex*.

A *Stein cobordism* (W, J, Ψ) is a compact cobordism W between $\partial_- W$ and $\partial_+ W$ which admits a complex structure J and a J-convex function Ψ such that $\partial_\pm W$ are regular level sets of Ψ. A compact complex manifold (W, J) with boundary is called a *Stein domain* if it admits a J-convex function such that the boundary ∂W is a level set. Therefore, the phrase "compact Stein manifold" should be understood as Stein domain.

Remark 2.9. Any Stein structure (J, Ψ) on W determines a Weinstein structure on W given by $(-d(d\Psi \circ J), \nabla_\Psi \Psi, \Psi)$ where ∇_Ψ is the gradient operator determined by the metric $g(\cdot, \cdot) = -d(d\Psi \circ J)(\cdot, J\cdot)$. Therefore, any Stein manifold is also a Weinstein manifold. According to the results of Eliashberg, it is also possible to construct a Stein structure from a given Weinstein structure, and we will need such approach at the end of the paper.

Similar to the Weinstein case, there is a topological characterization of Stein domains:

Theorem 2.10 ([E1])**.** *Let (W, J') be a compact almost complex $2(n+1)$-manifold with a handle decomposition whose handles have index $\leq n+1$. Then W admits a Stein domain structure (W, J, ψ) where J is homotopic to J', and ψ is a suitable plurisubharmonic function inducing the handle decomposition. Conversely, any Stein domain of dimension $2(n+1)$ admits a handle decomposition whose handles have index $\leq n+1$.*

Definition 2.11. A Stein or Weinstein manifold (or domain) of dimension $2n+2$ is called *subcritical* if it admits a handle decomposition with handles of index $\leq n$.

The following splitting result for subcritical Weinstein domains also holds in noncompact case, and has similar versions for Stein category as well [C]. The compact Weinstein version will be enough for us.

Theorem 2.12 ([C], also Lemma 11.22 in [CE])**.** *Any subcritical Weinstein domain (W, Ω, Z, Ψ) splits as*

$$(X \times D^2, \Omega_X + dx \wedge dy, Z_X + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y}, \Psi_X + x^2 + y^2)$$

where $(X, \Omega_X, Z_X, \Psi_X)$ is a Weinstein domain of dimension $2n$ and D^2 is the standard unit disk in $\mathbb{R}^2$ with the coordinates (x, y).

Remark 2.13. Through out the paper, for trivial products we will denote pull-backs of forms or obvious lifts of vector fields by their original symbols. So given a subcritical Weinstein domain as above we will write

$$\Omega = \Omega_X + dx \wedge dy \quad (\text{or } = \Omega_X + r dr \wedge d\theta),$$

$$Z = Z_X + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y} \quad (\text{or } = Z_X + \frac{r}{2}\frac{\partial}{\partial r}), \quad \text{and} \quad \Psi = \Psi_X + x^2 + y^2 \quad (\text{or } = \Psi_Z + r^2)$$

where (r, θ) are the polar coordinates on D^2.

2.2. **Lefschetz fibrations.** Let W^{2n+2}, $n \geq 1$, be a compact oriented smooth manifold.

Definition 2.14. A smooth map $\pi : W \to D^2 \subset \mathbb{C}$ is called a *Lefschetz fibration* if π has finitely many critical points, such that in a neighborhood of each critical point W admits a coordinate neighborhood with complex coordinates $w = (z_1, z_2, ..., z_{n+1})$, consistent with the given orientation of W where $\pi(p)$ has the representation:

$$\pi(w) = z_0 + z_1^2 + z_2^2 + ... + z_{n+1}^2.$$

By definition, π has finitely many critical values, say $\{\lambda_1, \lambda_2, .., \lambda_\mu\}$. The fiber $\pi^{-1}(\lambda_i)$ is called the *singular fiber* of π corresponding to λ_i. Consider the base disk as $D^2 = \{z \in \mathbb{C} : |z| \leq 2\}$. Composing π with an orientation preserving diffeomorphism of D^2, we can assume that the critical values are μ roots of unity (and so $(0,0) \in D^2$ is a regular value of π). Such a map π is called *normalized*. If a is a regular value, then $X = \pi^{-1}(a)$ is called a *regular fiber*. Being an honest fibration away from critical values, all regular fibers are diffeomorphic to X. If π is normalized, it induces an open book $\mathcal{OB}_\pi = (\partial\pi^{-1}(0,0), \pi|_{\partial W - \partial\pi^{-1}(0,0)})$ on the boundary ∂W. The *monodromy* of π is defined to be the monodromy of the associated open book on ∂W.

According to the handlebody description of W associated to π (see [K]), W is obtained from $X \times D^2$ by attachhing the handles $H_i \cong D^{n+1} \times D^{n+1}$, $1 \leq i \leq \mu$, along the embedded n-spheres $\phi_i : S^n \hookrightarrow V_i \subset X \times \{p\}$ with $p \in \partial D^2$, which are called the *vanishing cycles*, using the framings identifying $TS^n \times [0,1] \cong S^n \times D^{n+1}$ with $N_X(V_i) \times [0,1]$ where for each i the identification is determined by a bundle isomorphism ϕ_i' (called the *normalization* of V_i) which maps the tangent bundle TS^n of S^n onto the normal bundle $N_X(V_i)$ of V_i in the fiber X. We will call H_i a *Lefschetz handle*. Indeed, using the canonical isomorphism $TS^n \cong T^*S^n$ and the tubular neighborhood theorem, one can see that attaching the Lefschetz handle H_i (using ϕ_i, ϕ_i') changes the (global) monodromy by composing it with the *right-handed Dehn twist* δ_i along V_i. Observe that each attachment creates a new singular fiber whose singularity occurs at the origin of H_i.

Theorem 2.15 ([K], [DK])**.** *The Lefschetz fibration $\pi : W \to D^2$ is uniquely (up to deformation) determined by a sequence of vanishing cycles $(V_1, V_2, ..., V_\mu)$ and a sequence of their normalizations $(\phi_1', \phi_2', ..., \phi_\mu')$. The monodromy h of the fibration is given by*

$$h = \delta_1 \circ \delta_2 \circ \cdots \circ \delta_\mu \in \mathrm{Diff}(X)$$

where δ_i is the right-handed Dehn twist determined by V_i and ϕ_i' for each i.

Next, we recall the following symplectic construction for Lefschetz fibrations: Let X^{2n} be a Liouville (or Weinstein) domain, and Ω_X denote the corresponding exact symplectic form on X. Fix a cyclically ordered set $\{z_1, ..., z_k\} \subset \partial D^2$ and suppose $\{V_1, ..., V_k\}$ is an ordered collection of Lagrangian n-spheres embedded in $(X \setminus \partial X, \Omega_X)$ via embedings

$$\phi_i : S^n \to V_i \subset X \times \{z_i\}$$

for $i = 1, ..., k$. Note that for each i there exists a bundle isomorphism $\phi_i' : TV_i \to N_X(V_i)$ as V_i is Lagrangian in X. Let W be the smooth $(2n+2)$-manifold obtained from $X \times D^2$ by attaching Lefschetz handles $H_{\phi_i, \phi_i'}$, for $i = 1, ..., k$. That is,

$$W = (X \times D^2) \cup \bigcup_{i=1}^{k} H_{\phi_i, \phi_i'}.$$

Then W admits a Lefschetz fibration $\pi : W \to D^2$ with regular fiber X by construction, and the monodromy $h \in \mathrm{Symp}(X, \Omega_X)$ of π is the composition of right-handed Dehn twist along the vanishing cycles V_i. On the other hand, the product $X \times D^2$ admits a Liouville (indeed, Weinstein) structure described as in Remark 2.13. Therefore, it is natural to ask if W admits a Liouville (or Weinstein) structure. The answer to this question is positive in both categories. One should note that up to Liouville [resp. Weinstein] isomorphism the resulting Liouville [resp. Weinstein] domain W depends on the Lagrangian isotopy classes of the vanishing cycles $\{V_1, ..., V_k\}$. An equivalent description of such Lefschetz fibrations was described in [S1, S2] under the name *convex* (or *exact*) *Lefschetz fibrations* which we recall next.

Let $\pi : W^{2n+2} \to S$ be a differentiable fiber bundle, denoted by (π, W), whose fibers and base are compact connected manifolds with boundary. The boundary of such an W consists of two parts: The vertical part $\partial_v W; = \pi^{-1}(\partial S)$, and the horizontal part $\partial_h W := \bigcup_{z \in S} \partial W_z$ where $W_z = \pi^{-1}(z)$ is the fiber over $z \in S$.

Definition 2.16 ([S1, S2])**.** A *compact convex symplectic fibration* $(\pi, W, \omega, \lambda)$ over a bordered surface S is a differentiable fiber bundle (π, W) equipped with a 2-form ω and a 1-form λ on W, satisfying $\omega = d\lambda$, such that

(i) each fiber W_z with $\omega_z = \omega|_{W_z}$ and $\lambda_z = \lambda|_{W_z}$ is a compact convex symplectic manifold (i.e., a Liouville domain),

(ii) the following triviality condition near $\partial_h W$ is satisfied: Choose a point $z_0 \in S$ and consider the trivial fibration $\tilde{\pi} : \tilde{W} := S \times W_{z_0} \to S$ with the forms $\tilde{\omega}, \tilde{\lambda}$ which are pullbacks of $\omega_{z_0}, \lambda_{z_0}$, respectively. Then there should be a fiber-preserving diffeomorphism $\Upsilon : N \to \tilde{N}$ between neighborhoods N of $\partial_h W$ in W and $\tilde{N}$ of $\partial_h \tilde{W}$ in $\tilde{W}$ which maps $\partial_h W$ to $\partial_h \tilde{W}$, equals the identity on $N \cap W_z$, and $\Upsilon^* \tilde{\omega} = \omega$ and $\Upsilon^* \tilde{\lambda} = \lambda$.

Definition 2.17 ([S1, S2])**.** A *compact convex Lefschetz fibration* is a tuple $(\pi, W, S, \omega, \lambda, J_0, j_0)$ which satisfies the following conditions:

(i) $\pi : W \to S$ is allowed to have finitely many critical points all of which lie in the interior of W.
(ii) π is injective on the set C of its critical points.
(iii) J_0 is an integrable complex structure defined in a neighborhood of C in W such that ω is a Kähler form for J_0.
(iv) j_0 is a positively oriented complex structure on a neighborhood of the set $\pi(C)$ in S of the critical values.
(v) π is (J_0, j_0)-holomorphic near C.
(vi) The Hessian of π at any critical point is nondegenerate as a complex quadratic form, in other words, π has nondegenerate complex second derivative at each its critical point.
(vii) $(\pi, W \setminus \pi^{-1}(\pi(C)), \omega, \alpha)$ is a compact convex symplectic fibration over $S \setminus \pi(C)$.

Remark 2.18. For the codimension two corners (i.e., $\partial_v W \cap \partial_h W$) of the total space W there is canonical way of smoothening (see Remark 3.2). Assuming such smoothening has been made, all total spaces will be assumed to be smooth through out the paper. Also the statements (ii)-(vi) guarantees that the singularities of π are of Lefschetz type as in Definition 2.14. Let us consider a pair (J, j) where J is an almost complex structure on W agreeing with J_0 near C and j is a positively oriented complex structure on S agreeing with j_0 near $\pi(C)$ such that π is (J, j)-holomorphic and $\omega(\cdot, J\cdot)|_{\mathrm{Ker}\,(\pi_*)}$ is symmetric and positive definite everywhere. As pointed out in [S1], the space of such pairs is always contractible, and in particular, always nonempty, and furthermore, once we fixed (J, j), we can modify ω by adding a positive 2-form on S so that it becomes symplectic and tames J everywhere on W. Indeed, a stronger fact can be stated as follows:

Theorem 2.19 ([Mc])**.** *Suppose that the base surface of a compact convex Lefschetz fibration $(\pi, W, S, \omega, \lambda, J_0, j_0)$ is a compact convex symplectic manifold (S, λ_S). Then there exists a constant $K > 0$ such that for all $k \geq K$ we have $\Omega := \omega + k\pi^*(d\lambda_S)$ is a symplectic form, and the Ω-dual Z of $\Lambda := \lambda + k\pi^*(\lambda_S)$ is transverse to ∂W and pointing outwards. (In other words, (W, Ω, Λ) is a compact convex symplectic manifold, i.e., a Liouville domain.)*

Definition 2.20 (& Notation)**.** ([AA1]) A Liouville domain (W, Ω, Λ) and a compact convex Lefschetz fibration $(\pi, W, S, \omega, \lambda, J_0, j_0)$ are said to be *compatible* if for a pair (J, j) as in Remark 2.18 there exists a positive volume form ω_S on S such that

$$\Omega = \omega + \pi^*(\omega_S)$$

and Ω tames J everywhere on E. From now on, we'll always take $S = D^2$, and assuming the above choice of (J, j) is already made, (J_0, j_0) will be dropped from the notation. We will denote a compatible compact convex Lefschetz fibration (over D^2) on the compact convex symplectic manifold (W, Ω, Λ) by the tuple $(\pi, W, \Omega, \Lambda, X, h)$.

Definition 2.21 ([AA1])**.** An open book $\mathcal{OB}_\pi$ induced by a compatible compact convex Lefschetz fibration $(\pi, W, \Omega, \Lambda, X, h)$ over the disk D^2 is called a *convex* (or *exact*) *open book*.

In order to simplfy the wording we make the following definition:

Definition 2.22. A *Liouville Lefschetz fibration* on a Liouville domain (W, Ω, Λ) is a compact convex Lefschetz fibration which is compatible with the Liouville structure (Ω, Λ) on W.

3. Proof of Theorem 1.1

The proof basically relies on the following theorem which can be considered as a Weinstein counterpart of Theorem 1.1.

Theorem 3.1. *For any Weinstein domain W of dimension $2n + 2 > 4$, there exists a Liouville domain W', diffeomorphic to W, such that with the underlying Liouville structure W is Liouville isomorphic to W', and W' admits a Liouville Lefschetz fibration with Weinstein regular fibers. The monodromy of the fibration is a symplectomorphism induced by compositions of right-handed Dehn twists along embedded Lagrangian n-spheres on a generic fiber. Moreover, the induced convex open book is compatible with the induced contact structure on $\partial W'$.*

Before proving Theorem 3.1, we first observe some preparatory results. Let us first see how to remove codimension two corner $\partial X \times S^1$ of the trivial Liouville Lefschetz fibration on $X \times D^2$. A nice way of achiving this in Liouville category can be obtained from the construction given in Section 8 of [BEE] as follows: Denote by $(\hat{X}, \hat{\beta})$ the completion of a Liouville domain (X, β). The *stabilization* of $(\hat{X}, \hat{\beta})$ is the Liouville manifold

$$(\hat{X}^{st}, \hat{\beta}^{st}) := \left(\hat{X} \times \mathbb{R}^2, \hat{\beta} + \tfrac{1}{2}(x\,dy - y\,dx)\right).$$

In [BEE], the autors constructed a Liouville domain (with no corners) $(\bar{X}^{st}, \bar{\beta}^{st})$ such that $\bar{X}^{st} \subset \hat{X}^{st}$ is homemorphic to $X \times D^2$ and $\bar{\beta}^{st} := \hat{\beta}^{st}|_{\bar{X}^{st}}$. Indeed, one can think of $\bar{X}^{st}$ as the smooth manifold obtained by smoothing the corners of $X \times D^2$ (see Figure 1).

The projection $\hat{X} \times \mathbb{R}^2 \to \hat{X}$ maps $\bar{X}^{st}$ onto some subset $\bar{X} \subset \hat{X}$ containing X, and indeed, $\bar{X}^{st}$ also admits a trivial Liouville Lefschetz fibration $\pi_T^{st} : \bar{X}^{st} \to D^2$ whose fibers are obtained from the corresponding fibers of the trivial fibration $\pi_T : X \times D^2 \to D^2$ by adding some suitable

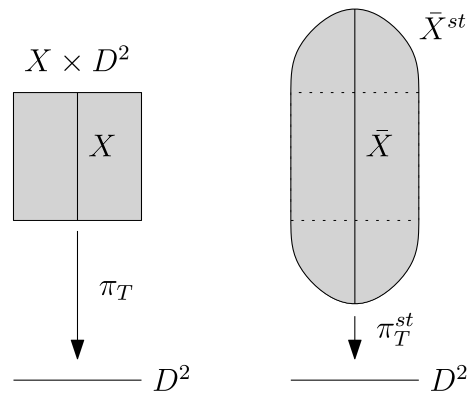

FIGURE 1. Smoothing corners.

amount of the completion $\hat{X}$ (Figure 1). Set $Y = \partial \bar{X}^{st}$. Then we have the decomposition $Y = Y_1 \cup Y_2$, where $Y_1 = Y \cap (X \times \mathbb{C})$ and $Y_2 = Y \setminus \mathrm{Int} Y_1$. Note that $Y_1 = X \times S^1$ and that the induced contact form $\bar{\beta}^{st}|_{Y_1}$ takes the form $\beta + d\theta$ where θ is the coordinate on $\mathbb{R}/2\pi\mathbb{Z} = S^1$. One should also point out that if X is a Weinstein domain, then the stabilization $\hat{X}^{st}$ (and so $\bar{X}^{st}$) also admits a Weinstein structure.

Remark 3.2. As promised in Remark 2.18, we now explain how to smoothen the corners $\partial_v W \cap \partial_h W$ of a given compact convex Lefschetz fibration $(\pi, W, D^2, \omega, \lambda, J_0, j_0)$. (One should note that the following argument works also for an arbitrary base surface S.) Let X be a regular fiber of π. By the local triviality condition near $\partial_h W$, we know that π looks like the trivial fibration π_T. More precisely, if N is a neighborhood of $\partial_h W$ where the local triviality condition is satisfied, then λ restricts to the same Liouville structure, say β, on each fiber of $\pi|_N$. Therefore, N can be identified with the neighborhood of $\partial X \times D^2$ above. Hence, we can attach $\bar{X}^{st} \setminus (X \times D^2)$ to W, and this will remove the corners of W. Note that as before each fiber of π is expanded by adding some suitable amount of the completion $\hat{X}$. To sum up, we can assume (indeed, this should be taken as what we have assumed so far) that total space of any $(\pi, W, D^2, \omega, \lambda, J_0, j_0)$ is smoothened in this way, and the fibration map π is already extended over $\bar{X}^{st} \setminus (X \times D^2)$ in the obvious way.

We will make use of the following proposition as the first step in the proof of the main theorem. It will be used to make the boundary convex open book of any Liouville Lefschetz fibration standard in the symplectic sense.

Proposition 3.3. *Let $(\pi_0, W_0, \Omega_0, \Lambda_0, X, h)$ be a Liouville Lefschetz fibration, and (X_β, β) be any fixed page of the boundary convex open book with the restricted Liouville form $\beta := \Lambda_0|_{X_\beta}$. Then there exits a smooth 1-parameter family $(\pi_t, W_t, \Omega_t, \Lambda_t, X, h)$ of Liouville Lefschetz fibrations such that*

(i) *$(W_t, \Omega_t, \Lambda_t)$'s are mutually Liouville isomorphic.*
(ii) *There exists a constant $K > 0$ such that the Liouville form Λ_1 restricts to $K\beta$ on every page X of the convex open book on ∂W_1 induced by $(\pi_1, W_1, \Omega_1, \Lambda_1, X, h)$.*

In order to prove this proposition, we will need a lemma:

Lemma 3.4. *Let $(\pi_0, W, \Omega_0, \Lambda_0, X, h)$ be any Liouville Lefschetz fibration on the Liouville domain (W, Ω_0, Z_0) where Z_0 is the Ω_0-dual of Λ_0. Denote by ξ_0 the induced contact structure $\mathrm{Ker}(\Lambda_0|_{\partial W})$ on ∂W. Let ξ_t, $t \in [0,1]$, be an isotopy of contact structures on ∂W. Then there exists a homotopy (W, Ω_t, Z_t) of Liouville domains such that the following hold:*

(i) *(W, Ω_t, Λ_t)'s are mutually Liouville isomorphic.*
(ii) $\mathrm{Ker}(\Lambda_t|_{\partial W}) = \xi_t$ *where* $\Lambda_t := \iota_{Z_t}\Omega_t$ *is the Liouville form on W associated to (Ω_t, Z_t).*
(iii) *There is a smooth family $(\pi_t, W, \Omega_t, \Lambda_t, X, h)$, $t \in [0, 1]$, of Liouville Lefschetz fibrations.*

Proof. By Gray's Stability (see, for instance, Theorem 2.2.2 of [Ge]), there exists a diffeotopy $\phi_t : \partial W \to \partial W$, $t \in [0, 1]$, such that $(\phi_t)_*(\xi_0) = \xi_t$ for each t. Set $\alpha_0 := \Lambda_0|_{\partial W}$ and define $\alpha_t := (\phi_t^{-1})^*(\alpha_0)$ for $t \in [0, 1]$. Then α_t is a contact form on ∂W and $\xi_t = \mathrm{Ker}(\alpha_t)$ for each t.

Using the flow of the Liouville vector field Z_0 we can (symplectically) identify a small collar neighbourhood, say N_ϵ, of ∂W in W with a region sitting in the negative-half of the symplectization $(\partial W \times \mathbb{R}, d(e^r\alpha_0), \partial/\partial r)$. So we have a model Liouville cobordism for N_ϵ given by

$$(N_\epsilon, \Omega_0|_{N_\epsilon}, Z_0|_{N_\epsilon}) = (\partial W \times (-\epsilon, 0], d(e^r\alpha_0), \partial/\partial r).$$

Next, we extend the diffeotopy $\phi_t : \partial W \to \partial W$ to a diffeotopy $\widetilde{\Phi}_t : W \to W$ as follows: Let $\mu : (-\epsilon, 0] \to [0, 1]$ be a smooth cut-off function whose graph is given as in Figure 2.

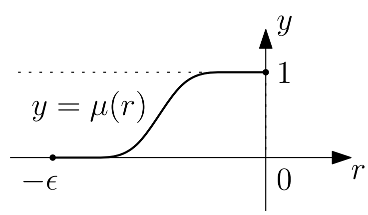

FIGURE 2. A smooth cut-off function $y = \mu(r)$.

Define a diffeotopy $\Phi_t : N_\epsilon \to N_\epsilon$, $t \in [0, 1]$, by the rule

$$\Phi_t(x, r) = (\phi_{t\mu(r)}, r)$$

where we use the identification $N_\epsilon = \partial W \times (-\epsilon, 0]$ described above. Observe that Φ_t is identity near $\partial W \times \{-\epsilon\}$, and so it can be extended to a diffeotopy $\widetilde{\Phi}_t : W \to W$ which is identity in the complement $W \setminus N_\epsilon$ for all t. Note that $\widetilde{\Phi}_t$ defines a diffeotopy for each contact hypersurface $\partial W \times \{r\} \subset N_\epsilon$.

Now we define a smooth 1-parameter family of Liouville structures $(\Omega_t, \Lambda_t, Z_t)$, $t \in [0, 1]$, on W using the diffeotopy $\widetilde{\Phi}_t$ by setting

$$\Lambda_t := (\widetilde{\Phi}_t^{-1})^*(\Lambda_0), \quad \Omega_t := d\Lambda_t,$$

and defining Z_t as the Ω_t-dual of Λ_t. We know, by Lemma 2.4, that (Ω_t, Z_t) is a Liouville homotopy, and that (W, Ω_t, Z_t)'s are mutually Liouville isomorphic. Therefore, part (i) follows. Moreover, the above construction implies that $\Lambda_t|_{\partial W \times \{0\}} = \alpha_t$ for each $t \in [0, 1]$. This can be easily seen from an alternative description of Λ_t given by

$$\Lambda_t(p) = \begin{cases} (\Phi_t^{-1})^*(e^r\alpha_0)(p) & \text{if} \quad p = (x, r) \in N_\epsilon = \partial W \times (-\epsilon, 0] \\ \Lambda_0(p) & \text{if} \quad p \in W \setminus N_\epsilon \end{cases}$$

where we use the fact that $\Lambda_0 = e^r\alpha_0$ on N_ϵ. As a result, we have $\xi_t = \mathrm{Ker}(\alpha_t) = \mathrm{Ker}(\Lambda_t|_{\partial W})$ as claimed in part (ii).

For part (iii), observe that one can deform the given fibration map $\pi_0 : W \to D^2$ using the diffeotopy $\widetilde{\Phi}_t$. More precisely, for each $t \in [0,1]$ consider the fibration $\pi_t : W \to D^2$ given by the composition map

$$\pi_t(p) := (\pi \circ \widetilde{\Phi}_t)(p), \quad p \in W.$$

Then clearly $\{\pi_t\}_{t\in[0,1]}$ is a smooth family of Lefschetz fibrations on W all of which have regular fiber X and the monodromy h. Moreover, since both the fibration π_t and the Liouville structure (Ω_t, Λ_t) were defined by using the pull-back operator of (the same) diffeotopy $\widetilde{\Phi}_t$, we know that (Ω_t, Λ_t) restricts to a convex symplectic structure on every regular fiber of π_t and the triviality condition near the horizontal boundary component of π_t is satisfied. Hence, we conclude that each member of the smooth family $(\pi_t, W, \Omega_t, \Lambda_t, X, h)$, $t \in [0,1]$, is a Liouville Lefschetz fibration. □

Before we start the next proof, let us recall that an open book $\mathcal{OB} = (B, \Theta)$ on a closed manifold Y determines an abtract open book (X, h): By definition B is a codimension-two subset of Y with a trivial normal bundle $B \times D^2$ and $\Theta : Y \setminus B \to S^1$ is a fiber bundle map agreeing the angular coordinate on the D^2-factor. To describe an abstract open book, we first pick a page $X = \Theta^{-1}(p)$ for some $p \in S^1$. Then the monodromy $h : X \to X$ can be read from the first return map of a vector field on Y transversal to the fibers of Θ. Now using the resulting abstract open book we can construct a closed manifold

$$M(X, h) := \Sigma(X, h) \cup_\partial (\partial X \times D^2)$$

where $\Sigma(X, h)$ is the mapping torus determined by h. Observe that the construction defines an open book decomposition $\mathcal{OB}_{(X,h)}$ on $M(X, h)$, and also that Y can be identified with $M(X, h)$ via some diffeomorphism respecting the fibration maps on $Y \setminus B$ and $M(X, h) \setminus (\partial X \times \{0\})$.

Proof of Proposition 3.3: Denote by ξ_0 the contact structure on ∂W_0 defined by the contact form $\alpha_0 := \Lambda_0|_{\partial W_0}$. Let us identify ∂W_0 with the manifold $M(X, h)$ using the abstract open book $(X = X_\beta, h)$ determined by the convex open book $\mathcal{OB}_{\pi_0}$ via some diffeomorphism

$$\Upsilon : M(X, h) \to \partial W_0$$

which maps the pages of $\mathcal{OB}_{(X,h)}$ to those of $\mathcal{OB}_{\pi_0}$. Here we consider that the pages of $\mathcal{OB}_{(X,h)}$ are all copies of X_β and equipped with the same Liouville structure β. By assumption $h \in \mathrm{Symp}(X_\beta, d\beta)$ and $h = \mathrm{id}$ near ∂X_β. Then from a collection of results from [Gi1] and [Gi2] (see also Section 7.3 of [Ge] for a detailed explanation), one can construct a contact structure ξ on $M(X, h)$ as the kernel of a contact form α_β (on $M(X, h)$) which restricts to β on every page of $\mathcal{OB}_{(X,h)}$. Let $\xi_1 := \Upsilon_*(\xi)$ be the contact structure on ∂W_0 obtained by pushing forward ξ using the identification map Υ. Denote by α the contact form for ξ_1 given by the pull-back $(\Upsilon^{-1})^*(\alpha_\beta)$. Note that $\xi_1 = \mathrm{Ker}(\alpha)$ as $\alpha = (\Upsilon^{-1})^*(\alpha_\beta)$, and also that α restricts to $(\Upsilon^{-1})^*(\beta) \equiv \beta$ on every page of the open book $\mathcal{OB}_{\pi_0}$ on ∂W_0. Since ξ_0 and ξ_1 are supported by the same open book $\mathcal{OB}_{\pi_0}$, we know, by Giroux's work, that there exists an isotopy ξ_t $(t \in [0,1])$ of contact structures on ∂W_0 connecting ξ_0 and ξ_1. Then Gray's Stability implies that there is a diffeotopy $\phi_t : \partial W_0 \to \partial W_0$, $t \in [0,1]$, such that $(\phi_t)_*(\xi_0) = \xi_t$ for each t. As in the previous proof, let us define the contact form $\alpha_t := (\phi_t^{-1})^*(\alpha_0)$ for each ξ_t.

By Lemma 3.4, we know that there exists a homotopy of Liouville structures on W_0 and a corresponding smooth family of (compatible) Liouville Lefschetz fibrations such that the final Liouville structure (on W_0) of the homotopy induces the contact structure ξ_1 on ∂W_0. Moreover, the construction of the homotopy also implies that the final Liouville form restricts to the contact form α_1 on ∂W_0. Therefore, assuming Lemma 3.4 has been already applied, we may assume that the Liouville structure (Ω_0, Λ_0) on W_0 defines the contact structure ξ_1 (or equivalently, $\alpha_1 = \Lambda_0|_{\partial W_0}$), and also that π_0 is the final fibration of the above smooth family of Liouville Lefschetz fibrations.

By the previous paragraph, the contact structure ξ_1 on ∂W_0 is the kernel of both α_1 and α which implies that there exists a strictly positive smooth function $g : \partial W_0 \to \mathbb{R}_+$ such that

$$\alpha = g\alpha_1.$$

Let $f : \partial W_0 \to \mathbb{R}$ be the smooth function uniquely determined by the equation $g = e^f$. Since ∂W_0 is compact, there exists a real number $m > 0$ such that $|f(x)| < m$ for all $x \in \partial W_0$. Then the smooth function $f + m : \partial W_0 \to \mathbb{R}_+$ is strictly positive. Denote by $(\widehat{W}_0, \widehat{\Omega}_0, \widehat{\Lambda}_0, \widehat{Z}_0)$ the completion of $(W_0, \Omega_0, \Lambda_0, Z_0)$, and consider the smooth 1-parameter family

$$\{t(f + m) : \partial W_0 \to \mathbb{R}_+ \cup \{0\} \mid t \in [0, 1]\}$$

of smooth functions on ∂W_0. Note that any member of this family is strictly positive except the one corresponding $t = 0$ which identically vanishes on ∂W_0. Then we obtain a smooth 1-parameter family $\{\Sigma_t \mid t \in [0, 1]\}$ where

$$\Sigma_t := \mathrm{Graph}(t(f + m)) = \{(x, r) \mid x \in \partial W_0, \quad r = t(f + m)(x)\}$$

is a hypersurface in the symplectization $\mathrm{Symp}(\partial W_0, \alpha_1) = (\partial W_0 \times \mathbb{R}, d(e^r\alpha_1), e^r\alpha_1, \partial/\partial r)$ which symplectically embeds into $(\widehat{W}_0, \widehat{\Omega}_0, \widehat{\Lambda}_0, \widehat{Z}_0)$. The flow lines of $\partial/\partial r = \widehat{Z}_0|_{\mathrm{Symp}(\partial W_0,\alpha_1)}$ define a diffeomorphism (called a *holonomy* map in [CE])

$$\Gamma_t : \partial W_0 \to \Sigma_t$$

such that $\Gamma_t^*(\widehat{\Lambda}_0|_{\Sigma_t}) = e^{t(f+m)}\alpha_1$. Therefore, for each $t \in [0, 1]$ we have a contactomorphism

$$\Gamma_t : (\partial W_0, \mathrm{Ker}(\alpha_1)) \to (\Sigma_t, \mathrm{Ker}(\widehat{\Lambda}_0|_{\Sigma_t})), \quad \Gamma_t^*(\widehat{\Lambda}_0|_{\Sigma_t}) = e^{t(f+m)}\alpha_1.$$

Next, we will construct a smooth 1-parameter family $(W_t, \Omega_t, \Lambda_t, Z_t)$ of Liouville domains inside the completion $(\widehat{W}_0, \widehat{\Omega}_0, \widehat{\Lambda}_0, \widehat{Z}_0)$: Let $\Delta_t \subset \widehat{W}_0$ be the region (which is, indeed, a Liouville cobordism) between the contact hypersurfaces $\Sigma_0 = \partial W_0$ and Σ_t as depicted in Figure 3. Then for each $t \in [0, 1]$ we set

$$W_t := W_0 \cup \Delta_t, \quad \Omega_t := \widehat{\Omega}_0|_{W_t}, \quad \Lambda_t := \widehat{\Lambda}_0|_{W_t}, \quad Z_t := \widehat{Z}_0|_{W_t}.$$

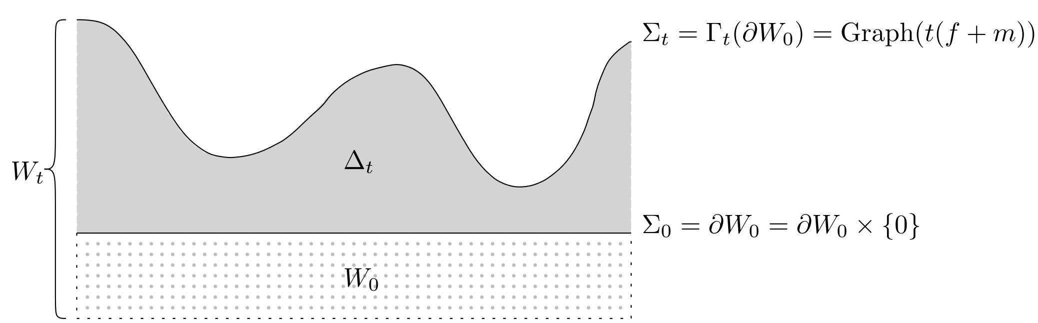

Figure 3. The construction of the Liouville domain $W_t := W_0 \cup \Delta_t$ inside the completion $(\widehat{W}_0, \widehat{\Omega}_0, \widehat{\Lambda}_0, \widehat{Z}_0)$.

Then part (i) of the proposition is immediate from the construction because all Liouville domains $(W_t, \Omega_t, \Lambda_t, Z_t)$ have the same completion $(\widehat{W}_0, \widehat{\Omega}_0, \widehat{\Lambda}_0, \widehat{Z}_0)$. For the claim in part (ii), we first need to define a Liouville Lefschetz fibration $\pi_t : W_t \to D^2$ compatible with the above construction. Since the Liouville Lefschetz fibration $\pi_0 : W_0 \to D \approx D^2$ is already given and $W_0 \subset W_t$, it is enough to define π_t as an appropriate extension of π_0 over the region Δ_t. To this end, consider the disk $D_t := D \cup A_t$ where

$$A_t := \{(z, s) \mid z = e^{i\theta}, \quad s \in [0, t]\} \subset S^1 \times \mathbb{R}$$

which we identify with $\partial D \times [0, t]$. Write $\mathcal{OB}_0$ for the convex open book $\mathcal{OB}_{\pi_0}$ on ∂W_0 (induced by π_0). Considering the image of $\mathcal{OB}_0$ under the holonomy map Γ_s for each $s \in [0, t]$, we obtain the smooth 1-parameter family

$$\{\mathcal{OB}_s \mid \mathcal{OB}_s := \Gamma_s(\mathcal{OB}_0), \quad s \in [0, t]\}.$$

Clearly, each $\mathcal{OB}_s$ is an open book on Σ_s supporting $\text{Ker}(\widehat{\Lambda}_0|_{\Sigma_s})$, and we have

$$\Delta_t = \bigcup_{s=0}^{t} \Sigma_s = \bigcup_{s=0}^{t} \mathcal{OB}_s.$$

Let $B_s \subset \Sigma_s$ denote the binding of $\mathcal{OB}_s$, and so $B_s = \Gamma_s(B_0)$ where $B_0 \subset \partial W_0$ is the binding of $\mathcal{OB}_0$. Also let $\theta_s : \Sigma_s \setminus B_s \to S^1 \times \{s\} \subset A_t$ be the fibration determined by the open book $\mathcal{OB}_s$. Then we define $\pi_t : W_t = W_0 \cup \Delta_t \to D_t$ as

$$\pi_t(p) = \begin{cases} \pi_0(p) & \text{if} \quad p \in W_0 \\ \pi_0((\Gamma_s)^{-1}(p)) & \text{if} \quad p \in B_s = \Gamma_s(B_0) \\ (z, s) & \text{if} \quad p \in \theta_s^{-1}(z, s) \end{cases}$$

By construction, π_t is a smooth extension of π_0 and it defines a Lefschetz fibration on W_t with regular fiber X and the monodromy h. Also the open book $\mathcal{OB}_t$ on $\Sigma_t = \partial W_t$ is induced by π_t (i.e., $\theta_t = \pi_t|_{\partial W_t \setminus B_t}$). Next, we will verify that the tuple $(\pi_t, W_t, \Omega_t, \Lambda_t, X, h)$ is a Liouville Lefschetz fibration by showing that for each $s \in [0, t]$ the contact form $\Lambda_t|_{\Sigma_s} = \widehat{\Lambda}_0|_{\Sigma_s}$ restricts to a Liouville form on each page of the open book $\mathcal{OB}_s$. To this end, for $(z, s) \in S^1 \times \{s\}$ consider a typical page $\theta_s^{-1}(z, s)$ of the open book $\mathcal{OB}_s$. Note that $\theta_s^{-1}(z, s)$ is equal to the image $\Gamma_s(\theta_0^{-1}(z, 0))$ of the page $\theta_0^{-1}(z, 0)$ (of the open book $\mathcal{OB}_0$) under the holonomy map Γ_s. The restriction of $\widehat{\Lambda}_0|_{\Sigma_s}$ to the page $\theta_s^{-1}(z, s)$ satisfies the equation

$$\Gamma_s^*((\widehat{\Lambda}_0|_{\Sigma_s})|_{\theta_s^{-1}(z,s)}) = (e^{s(f+m)})|_{\theta_0^{-1}(z,0)} \alpha_1|_{\theta_0^{-1}(z,0)}.$$

Therefore, showing that $\widehat{\Lambda}_0|_{\Sigma_s}$ restricts to a Liouville form on $\theta_s^{-1}(z, s)$ is equivalent to showing that $(e^{s(f+m)})|_{\theta_0^{-1}(z,0)} \alpha_1|_{\theta_0^{-1}(z,0)}$ is a Liouville form on $\theta_0^{-1}(z, 0)$. To verify the latter, first recall that $\alpha_1|_{\theta_0^{-1}(z,0)}$ is a Liouville form on $\theta_0^{-1}(z, 0)$ (by the assumption that π_0 is a Liouville Lefschetz fibration). Let $\chi_{\theta_0^{-1}(z,0)}$ be the Liouville vector field which is $d(\alpha_1|_{\theta_0^{-1}(z,0)})$-dual to $\alpha_1|_{\theta_0^{-1}(z,0)}$. Then we just observe that $(e^{s(f+m)})|_{\theta_0^{-1}(z,0)} \alpha_1|_{\theta_0^{-1}(z,0)}$ is an expansion of the Liouville form $\alpha_1|_{\theta_0^{-1}(z,0)}$ along $\chi_{\theta_0^{-1}(z,0)}$. But any such expansion still defines a Liouville form on $\theta_0^{-1}(z, 0)$.

To finish the proof, we need to show that there exists a constant $K > 0$ such that $\Lambda_1|_{\partial W_1}$ restricts to $K(\Upsilon^{-1})^*(\beta) \equiv K\beta$ on every page X of the open book $\mathcal{OB}_1$. For $(z, 1) \in S^1 \times \{1\}$ consider a typical page $\theta_s^{-1}(z, 1) \approx X$ of the open book $\mathcal{OB}_1$. To this end, note that when $t = 1$ we have the contactomorphism $\Gamma_1 : (\partial W_0, \text{Ker}(\alpha_1)) \to (\Sigma_1, \text{Ker}(\widehat{\Lambda}_0|_{\Sigma_1}))$ which satistifies

$$\Gamma_1^*(\widehat{\Lambda}_0|_{\Sigma_1}) = e^{f+m}\alpha_1 = e^m e^f \alpha_1 = e^m g \alpha_1 = K\alpha$$

where $K := e^m > 0$ is a fixed constant. Thus, by restricting this equation to the page $\theta_1^{-1}(z,1) = \Gamma_1(\theta_0^{-1}(z,0))$, and using the facts that $\Sigma_1 = \partial W_1$ and $\widehat{\Lambda}_0|_{\Sigma_1} = \Lambda_1|_{\partial W_1}$, we conclude that

$$\Gamma_1^*((\widehat{\Lambda}_1|_{\partial W_1})|_{\theta_1^{-1}(z,1)}) = K\alpha|_{\theta_0^{-1}(z,0)}$$

from which the claim follows as α restricts to $(\Upsilon^{-1})^*(\beta) \equiv \beta$ on every page of $\mathcal{OB}_0$. □

For completeness it is better to recall the construction of the contact form α_β on the manifold $M(X,h)$ in the following remark.

Remark 3.5. Given an abstract open book (X,h), consider the closed manifold $M(X,h) = \Sigma(X,h) \cup_\partial \partial X \times D^2$. Suppose that there exists a Liouville form β on X and $h \in \mathrm{Symp}(X, d\beta)$. Then as mentioned in the proof of Proposition 3.3 one can construct a contact structure on $M(X,h)$ by the following explicit construction of the defining contact form: Since $h \in \mathrm{Symp}(X, d\beta)$, the form $h^*(\beta) - \beta$ is closed, and it can be made exact by deforming h through symplectomorphisms which are identity near ∂X. Such a deformation of h changes $M(X,h)$ in its diffeomorphism class, and so we may assume that $h^*(\beta) - \beta = -d\rho$ for some smooth function $\rho : X \to \mathbb{R}$. Adding a large enough constant, we may assume that ρ is strictly positive, and so we can use ρ to construct a smooth mapping torus

$$\Sigma(X,h)_\rho := X \times \mathbb{R}/\sim_\rho \quad \text{where} \quad (x,z) \sim_\rho (h(x), z + \rho(x)\,).$$

Consider the contact form $\alpha = \beta + dz$ on $X \times \mathbb{R}$, and let $\sigma_\rho : X \times \mathbb{R} \to X \times \mathbb{R}$ be the diffeomorphism defining $\sim_\rho$, that is, $\sigma_\rho(x,z) = (h(x), z + \rho(x)\,)$. Then we compute

$$\sigma_\rho^*(\beta + dz) = h^*(\beta) + d(z+\rho) = \beta - d\rho + dz + d\rho = \beta + dz$$

which shows that $\alpha = \beta + dz$ decends to a contact form, say $\tilde{\alpha}$, on $\Sigma(X,h)_\rho$. Then using appropriate cut-off functions, one can construct a contact form α_β on

$$\Sigma(X,h)_\rho \cup_\partial \partial X \times D^2 \approx M(X,h)$$

by smoothly gluing $\tilde{\alpha}$ with the contact form $\beta|_{\partial X} + r^2 d\theta$ on the normal bundle $\partial X \times D^2$.

One of the key tools for the proof of Theorem 3.1 is the following:

Theorem 3.6. *Let X be a Weinstein domain of dimension $2n \geq 4$ whose underlying Liouville structure is given by the Liouville form β. For $h \in \mathrm{Symp}(X, d\beta)$, consider the contact manifold $(M(X,h), \mathrm{Ker}(\alpha_\beta))$ as in Remark 3.5. Let $\{\phi_i : S^n \hookrightarrow (M(X,h), \mathrm{Ker}(\alpha_\beta))\}_{i=1}^k$ be a family of disjointly embedded Legendrian spheres. Then the Legendrian link $\mathbb{S} = \bigsqcup_{i=1}^k \phi_i(S^n)$ can be Legendrian isotoped (through embedded Legendrian links) to another embedded Legendrian link $\mathbb{S}'$ which is disjoint from a page of the of the open book $\mathcal{OB}_{(X,h)}$ on $M(X,h)$ associated to the abstract open book (X,h).*

In order to prove this theorem we will make use of contact vector fields. A vector field on a contact manifold is said to be *contact* if its flow preserves the contact distrubution. The following fundamental lemma in contact geometry characterizes contact vector fields on a given contact manifold. More details can be found, for instance, in [Ge].

Lemma 3.7. *Let $(M, \mathrm{Ker}(\alpha))$ be any contact manifold and R_α denote the Reeb vector field of α. Then there is a one-to-one correspondence between the set $\{Z \in \Gamma(M) \,|\, Z \text{ is contact}\}$ of all contact vector fields on M and the set $\{H : M \to \mathbb{R} \,|\, H \text{ is smooth}\}$ of all smooth functions on M. The correspondence is given by $Z \to H_Z := \alpha(Z)$ (H_Z is called the "contact Hamiltonian" of the contact vector field Z), and $H \to Z_H$ where Z_H is the contact vector field uniquely determined by the equations $\alpha(Z_H) = H$ and $\iota_{Z_H} d\alpha = dH(R_\alpha)\alpha - dH$.* □

We note that a similar statement also holds between locally defined contact vector fields and locally defined smooth functons.

Proof of Theorem 3.6. For simplicity we will write $M(X,h) = Y$ and $\alpha_\beta = \alpha$. We may assume that $\mathbb{S}$ consists of a single Legendrian sphere $S = \phi(S^n)$ with the given Legendrian embedding $\phi : S^n \hookrightarrow (Y, \mathrm{Ker}(\alpha))$ since all the argument used below can be adapted (or generalized) to the case where $\mathbb{S}$ has more than one components. So we have an embeding of a Legendrian sphere S in the open book $\mathcal{OB}_{(X,h)} = (B, \Theta)$ on Y corresponding to the abstract open book (X, h). That is, we have

$$\phi : S^n \hookrightarrow Y = Y_1 \cup Y_2 = \Sigma(X,h)_\rho \cup (B \times D^2)$$

where $B = B \times \{0\} \subset B \times D^2 \subset Y$ is the binding, $Y_2 := B \times D^2$, and $Y_1 := \Sigma(X,h)_\rho$ is the mapping torus as above. We may consider $(Y_2, \mathrm{Ker}(\alpha|_{Y_2}))$ as the contact manifold

$$(B \times D^2, \mathrm{Ker}(\beta + (x/2)\, dy - (y/2)\, dx))$$

where (x, y) are the coordinates on the unit disk D^2. Consider the following vector fields:

$$Z_1 = \partial x - (y/2)R_\beta, \quad Z_2 = \partial y + (x/2)R_\beta, \quad Z_3 = \chi + \theta\, \partial\theta, \quad Z_4 = R_\alpha \tag{1}$$

where χ is the $d\beta$-dual vector field of β, R_α (resp. R_β) denotes the Reeb vector field of α (resp. $\beta|_B$), and θ is the S^1-coordinate in the fibration $\Theta : Y \setminus B \to S^1$ determined by $\mathcal{OB}_{(X,h)}$. Here $\partial x = \partial/\partial x, \partial y = \partial/\partial y$, and so on.... Note that the first two are defined on $Y_2 = B \times D^2$, and the third is defined on the contact manifold $(Y_1 \setminus X, \mathrm{Ker}(\alpha|_{Y_1 \setminus X}))$ where X is any fixed page of $\mathcal{OB}_{(X,h)}$. Here we note that α restricts to β on every page of $\mathcal{OB}_{(X,h)}$ (from its construction described in Remark 3.5) and $Y_1 \setminus X = X \times \mathbb{R}$, and so, in particular, we may also write $Z_3 = \chi + z\, \partial z$ where z is the $\mathbb{R}$-coordinate on $X \times \mathbb{R}$. It is easy to check that:

$$\mathcal{L}_{Z_i}\alpha = \begin{cases} 0 & \text{if } i = 1, 2, 4 \\ \alpha & \text{if } i = 3 \end{cases}$$

So they are all contact vector fields on the regions where they are defined. In fact, if H_i denotes the contact Hamiltonian function corresponding to Z_i as in Lemma 3.7, then we have

$$H_1 = -y, \quad H_2 = x, \quad H_3 = \theta, \quad H_4 \equiv 1.$$

Although we state and prove only for our case, the following lemma holds in the most generality, i.e., it is true for any Legendrian submanifold embedded into a supporting open book.

Lemma 3.8. *In any pre-given ϵ-neighbourhood of S, one can Legendrian isotope S (in a small neighbourhood of the binding B) so that it becomes everywhere transverse to B.*

Proof. Let N_ϵ be any ϵ-neighbourhood of S. We will use the fact that Legendrian (more generally, isotropic) submanifolds stays Legendrian (isotropic) under the flows of contact vector fields. Note that for any constants $a_1, a_2 \in \mathbb{R}$ the vector field $Z = a_1 Z_1 + a_2 Z_2$ is contact with the contact Hamiltonian $H_Z = a_1 H_1 + a_2 H_2$. If S intersects B transversally, then there is nothing to prove. If not, let $K \subset S \cap (B \times \{(0,0)\})$ be the region where they don't intersect transversally. Consider $Z_1|_{B\times\{(0,0)\}} = \partial x, Z_2|_{B\times\{(0,0)\}} = \partial y$ on $B = B \times \{(0,0)\}$. For any $p \in K$, the tangent space

$$(\{\mathbf{0}\} \times TD^2)|_p \subset (TB \times TD^2)|_p$$

does not lie in $TS|_p$ (otherwise S and B would intersect transversally at $p \in K$). Therefore, there exists a vector $v = a_1\partial x + a_2\partial y \in (\{\mathbf{0}\} \times TD^2)|_{B\times\{(0,0)\}}$ (for some constants $a_1, a_2 \in \mathbb{R}$) which is everywhere transverse to $S \cap B$. We consider $Z = a_1 Z_1 + a_2 Z_2$ as the smooth extension of v to the whole $B \times D^2$. Note that Z will stay transverse to S in a small neighbourhood

$N_\delta := B \times \{(x,y) \in D^2 \,|\, x^2+y^2 < \delta\}$ for some $\delta > 0$. Let $N_S K \subset N_\delta$ be a small neighbourhood of K in S. Now choose a regular value $q \in \{(x,y) \in D^2 \,|\, x^2+y^2 < \delta\} \subset D^2$ of the composition

$$\phi^{-1}(S) = S^n \supset U \xrightarrow{\phi} B \times D^2 \xrightarrow{\pi_2} D^2,$$

where π_2 is the projection and $U = \phi^{-1}(B \times \mathrm{int}(D^2))$, such that $q \in \pi_2(N_S K)$ lies on the line segment joining $(0,0)$ and (a_1,a_2), and the above composition has no critical value other than $(0,0)$ on the line segment, say l_q, joining $(0,0)$ and q. Note that, by construction, S intersects the identical copy $B_q = \pi_2^{-1}(q)$ of B transversally, and Z is everywhere transverse to $U \cap \pi_2^{-1}(l_q)$. Let $\mu : B \times D^2 \to \mathbb{R}$ be a cut-off function such that $\mu \equiv 1$ on a neigbourhood $N \subset N_\epsilon$ of $U \cap \pi_2^{-1}(l_q)$ in N_δ, and $\mu \equiv 0$ on a slightly larger neigbourhood $N' \subset N_\epsilon \cap N_\delta$ (see Figure 4).

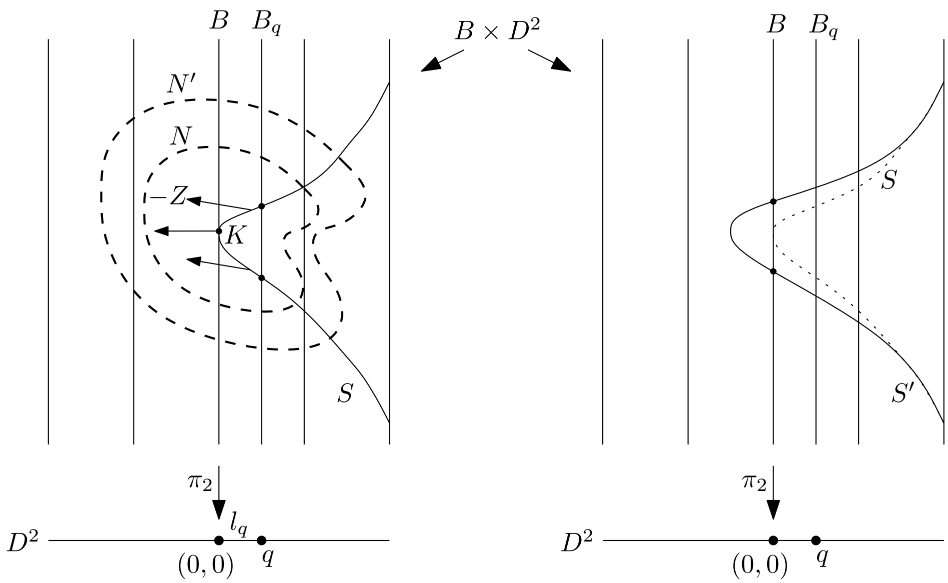

FIGURE 4. Isotoping the Legendrian sphere S to another Legendrian sphere S' which is transverse to the binding $B = B \times \{(0,0)\}$.

Next consider the contact vector field Z_μ corresponding to the contact Hamiltonian $H_\mu := \mu H_Z$. By the choice of μ, Z_μ agrees with Z on N and it is identically zero outside N'. Using the backward flow Z_μ^{-t} of Z_μ we define the following 1-parameter smooth family:

$$\Phi_t : S^n \longrightarrow Y, \quad \Phi_t(p) = Z_\mu^{-t}(\phi(p)), \quad t \in [0,T]$$

where T is the time elapsed during the points of $S \cap B_q$ are moved to their final images in $B = B \times \{(0,0)\}$ under the backward flow of Z_μ. (Note that all the points of $S \cap B_q \subset N$ reach B at the same time because the "horizontal" components of $Z_\mu|_N = Z|_N$ is defined by the constants a_1, a_2.) Observe that $\Phi_0 = \phi$, for each $t \in [0,T]$ we have $\Phi_t = \phi$ outside N' and $\Phi_t(S^n)$ is Legendrian, and $S' := \Phi_1(S^n) \subset Y$ is everywhere transverse to the binding B as depicted in Figure 4. Finally, by choosing δ small enough, one can guarantee that the isotopy Φ_t stays in the pre-given ϵ-neighbourhood N_ϵ of S. □

By the above lemma we may assume that the Legendrian sphere $S = \phi(S^n)$ is transverse to the binding B. Next, by picking a regular value $p \in S^1$ for the projection $\pi_2 : \phi(S^n) \cap \Sigma(X,h)_\rho \longrightarrow$

S^1 we can assume that for the page $\Theta^{-1}(p) \approx X$ of $\mathcal{OB}_{(X,h)}$, the intersection

$$L := \phi(S^n) \cap \Theta^{-1}(p)$$

is a properly imbedded $n-1$ dimensional submanifold $(L, \partial L) \subset (X, B = \partial X)$ meeting the binding along an $n-2$ dimensional submanifold ∂L (for simplicity we will write X for $\Theta^{-1}(p)$). As for the previous lemma, we point out that the next result holds not only for our case but it is also true in a more general situation. In other words, it is true for any Legendrian submanifold intersecting transversally a particular Weinstein page of a supporting open book.

Lemma 3.9. *In any pre-given ϵ-neighbourhood of S, one can Legendrian isotope S (in a small neighbourhood of the page X) so that $L = S \cap X$ becomes disjoint from $\Delta = \mathrm{Core}(X, \beta)$.*

Proof. Since X^{2n} is Weinstein (by assumption), $\dim(\Delta) = n$ by Theorem 2.8. Hence, by the general position in X, we can (topologically) isotope L^{n-1} to a nearby copy which is disjoint from Δ. This means that there exists a vector field Z on X which is transverse to both L and Δ along their intersection $L \cap \Delta$. In what follows, using contact vector fields which are compactly supported near $L \cap \Delta$ (and which are generated from Z), we will construct an isotopy which transforms L to some nearby copy L' (disjoint from Δ), and recognize this isotopy (in X) as the restriction of a local Legendrian isotopy moving S to another Legendrian sphere S'.

Recall that there is a canonical contact model for the tubular neighbourhood $N_Y(S)$ of the Legendrian sphere S in Y. That is, there exists a contactomorphism

$$\Upsilon : (T^*S^n \times \mathbb{R}, \mathrm{Ker}(\mathbf{q}d\mathbf{p} + dz)) \longrightarrow (N_Y(S), \mathrm{Ker}(\bar{\beta}^{st}|_{N_Y(S)}))$$

from the 1-jet bundle $(T^*S^n \times \mathbb{R}, \mathrm{Ker}(\mathbf{q}d\mathbf{p} + dz))$ where $\mathbf{p} = (p_1, ..., p_n), \mathbf{q} = (q_1, ..., q_n)$ are the standard coordinates on T^*S^n and z is the real coordinate. (Here Υ maps the zero section $\mathbf{S}_0 = \{\mathbf{q} = \mathbf{0}\} \times \{0\} \subset T^*S^n \times \mathbb{R}$ onto S.) Observe that, on $T^*S^n \times \mathbb{R}$, there are $2n+1$ linearly independent contact vector fields:

$$(2) \qquad Z'_1 = \partial p_1, ..., Z'_n = \partial p_n, \quad Z'_{n+1} = \partial q_1 - p_1 \partial z, ..., Z'_{2n} = \partial q_n - p_n \partial z, \quad Z'_{2n+1} = \partial z$$

The corresponding contact Hamiltonian functions (as in Lemma 3.7), respectively, are

$$(3) \qquad H'_1 = q_1, ..., H'_n = q_n, \quad H'_{n+1} = -p_1, ..., H'_{2n} = -p_n, \quad H'_{2n+1} = 1.$$

We will use these contact vector fields for local Legendrian isotopies in $N_Y(S)(\cong T^*S^n \times \mathbb{R})$ that we need for our purpose.

Let $L \cap \Delta = \sqcup_{i=1}^s L_i$ where L_i's are (disjoint) connected components. Note that L is compact as both $\phi(S^n)$ and X are compact. Moreover, the core Δ is compact, and so $L \cap \Delta$ is also compact from which we conclude that each L_i is a compact CW-complex of finite type. Denote by L_i^K the K-skeleton of L_i for $K = 0, 1, ..., n-1$. In particular, we have $L_i = L_i^{n-1}$ (Figure 5-a). Using the isotopies mentioned above, we will first make the closure of every $(n-1)$-cell in each L_i disjoint from Δ, and then do the same for $(n-2)$-cells, and so on...

Let $\{E^K_{i,1}, ..., E^K_{i,l_i^K}\}$ be the set of all K-cells in L_i^K. For any $1 \leq j \leq l_i^K$, consider the following open cover for the closure $\overline{E^K_{i,j}}$:

$$\mathcal{U}^K_{i,j} := \{U_x \mid x \in E^K_{i,j}, U_x \text{ is a nbhd of } x \text{ in } \overline{E^K_{i,j}} \text{ s.t. } Z(x) \pitchfork \overline{U_x}\}.$$

Clearly, $\mathcal{U}^K_{i,j}$ covers $\overline{E^K_{i,j}}$ which is compact. So, $\mathcal{U}^K_{i,j}$ has a finite subcover

$$\{U_{x^K_{i,j,1}}, U_{x^K_{i,j,2}}, ..., U_{x^K_{i,j,m^K_{i,j}}}\}$$

for some finite number of points $\{x^K_{i,j,1}, x^K_{i,j,2}, ..., x^K_{i,j,m^K_{i,j}}\}$ in $E^K_{i,j}$. We label these points in such a way that the neighbourhood of any point has nonempty intersection with the union of the neighbourhoods of the preceding points in the list as depicted in Figure 5-b (for the case $K = n-1$).

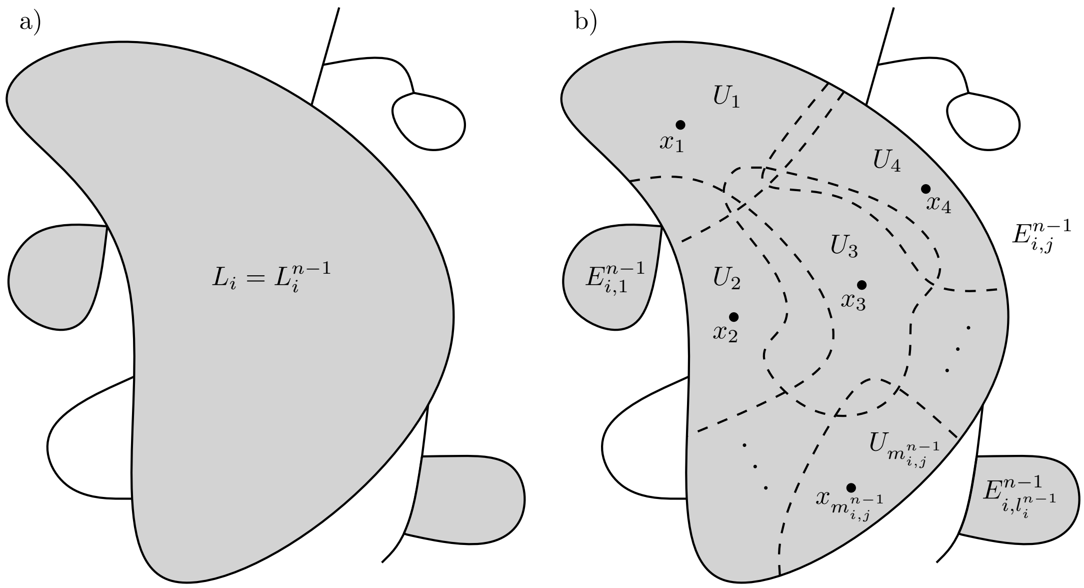

FIGURE 5. a) A typical connected component $L_i = L^{n-1}_i$ of $L \cap \Delta$, b) Finite subcover of $\mathcal{U}^{n-1}_{i,j}$ (where we write $x^{n-1}_{i,j,1} = x_1, ..., x^{n-1}_{i,j,m^{n-1}_{i,j}} = x_{m^{n-1}_{i,j}}$, and similarly $U_{x^{n-1}_{i,j,1}} = U_1, ..., U_{x^{n-1}_{i,j,m^{n-1}_{i,j}}} = U_{m^{n-1}_{i,j}}$ for simplicity).

We will first make $\overline{U_{x^{n-1}_{i,j,1}}}$ disjoint from Δ. From the definition of $\mathcal{U}^{n-1}_{i,j}$, we have a constant vector field $Z(x^{n-1}_{i,j,1}) \in TX|_{\overline{U_{x^{n-1}_{i,j,1}}}}$ which is everywhere transverse to $\overline{U_{x^{n-1}_{i,j,1}}}$. (Here at every point of $\overline{U_{x^{n-1}_{i,j,1}}}$, we have the same vector $Z(x^{n-1}_{i,j,1})$.) Observe that, since $\{Z'_i\}_{i=1}^{2n+1}$ is a linearly independent set, the pull-back vector field $\Upsilon^*(Z(x^{n-1}_{i,j,1}))$ can be written as the unique linear combination

$$\Upsilon^*(Z(x^{n-1}_{i,j,1})) = c_1 Z'_1 + \cdots c_{2n+1} Z'_{2n+1}$$

for some unique constants $c_1, ..., c_{2n+1} \in \mathbb{R}$. Since these constants depend on the point $x^{n-1}_{i,j,1}$, we set the notation

$$\mathbf{Z}_{x^{n-1}_{i,j,1}} := c_1 Z'_1 + \cdots c_{2n+1} Z'_{2n+1} (= \Upsilon^*(Z(x^{n-1}_{i,j,1}))).$$

Let us write $V_{x^{n-1}_{i,j,1}}$ for the pre-image $\Upsilon^{-1}(U_{x^{n-1}_{i,j,1}})$. Observe that $\mathbf{Z}_{x^{n-1}_{i,j,1}}$ is a contact vector field on $T^*S^n \times \mathbb{R}$ with the contact Hamiltonian

$$\mathbf{H}_{x^{n-1}_{i,j,1}} := c_1 H'_1 + \cdots c_{2n+1} H'_{2n+1},$$

and also that it is everywhere transverse to $\overline{V_{x^{n-1}_{i,j,1}}} = \Upsilon^{-1}(\overline{U_{x^{n-1}_{i,j,1}}}) \subset \Upsilon^{-1}(L) \subset \mathbf{S}_0$.

Denote by $\widetilde{V_{x^{n-1}_{i,j,1}}}$ a small neighbourhood of $\overline{V_{x^{n-1}_{i,j,1}}}$ in $\Upsilon^{-1}(N_Y(S) \cap X) \subset T^*S^n \times \mathbb{R}$. Let $\mu_{x^{n-1}_{i,j,1}} : \Upsilon^{-1}(N_Y(S) \cap X) \to \mathbb{R}$ be a smooth cut-off function such that $\mu_{x^{n-1}_{i,j,1}} \equiv 1$ near $\overline{V_{x^{n-1}_{i,j,1}}}$, and $\mu_{x^{n-1}_{i,j,1}} \equiv 0$ on the complement $\Upsilon^{-1}(N_Y(S) \cap X) \setminus \widetilde{V_{x^{n-1}_{i,j,1}}}$. Now consider the contact vector field $\mathbf{Z}_{\mu_{x^{n-1}_{i,j,1}}}$ whose corresponding contact Hamiltonian is equal to $\mu_{x^{n-1}_{i,j,1}}\mathbf{H}_{x^{n-1}_{i,j,1}}$. By the choice of the cut-off function $\mathbf{Z}_{\mu_{x^{n-1}_{i,j,1}}}|_{\overline{V_{x^{n-1}_{i,j,1}}}} = \mathbf{Z}_{x^{n-1}_{i,j,1}}$, and so $\mathbf{Z}_{\mu_{x^{n-1}_{i,j,1}}}$ is also transverse to $\overline{V_{x^{n-1}_{i,j,1}}}$. Using the flow $\mathbf{Z}^t_{\mu_{x^{n-1}_{i,j,1}}}$ we isotope $\overline{V_{x^{n-1}_{i,j,1}}}$ to its nearby copy $\mathbf{Z}^t_{\mu_{x^{n-1}_{i,j,1}}}(\overline{V_{x^{n-1}_{i,j,1}}})$ for some fixed time t. Note that pushing along a transverse contact vector field imples that $\mathbf{Z}^t_{\mu_{x^{n-1}_{i,j,1}}}(\overline{V_{x^{n-1}_{i,j,1}}})$ is disjoint from Δ, and is still isotropic in $(T^*S^n \times \mathbb{R}, \mathrm{Ker}(\mathbf{q}d\mathbf{p} + dz))$. Indeed, since the transversality is an open condition, we know that the isotropic image $\mathbf{Z}^t_{\mu_{x^{n-1}_{i,j,1}}}(\widehat{V_{x^{n-1}_{i,j,1}}})$ is disjoint from Δ where $\widehat{V_{x^{n-1}_{i,j,1}}} \supset \overline{V_{x^{n-1}_{i,j,1}}}$ is a neighbourhood of $\overline{V_{x^{n-1}_{i,j,1}}}$ in L_i such that

$$\widehat{V_{x^{n-1}_{i,j,1}}} \subset \widetilde{V_{x^{n-1}_{i,j,1}}} \cap L_i.$$

Similarly, we can make the closure of all the other open sets in the above finite subcover of $\mathcal{U}^{n-1}_{i,j}$ disjoint from Δ (this will isotope the whole closed $(n-1)$-cell $\overline{E^{n-1}_{i,j}}$ to some isotropic copy which is disjoint from Δ). However, for each such closure, the choice of how much we push it (using the flow of the corresponding contact vector field) needs a litle bit of more care: Let us discuss this in an inductive way: Suppose that we have already isotoped the union

$$\overline{V_{x^{n-1}_{i,j,1}}} \cup \overline{V_{x^{n-1}_{i,j,2}}} \cup \cdots \cup \overline{V_{x^{n-1}_{i,j,k}}}, \quad \text{for some } k \in \{1, ..., m^{n-1}_{i,j} - 1\}$$

along the contact vector fields $\{\mathbf{Z}_{\mu_{x^{n-1}_{i,j,m}}}\}_{m=1}^k$ (where the smooth cut-off functions $\mu_{x^{n-1}_{i,j,m}}$: $\Upsilon^{-1}(N_Y(S) \cap X) \to \mathbb{R}$ are constructed in the same way as above) so that the image of the union

$$\bigcup_{m=1}^{k} \widehat{V_{x^{n-1}_{i,j,m}}}$$

is isotropic in $(T^*S^n \times \mathbb{R}, \mathrm{Ker}(\mathbf{q}d\mathbf{p} + dz))$ and is disjoint from Δ where $\widehat{V_{x^{n-1}_{i,j,m}}} \subset \widetilde{V_{x^{n-1}_{i,j,m}}} \cap L_i$ is a small neighbourhood of $\overline{V_{x^{n-1}_{i,j,m}}}$ in L_i. Now we would like to push (i.e., isotope) $\overline{V_{x^{n-1}_{i,j,k+1}}}$ using $\mathbf{Z}_{\mu_{x^{n-1}_{i,j,k+1}}}$. Observe that the region

$$\overline{V_{x^{n-1}_{i,j,k+1}}} \cap (\widehat{V_{x^{n-1}_{i,j,1}}} \cup \cdots \cup \widehat{V_{x^{n-1}_{i,j,k}}})$$

has been already made disjoint from Δ, and also that $\mathbf{Z}_{\mu_{x^{n-1}_{i,j,k+1}}}$ might be tangent to the image of this region at some points, or even its flow might transform some points in the region back to Δ (if we let them flow much). On the other hand, since transversality is an open condition there exists a codim-0 subset $\overline{\overline{V_{x^{n-1}_{i,j,k+1}}}} \subset \overline{V_{x^{n-1}_{i,j,k+1}}}$ with a codim-0 nonempty intersection

$$\overline{\overline{V_{x^{n-1}_{i,j,k+1}}}} \cap (\widehat{V_{x^{n-1}_{i,j,1}}} \cup \cdots \cup \widehat{V_{x^{n-1}_{i,j,k}}})$$

to which $\mathbf{Z}_{\mu_{x^{n-1}_{i,j,k+1}}}$ is everywhere transverse. Therefore, we can make the union

$$\overline{V_{x^{n-1}_{i,j,1}}} \cup \cdots \cup \overline{V_{x^{n-1}_{i,j,k+1}}}$$

disjoint from Δ by pushing (in an appropriate amount) along $\mathbf{Z}_{\mu_{x^{n-1}_{i,j,k+1}}}$ as shown in Figure 6.

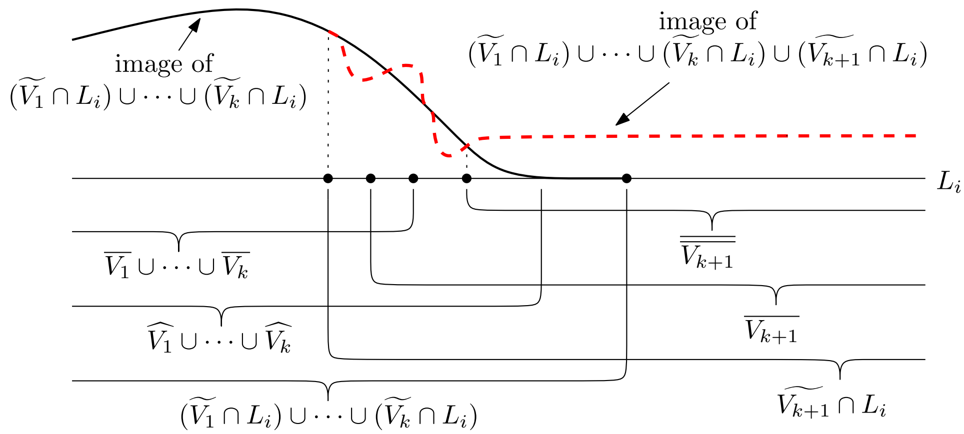

FIGURE 6. Making $\overline{V_{x^{n-1}_{i,j,k+1}}}$ disjoint from Δ (where for each $m = 1, ..., k+1$ we write $\overline{V_{x^{n-1}_{i,j,m}}} = \overline{V_m}$, $\widehat{V_{x^{n-1}_{i,j,m}}} = \widehat{V_m}$, $\widetilde{V_{x^{n-1}_{i,j,m}}} = \widetilde{V_m}$, and also $\overline{\overline{V_{x^{n-1}_{i,j,k+1}}}} = \overline{\overline{V_{k+1}}}$ for simplicity).

Repeating the above process we can make the union $\overline{E^{n-1}_{i,1}} \cup \cdots \cup \overline{E^{n-1}_{i,l^{n-1}_i}}$ of the closures of all $(n-1)$-cells disjoint from Δ by pushing to a nearby isotropic copy in $\Upsilon^{-1}(N_Y(S) \cap X)$. Note that for any particular closure in the union if some part of it has been already pushed (this might happen if it has a common boundary part with another $(n-1)$-cell which has been pushed earlier), then we isotope it in an appropriate amount (as in the above discussion) so that previously pushed regions in the cell would not be moved back to Δ.

Similarly, we can deal with the union $\{\overline{E^K_{i,1}} \cup \cdots \cup \overline{E^K_{i,l^K_i}}\}$ of all closed K-cells in L_i (for $0 \leq K \leq n-2$) in the same way (under the assumption that all closed $(K+1)$-cells have been already made disjoint from Δ. Note that we do not need to push any K-cell which appears as a part of the boundary of some $(K+1)$-cell(s) because such K-cells have been already made disjoint from Δ in the previous step. This process deforms the connected component L_i to its image L'_i which is isotropic and disjoint from Δ. Repeating this for each connected component, we conclude that there exists an isotopy of the embeddings in X from L to its nearby isotropic copy L' which is disjoint from Δ. This isotopy is generated by contact vector fields $\mathbf{Z}_{\mu_{x^K_{i,j,k}}}$ and compactly supported in the neighbourhood

$$\widetilde{V} := \bigcup_{0 \leq i \leq s} \ \bigcup_{0 \leq K \leq n-1} \ \bigcup_{0 \leq j \leq l^K_i} \ \bigcup_{0 \leq k \leq m^K_{i,j}} \widetilde{V_{x^K_{i,j,k}}} \subset \Upsilon^{-1}(N_Y(S) \cap X)$$

Next, we want to extend this local isotopy of L in $\Upsilon^{-1}(N_Y(S) \cap X) \times \{0\}$ to a local Legendrian isotopy of S compactly supported in $\widetilde{V} \times [-1,1] \subset \Upsilon^{-1}(N_Y(S) \cap X) \times [-1,1] \subset T^*S^n \times \mathbb{R}$. (Here since L is codimension 1 submanifold of S, we can consider the tubular neighbourhood of L in S as the product $L \times [-1,1]$ such that L corresponds to $L \times \{0\}$.) Consider the smooth cut-off function

$$f : \Upsilon^{-1}(N_Y(S) \cap X) \times [-1,1] \to \mathbb{R}, \quad f(x,t) = \mu(t)$$

where $\mu : [-1,1] \to \mathbb{R}$ is a smooth cut-off function which is equal to 1 near $t=0$, and 0 near $t=\pm 1$. Also let $\widetilde{\mu}_{x^K_{i,j,k}} : \Upsilon^{-1}(N_Y(S)\cap X)\times[-1,1] \to \mathbb{R}$ be the extension of $\mu_{x^K_{i,j,k}}$ given by

$$\widetilde{\mu}_{x^K_{i,j,k}}(x,t) = \mu_{x^K_{i,j,k}}(x).$$

Denote by $\widetilde{\mathbf{Z}}_{\mu_{x^K_{i,j,k}}}$ the contact vector field on $\Upsilon^{-1}(N_Y(S)\cap X)\times[-1,1]$ whose coresponding contact Hamiltonian (as in Lemma 3.7) is equal to $f\widetilde{\mu}_{x^K_{i,j,k}}\mathbf{H}_{x^K_{i,j,k}}$. Now we isotope S to its nearby Legendrian copy S' by applying the flow maps of $\widetilde{\mathbf{Z}}_{\mu_{x^K_{i,j,k}}}$ in the same order and amount that we apply the flow maps of $\mathbf{Z}_{\mu_{x^K_{i,j,k}}}$ to isotope L to L'. By construction, this isotopy is compactly supported in $\widetilde{V}\times[-1,1]$, and its restriction to $\Upsilon^{-1}(N_Y(S)\cap X)\times\{0\}$ is the isotopy taking L to L' (constructed above) as $\widetilde{\mathbf{Z}}_{\mu_{x^K_{i,j,k}}}|_{\Upsilon^{-1}(N_Y(S)\cap X)\times\{0\}} = \mathbf{Z}_{\mu_{x^K_{i,j,k}}}$. In particular, for the new Legendrian sphere S', its intersection $L' = S'\cap X$ is disjoint from the core Δ of X.

Finally, we note that by working on a small enough neighbourhood $N_Y(S)$ one can guarantee that S' lies in any pre-given ϵ-neighbourhood of S in Y. □

By the last lemma we may assume that the transverse intersection $L = S\cap X$ of the Legendrian sphere $S=\phi(S^n)$ is disjoint from the core Δ of $X=\Theta^{-1}(p)$. To finish the proof of Theorem 3.6, we will construct an isotopy of Legendrian embeddings of S which will be compactly supported in a neighbourhood of X in Y and will push L completely outside X. To this end, we will first isotope S along a contact vector field generated from Z_3 given in the list (1) above so that the part $L\cap Y_1$ of L (recall $Y_1 = \Sigma(X,h)_\rho$ is the mapping torus of the open book $\mathcal{OB}_{(X,h)}$) is completely pushed into the interior of $Y_2 = B\times D^2$ (tubular neighbourhood of the binding $B=\partial X$). Then using Z_1, Z_2 of the list (1) we will isotope S until $L\cap Y_2$ completely crosses the binding $B = B\times\{0\}$.

Remark 3.10. So far, when we write X we meant the whole page (in particular, ∂X was the binding B). However, for what follows it is better to use the abstract open book desription as in the previous paragraph. Therefore, from now on X will denote the complement of the collar neighbourhood of the binding B in the corresponding page.

From its construction the contact manifold $(Y_1 = \Sigma(X,h)_\rho, \alpha|_{Y_1})$ is obtained as the quotient space of $(X\times\mathbb{R}, \beta+dz)$ using the equivalence relation $\sim_\rho$. Recall that we have

$$Y_1 = X\times\mathbb{R}/\sim_\rho \quad \text{where} \quad (x,z)\sim_\rho (h(x), z+\rho(x)).$$

By translating (i.e., isotoping) S along the Reeb direction $Z_4 = R_\alpha$ (which corresponds to ∂z on $X\times\mathbb{R}$), we may assume that $X=\Theta^{-1}(p)$ corresponds to $X\times\{0\}$ under this identification. Therefore, we can identify a neighbourhood of X in Y_1 with $X\times[-a,a]\subset X\times\mathbb{R}$ for some real number $0<a<K$ where $K>0$ is a constant satisfying

$$K < \rho(x), \quad \forall x\in X.$$

(Note that such K exists since X is compact and ρ is a strictly positive continuous function). The contact form α on Y is equal to $\beta+dz$ on $X\times[-a,a]$ where Z_3 takes the form $Z := \chi + z\partial z$ as mentioned earlier. By choosing a small enough, we may guarantee that the intersection $S\cap(X\times[-a,a])$ is equal to $L\times[-a,a]$ (S and $X = X\times\{0\}$ intersect transversally), and also that $L\times[-a,a]$ is disjoint from $\Delta\times[-a,a]$ (this is because all cut-off functions which we used to isotope S to S' in the proof of Lemma 3.9 are all equal to 1 near $(L\times\{0\})\cap(\Delta\times\{0\})$.)

One can think of Y_1 slightly larger by considering $Y_2 = B \times D^2$ slightly smaller. More precisely, let $N = B \times D \subset Y_2$ be a smaller neighbourhood of B where $D \subset D^2$ is a smaller disk in $\mathbb{R}^2$ around the origin. By expanding each (X, β) in Y_1 to a larger domain $(\tilde{X}, \tilde{\beta})$, we get another decomposition $Y = Y_1' \cup N$ where $Y_1' = Y \setminus N$. Note that extending the above identification, a neighbourhood of $\tilde{X}$ in Y_1' can be identified with $\tilde{X} \times [-a, a]$ ($\tilde{X} = \tilde{X} \times \{0\}$) where we have $\alpha = \tilde{\beta} + dz$ and the extension $\tilde{Z} = \tilde{\chi} + z\partial z$ of Z ($\tilde{\chi}$ is the $d\tilde{\beta}$-dual of $\tilde{\beta}$). We remark that $\tilde{Z}$ is contact with the contact Hamiltonian $H = z$. Let $\mu_1 : Y \to \mathbb{R}$ be a smooth cut-off function such that $\mu_1 \equiv 1$ near $\tilde{X} \times \{0\}$ and $\mu_1 \equiv 0$ in the complement $Y \setminus \tilde{X} \times (-\epsilon, \epsilon)$ for some $0 < \epsilon < a$ which will be determined later. Denote by Z_{μ_1} the contact vector field on Y which corresponds to the contact Hamiltonian $\mu_1 H$.

Now we first push S using the flow $Z^t_{\mu_1}$ until the image of L lies completely in the interior $\mathrm{int}(Y_2)$ of Y_2 as follows: First note that $Z_{\mu_1} = \tilde{\chi}$ on $\tilde{X} = \tilde{X} \times \{0\}$ (as $z = 0$ there). Since L is disjoint from the core of $\tilde{X}$ (which is the same as that of X), for every point $x \in L \cap \tilde{X}$ there exists a unique flow line of $\tilde{\chi}$ (i.e., of $Z_{\mu_1}|_{\tilde{X}}$) passing through x. All such flow lines reach the region $\tilde{X} \cap \mathrm{int}(Y_2 \setminus N)$. Consider the set

$$A = \{t \in (0, \infty) \mid Z^t_{\mu_1}(x) \in \tilde{X} \cap \mathrm{int}(Y_2 \setminus N) \quad \text{for all } x \in L \cap \tilde{X}\}.$$

Since $L \cap \tilde{X}$ is a compact, the set A is non-empty. Choose a finite number $T > 0$ from A. We isotope the Legendrian sphere $S \subset Y$ using the flow maps $\{Z^t_{\mu_1} \mid 0 \leq t \leq T\}$. Indeed, by the construction of Z_{μ_1}, we only push the region $(L \cap \tilde{X}) \times (-\epsilon, \epsilon) \subset S$ during the isotopy. Observe that by choosing $\epsilon > 0$ small enough, we can guarantee that the image $Z^T_{\mu_1}(L \times (-\epsilon, \epsilon))$ completely lies in $\tilde{X} \times [-a, a]$. Therefore, $Z^T_{\mu_1}(S)$ is an embedded Legendrian which is Legendrian isotopic to S (see Figure 7).

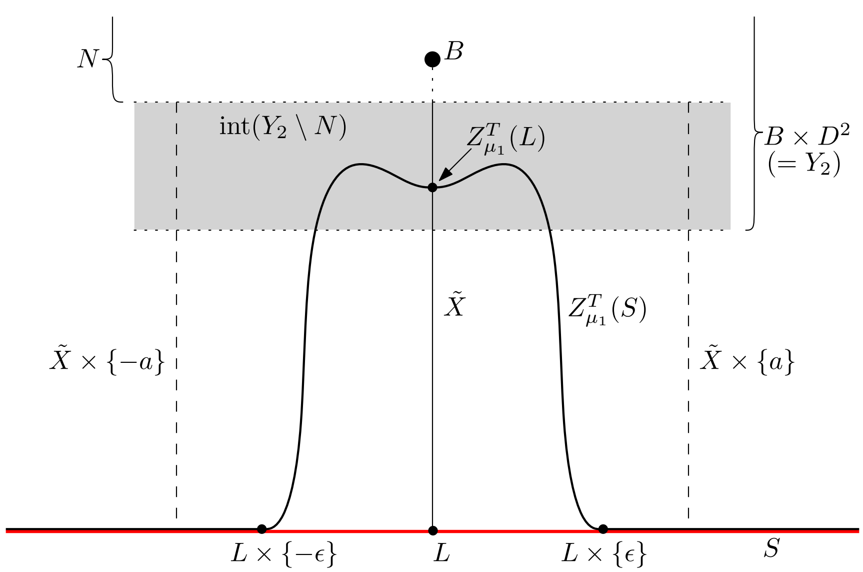

Figure 7. Making L closer to the binding via a Legendrian isotopy of S.

By the choice of the flow parameter T above, we know that $Z^T_{\mu_1}(L)$ is completely lie in the region $\mathrm{int}(Y_2)$. For $0 < r < 1$ consider the disk

$$D_r = \{(x, y) \in D^2 \mid x^2 + y^2 \leq r^2\} \subset D^2$$

and the neighborhood $N_r = B \times D_r \subset \mathrm{int}(B \times D^2) = \mathrm{int}(Y_2)$ of the binding. Then after the above isotopy we know that $Z^T_{\mu_1}(L) \subset N_r$ for some r. Let $(a_1, a_2) \in D^2$ be a point which corresponds to the angular coordinate $p \in S^1$ (recall that $\tilde{X} = \tilde{X} \times \{p\}$). Recall the contact vector fields Z_1, Z_2 from the list (1) and their contact Hamiltonians H_1, H_2. Then the vector field $Z' = a_1Z_1 + a_2Z_2$ is also contact whose contact Hamiltonian is given by $H' = a_1H_1 + a_2H_2$. Let $\mu_2 : Y \to \mathbb{R}$ be a smooth cut-off function such that $\mu_2 \equiv 1$ near N_r and $\mu_2 \equiv 0$ in the complement $Y \setminus Y_2$. Denote by Z_{μ_2} the contact vector field on Y which corresponds to the contact Hamiltonian $\mu_2 H'$. Now we isotope $Z^T_{\mu_1}(S)$ using the flow maps of the contact vector field Z_{μ_2} until $Z^T_{\mu_1}(L)$ completely crosses the binding. Say for $T' > 0$ the image of $Z^T_{\mu_1}(L)$ under the flow map $Z^{T'}_{\mu_2}$ completely crosses the binding. As a result, the page $\Theta^{-1}(p)$ of the open book $\mathcal{OB}_{(X,h)}$ is disjoint from the final image $S' := Z^{T'}_{\mu_2}(Z^T_{\mu_1}(S))$ which is Legendrian isotopic to $S = \phi(S^n)$ as claimed. This finishes the proof of Theorem 3.6. □

Corollary 3.11. *Let $\mathcal{OB}$ be an open book decomposition carrying a contact structure ξ on a (closed) manifold Y of dimension $2n + 1 > 3$. In particular, there exists a contact form α for ξ such that α restricts to a Liouville form on every page of $\mathcal{OB}$. Suppose that for any page X of $\mathcal{OB}$, the resriction $\alpha|_X$ is, indeed, the underlying Liouville form of a Weinstein structure on X. Let $\{\psi_i : S^n \hookrightarrow (Y, \xi)\}_{i=1}^k$ be a family of disjointly embedded Legendrian spheres. Then the Legendrian link $\bigsqcup_{i=1}^k \psi_i(S^n)$ can be Legendrian isotoped (through embedded Legendrian links) to another embedded Legendrian link which is disjoint from a page of the of the open book $\mathcal{OB}$.*

Proof. Pick a page X equipped with the Liouville form $\beta := \alpha|_X$ which is , by assumption, the underlying Liouville form of a Weinstein structure on X. Denote by h the monodromy of $\mathcal{OB}$ and consider the manifold $M(X, h)$ equipped with the contact structure $\xi' = \mathrm{Ker}(\alpha_\beta)$ where α_β is the contact form on $M(X, h)$ constructed as in Remark 3.5. Then we have a contactomorphism

$$\Upsilon : (M(X, h), \xi') \to (Y, \xi)$$

because $\Upsilon_*(\xi')$ and ξ are isotopic contact structure on Y. (The existence of this isotopy follows from similar lines used at the beginning of the proof of Proposition 3.3.) By pulling back ψ_i using this contactomorphism, we obtain the embedding $\phi_i := \Upsilon^{-1} \circ \psi_i : S^n \hookrightarrow (M(X, h), \xi')$ of a Legendrian sphere $\phi_i(S^n)$ for each $i = 1, ..., k$. Consider the embedded Legendrian link $\mathbb{S} = \bigsqcup_{i=1}^k \phi_i(S^n) \subset (M(X, h), \xi')$. Then by Theorem 3.6 we have a smooth 1-parameter family

$$\Phi_t : \bigsqcup_{i=1}^{k} S^n \hookrightarrow (M(X, h), \xi'), \quad t \in [0, 1]$$

of Legendrian embeddings such that $\Phi_0(\bigsqcup_{i=1}^k S^n) = \mathbb{S}$ and the embedded Legendrian link $\Phi_1(\bigsqcup_{i=1}^k S^n)$ is disjoint from a page of the open book $\mathcal{OB}_{(X,h)}$ on $M(X, h)$ associated to (X, h). By pushing forward Φ_t using the contactomorphism Υ, we obtain a smooth 1-parameter family

$$\Psi_t := \Upsilon \circ \Phi_t : \bigsqcup_{i=1}^{k} S^n \hookrightarrow (Y, \xi), \quad t \in [0, 1]$$

of Legendrian embeddings such that $\Psi_0(\bigsqcup_{i=1}^k S^n) = \bigsqcup_{i=1}^k \psi_i(S^n)$. Now since Υ is page preserving (i.e., mapping pages of $\mathcal{OB}_{(X,h)}$ to the corresponding pages of $\mathcal{OB}$), the Legendrian link $\Psi_1(\bigsqcup_{i=1}^k S^n)$ is disjoint from a page of the open book $\mathcal{OB}$. □

Next we have another lemma which will be also used in the proof of Theorem 3.1. Consider the contact manifold $(X \times \mathbb{R}, \mathrm{Ker}(\beta + dz))$ where (X, β) is any Liouville domain and z is the coordinate on $\mathbb{R}$. Then we have the *Lagrangian projection* $\Pi : X \times \mathbb{R} \to X$. The image of a Legendrian sphere in $X \times \mathbb{R}$ under Π is an immersed Lagrangian sphere in $(X, d\beta)$. Indeed, with a little bit more care, more is true:

Lemma 3.12. *Suppose that X is a Liouville domain of dimension $2n \geq 4$ whose Liouville structure is given by the Liouville form β. Denote by χ the $d\beta$-dual of β. Let S be any embedded Legendrian sphere in $(X \times \mathbb{R}, \mathrm{Ker}(\beta + dz))$. Then after an ϵ-Legendrian isotopy of S we may assume that $\Pi(S)$ is an immersed Lagrangian sphere in $(X, d\beta)$ such that*

- (i) *Any self intersection of $\Pi(S)$ is a transverse double point.*
- (ii) *The set of double points of $\Pi(S)$ is disjoint from the core* $\mathrm{Core}(X, \beta)$.
- (iii) *For any double point P of $\Pi(S)$, there exits a trajectory σ of χ connecting P to a point $R \subset \partial X$ such that the part of σ between P and R intersects $\pi(S)$ only at P.*

Proof. The following definitions and results are from [EES1] and [EES2]. Let c denote a *Reeb chord* of $S \subset X \times \mathbb{R}$, that is, a trajectory of the Reeb vector field ∂z starting and ending at points on S. Then $P = \Pi(c)$ is a double point of $\Pi(S)$. S is said to be *chord generic* if the only self intersections of the Lagrangian immersion $\Pi(S)$ are transverse double points. Note that this is an open and dense condition. Indeed, Lemma 2.7 of [EES2] implies that, by an arbitrarily small Legendrian isotopy we can isotope S so that it becomes chord generic. Hence, the part (i) follows.

For (ii) and (iii), basically we follow a similar argument. Let us do this in a more precise way: Let $\epsilon > 0$ be a given arbitrarily small number. Fix a transverse double point $P = \Pi(c)$ of $\Pi(S)$ with the Reeb cord c. The boundary ∂c consists of a pair $\{c_1, c_2\}$ of points in S. For each $j = 1, 2$, let N_j be a small neighbourhood of c_j in some small neighbourhood of S in the given ϵ range. Here we consider N_j as a subset of 1-jet bundle $T^*S \times \mathbb{R}$ (which can be identified with the ϵ-neighbourhood of S in $X \times \mathbb{R}$). Consider a small $2n$-ball $\mathbf{D} \subset X$ around P such that $\mathbf{D} \subset \Pi(N_1) \cap \Pi(N_2)$.

Let $Q \in \mathbf{D} \subset X$ be a point near the double point P such that

- (I) $Q \in X \setminus \mathrm{Core}(X, \beta)$,
- (II) the constant vector field Z on $\mathbf{D}$ in the direction from P to Q is transverse to both of the Lagrangian disks $D_1 := \Pi(N_1 \cap S)$, $D_2 := \Pi(N_2 \cap S)$, and
- (III) there exits a (unique) trajectory σ of χ which lies in the symplectization $\partial X \times (-\infty, 0] = X \setminus \mathrm{Core}(X, \beta)$ starting at Q and ending at a point $R \subset \partial X$ such that $\sigma \cap \pi(S) = \emptyset$.

Note that, by general position in X and taking $\mathbf{D}$ small enough, and the fact that interior of $\mathrm{Core}(X, \beta)$ is an empty set (Remark 2.2), such Q exits in $\mathbf{D}$ (again this is an open and dense condition as depicted in Figure 8 and recall that $n > 1$).

For $j = 1, 2$, we pick a preimage $Q_j \in \Pi^{-1}(Q) \cap N_j$ such that the lift $\tilde{Z}_j$ of Z pointing from c_j to Q_j is transverse to the sheet $S_j := S \cap N_j$ as depicted in Figure 9. Considering $\tilde{Z}_j$ as a constant vector field on $T^*S \times \mathbb{R}$, we can write it as a linear combination

$$\tilde{Z}_j = A_1^j Z_1' + \cdots + A_{2n+1}^j Z_{2n+1}'$$

where $\{Z_i'\}_{i=1}^{2n+1}$ are linearly independent contact vector fields (defined on $T^*S \times \mathbb{R}$) given in the list (2). Therefore, $\tilde{Z}_j$ is also a contact vector field with the contact Hamiltonian

$$\tilde{H}_j = A_1^j H_1' + \cdots + A_{2n+1}^j H_{2n+1}'$$

where each H_i' is the contact Hamiltonian of Z_i' as in the list (3). Consider a smaller neighbourhood $N_j' \subset N_j$ of c_j in $T^*S \times \mathbb{R}$ such that $Q_j \in N_j'$. Let $\mu_j : T^*S \times \mathbb{R} \to \mathbb{R}$ be a smooth cut-off

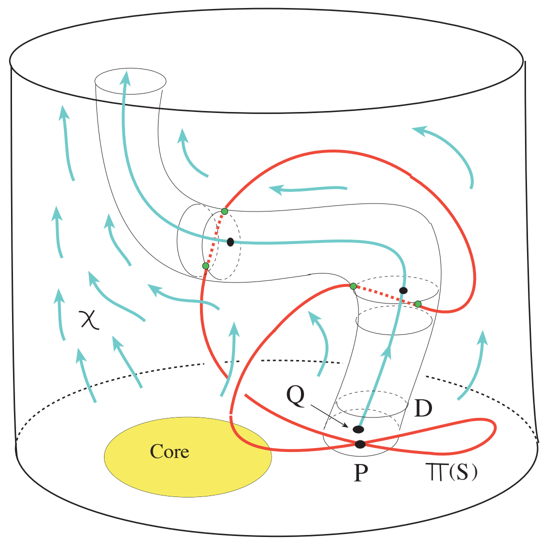

FIGURE 8. Choosing a nearby point Q by genericity.

function such that $\mu_j \equiv 1$ near N_j' and $\mu_j \equiv 0$ in the complement $T^*S \times \mathbb{R} \setminus N_j$. Denote by $\tilde{Z}_{\mu_j}$ the contact vector field on $T^*S \times \mathbb{R}$ which corresponds to the contact Hamiltonian $\mu_j \tilde{H}_j$. Note that by the choice of μ_j, the trajectory of $\tilde{Z}_{\mu_j}$ starting from c_j passes through Q_j.

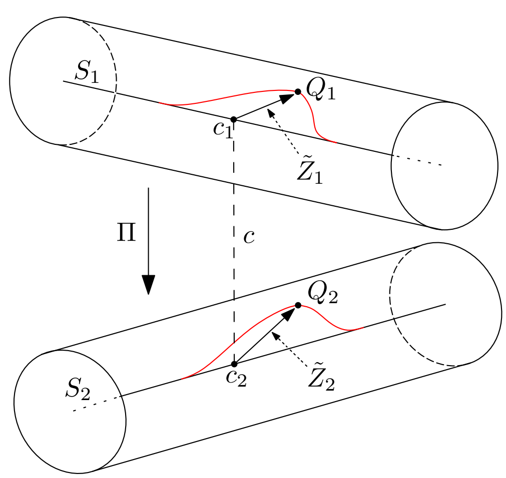

FIGURE 9. Local Legendrian isotopy of S_j moving c_j to a nearby point Q_j.

Now we locally (Legendrian) isotope the embedding of S in $T^*S \times \mathbb{R}$ (near $S_j \subset S$) along the contact vector field $\tilde{Z}_{\mu_j}$ until c_j reaches Q_j as depicted in Figure 9. By construction, it is clear that the corresponding isotopies in X (along $\Pi_*(\tilde{Z}_{\mu_j})$) deform the Lagrangian n-disks D_1, D_2 (through Lagrangian disks) so that the double point P is replaced with the new one Q. □

Before starting the proof of Theorem 3.1, we need one more result which is, indeed, a more general version of Proposition 8.1 in [BEE].

Proposition 3.13. *Let $\pi : W \to D^2$ be a Liouville Lefschetz fibration on a Liouville domain (W, Ω, Z) with regular fiber $(X, \omega = \Omega|_X)$. Suppose $S \subset (\partial W, \xi = \mathrm{Ker}(\iota_Z(\Omega))$ is an embedded Legendrian sphere which misses a page of the boundary convex open book. Also suppose that S projects onto an embedded Lagrangian sphere L sitting on a page (X, ω) of the boundary convex open book. Let $W(S)$ (resp. $W(L)$) denote the Liouville domain obtained from W by attaching a Weinstein (resp. Lefschetz) handle along S (resp. L). Then $W(S)$ and $W(L)$ are Liouville isomorphic.*

Proof. The proof basically follows similar lines used in Proposition 8.1 of [BEE]: Consider the decomposition of ∂W given by the boundary convex open book. That is, we have

$$\partial W = \Sigma(X, h) \,\cup\, B \times D^2$$

where $\Sigma(X, h) = X \times [0, 1]/(p, 0) \sim (h(p), 1)$ is the mapping torus determined by the monodromy h of π and $B = \partial X$ is the binding. Note that since S is disjoint from a page $X \times \{p\}$, we may assume that it is also disjoint from $B \times D^2$, and therefore, S is contained in $X \times [0, 1]$ which we consider as a subset of $\Sigma(X, h) \subset \partial W$. In terms of the terminology given in Section 8 of [BEE], the set $\{S\}$ is a basis for Lefschetz type Legendrian surgery, and hence the claim follows from Proposition 8.1 of [BEE]. □

Proof of Theorem 3.1. Let (W, Ω, Z, Ψ) be a Weinstein domain of dimension $2n + 2 \geq 6$. By Theorem 2.8 (this also follows from Theorem 2.10), we know that W^{2n+2} is characterized by

$$W^{2n+2} \cong (2n + 2)\text{-ball } B^{2n+2} + \{\text{Weinstein handles}\}.$$

Weinstein handles (introduced in [W]) can have indices at most $n+1$ as they are attached along isotropic spheres. Therefore, we may assume that W^{2n+2} is obtained from attaching Weinstein handles of index $(n + 1)$ to a subcritical Weinstein domain. By Theorem 2.12 every subcritical Weinstein domain splits, and so using Remark 2.13 we may describe W as

$$W \cong (X \times D^2, \Omega_X + r dr \wedge d\theta, Z_X + \frac{r}{2}\partial r, \Psi_X + r^2) + \mathcal{H}_1 + ... + \mathcal{H}_k$$

where $(X, \Omega_X, Z_X, \Psi_X)$ is a Weinstein domain of dimension $2n$, and $\mathcal{H}_i$ $(1 \leq i \leq k)$ is a Weinstein handle of index $n+1$. Instead of the Weinstein domain $X \times D^2$, we will consider the smooth (no codimension-2 corners) version $\bar{X}^{st}$ which admits a trivial (no singular fibers) Liouville Lefschetz fibration $(\pi^{st}_T, \bar{X}^{st}, d\bar{\beta}^{st}, \bar{\beta}^{st}, X, \mathrm{id})$. (Here we apply the construction at the beginning of Section 3 by taking $(X, \beta) = (X, \iota_{Z_X}\Omega_X)$.) Therefore, we may assume that

$$W = \bar{X}^{st} + \mathcal{H}_1 + ... + \mathcal{H}_k$$

where $\mathcal{H}_i \cong D^{n+1} \times D^{n+1}$ is a Weinstein handle (attached to the Weinstein domain $\bar{X}^{st}$) attached along a Legendrian sphere $S_i \subset (\partial \bar{X}^{st}, \mathrm{Ker}(\bar{\beta}^{st}|_{\partial \bar{X}^{st}}))$ with an embedding

$$\phi_i : S^n \hookrightarrow \phi_i(S^n) = S_i \subset \partial \bar{X}^{st}$$

which determines a canonical normal framing. Starting from the Weinstein domain $W_0 := \bar{X}^{st}$ with the convex boundary $(\partial W_0, \xi_0) = (\partial \bar{X}^{st}, \mathrm{Ker}(\bar{\beta}^{st}|_{\partial \bar{X}^{st}}))$, for $i = 1, ..., k$ we iteratively define W_i to be the Weinstein domain obtained from the Weinstein domain W_{i-1} by attaching the Weinstein handle $\mathcal{H}_i$ along the Legendrian sphere S_i in the convex boundary ∂W_{i-1} of W_{i-1}. So, we have

$$W_0 = \bar{X}^{st}, \;\; ... \;, \;\; W_i = W_{i-1} \cup \mathcal{H}_i, \;\; ... \;, \;\; W_k = W_{k-1} \cup \mathcal{H}_k = W.$$

We will prove the theorem by induction on the number k of the Weinstein handles attached to the Weinstein domain $W_0 = \bar{X}^{st}$. We remark that the second and the third statements in the theorem immediately follow from the first one because for the first statement we show the existence of a Liouville Lefschetz fibration and such a fibration satisfies the second and third statements. Therefore, during each step of the induction it will be enough for us to prove the first statement.

The case $k = 0$: The theorem follows from the fact that the Weinstein domain $W = W_0 = \bar{X}^{st}$ admits the trivial Liouville Lefschetz fibration $(\pi_T^{st}, \bar{X}^{st}, d\bar{\beta}^{st}, \bar{\beta}^{st}, X, \mathrm{id})$.

Induction hypothesis: Suppose that the theorem holds for the Weinstein domain W_{i-1}.

Claim 3.14. *The theorem also holds for the Weinstein domain* $W_i = W_{i-1} \cup \mathcal{H}_i$.

By the induction hypothesis, there exists a Liouville domain W'_{i-1}, diffeomorphic to W_{i-1}, such that W_{i-1} is Liouville isomorphic to W'_{i-1}, and W'_{i-1} admits a Liouville Lefschetz fibration with Weinstein regular fibers. The Liouville isomorphism between the completions $\widehat{W}_{i-1}$ and $\widehat{W'}_{i-1}$ defines a contactomorphism $f : \partial W_{i-1} \to \partial W'_{i-1}$ between the convex boundaries ∂W_{i-1}, $\partial W'_{i-1}$ equipped with the contact structures induced, respectively, by the Liouville structures on W_{i-1}, W'_{i-1}. Therefore, up to Liouville isomorphism, we can write

$$W_i = W'_{i-1} \cup \mathcal{H}_i$$

where the Weinstein handle $\mathcal{H}_i$ is attached along the Legendrian sphere $f(S_i)$ in the convex boundary $\partial W'_{i-1}$. Thus, Claim 3.14 follows from the proposition below.

Proposition 3.15. *Let* $(\widetilde{\pi}, \widetilde{W}, \widetilde{\Omega}, \widetilde{\Lambda}, \widetilde{X}, \widetilde{h})$ *be a Liouville Lefschetz fibration on a Liouville domain* $(\widetilde{W}, \widetilde{\Omega}, \widetilde{Z})$. *Suppose that for any regular fiber* $\widetilde{X}$ *of* $\widetilde{\pi}$, *the restriction* $\widetilde{\Lambda}|_{\widetilde{X}}$ *is the underlying Liouville form of some Weinstein structure on* $\widetilde{X}$. *Also let* (W, Ω, Z) *be the Liouville domain described by* $W = \widetilde{W} + \mathcal{H}$ *where* $\mathcal{H}$ *is a Weinstein handle attached along a Legendrian sphere* $S \subset (\partial\widetilde{W}, \widetilde{\xi} = \mathrm{Ker}(\widetilde{\Lambda}|_{\partial\widetilde{W}}))$. *Then there exists a Liouville Lefschetz fibration* $(\pi', W', \Omega', \Lambda', X', h')$ *on a Liouville domain* (W', Ω', Z') *which is Liouville isomorphic to* (W, Ω, Z).

Proof of Proposition 3.15. Consider the boundary convex open book $\mathcal{OB}_{\widetilde{\pi}} = (\Theta_{\widetilde{\pi}}, B)$ on $\partial\widetilde{W}$ induced by $\widetilde{\pi}$. We have an embedded Legendrian sphere $S \subset (\partial\widetilde{W}, \widetilde{\xi} = \mathrm{Ker}(\widetilde{\Lambda}|_{\partial\widetilde{W}}))$. By Corollary 3.11, we may assume that S is disjoint from a page $\Theta_{\widetilde{\pi}}^{-1}(p)$ for some $p \in S^1$. In order to simplify the notation, we will write $\Theta_{\widetilde{\pi}}^{-1}(p) = X$. Denote by β' the restriction $\widetilde{\Lambda}|_X$. Then by assumption β' is the underlying Liouville form of some Weinstein structure on X. On the other hand, Proposition 3.3 implies that (up to a Liouville isomorphism and a 1-parameter smooth deformation of Liouville Lefschetz fibrations) we may assume that there exists a constant $K > 0$ such that the Liouville form $\widetilde{\Lambda}$ on $\widetilde{W}$ restricts to $\beta := K\beta'$ on every page $\widetilde{X}$ of the boundary convex open book $\mathcal{OB}_{\widetilde{\pi}}$. Similar to the above discussion, since the Liouville isomorphism (between the completions) defines a contactomorphism between the convex boundaries, we may still assume that S is a Legendrian sphere sitting in $(\partial\widetilde{W}, \widetilde{\xi} = \mathrm{Ker}(\widetilde{\Lambda}|_{\partial\widetilde{W}}))$ by considering it as its image under this contactomorphism.

By cutting out the page $\Theta_{\widetilde{\pi}}^{-1}(p) = X$ (that S misses) from $\mathcal{OB}_{\widetilde{\pi}}$ (this also cuts out the binding $B = \partial X$), we obtain the manifold $X \times (0, 1) \approx X \times \mathbb{R}$. From the construction in the proof of Proposition 3.3, the contact form $\widetilde{\Lambda}|_{\partial\widetilde{W}}$ is equal to the pull back of the contact form $K\alpha_{\beta'}$ on $M(X, h)$ where h is the monodromy of $\mathcal{OB}_{\widetilde{\pi}}$ obtained from the first return map of a suitable vector field on $\partial\widetilde{W}$ transversal to fibers of $\Theta_{\widetilde{\pi}}$. As a result, we can consider S as an embedded

Legendrian sphere in the contact manifold

$$(\partial\widetilde{W} \setminus \Theta_{\widetilde{\pi}}^{-1}(p), \mathrm{Ker}(\widetilde{\Lambda}|_{\partial\widetilde{W}\setminus\Theta_{\widetilde{\pi}}^{-1}(p)})) = (X \times \mathbb{R}, \mathrm{Ker}(K\beta' + Kdz)) = (X \times \mathbb{R}, \mathrm{Ker}(\beta + Kdz))$$

where z is the coordinate on $\mathbb{R}$. We note that since S also misses the pages near $\Theta_{\widetilde{\pi}}^{-1}(p)$, there exists a number $0 < K' \in \mathbb{R}$ such that

$$S \cap (X \times \{z\}) = \emptyset, \quad \text{for all } |z| \geq K'.$$

Observe that any Legendrian in $X \times \mathbb{R}$ projects onto Lagrangian in X since the coefficient K in front of dz is constant, and so we have a well-defined Lagrangian projection map. Next, we project S onto $X \times \{z_0\}$ for some fixed z_0 with $|z_0| > K'$ using the Lagrangian projection $\Pi : X \times \mathbb{R} \to X \times \{z_0\}$. For simplicity, we will write X for $X \times \{z_0\}$. By Lemma 3.12, we know that $L := \Pi(S)$ is an immersed Lagrangian sphere in $(X, d\beta)$ whose self-intersections are all transverse double points and the properties in the lemma are satistified. More precisely, if $\{P_1, ..., P_d\}$ is the set of all transverse double points of L, then for each $j = 1, ..., d$, by Lemma 3.12, P_j lies in the complement of the core $\mathrm{Core}(X, \beta)$ of X. By asumption X is equipped with a Weinstein structure $(d\beta, \chi, \psi)$. Then there is a (unique) trajectory σ_j of the Liouville vector field χ from P_j to a point $R_j \in \partial X$ such that $\sigma_j \cap L = \{P_j\}$. Consider a small neighbourhood $\mathbf{N}(R_j) \approx D^{2n-1}$ of R_j in ∂X, and let $\mathbf{B}(P_j) \approx D^{2n}$ be a small Darboux ball around P_j in $X \setminus \mathrm{Core}(X, \beta)$ such that the the flow of χ maps the entire ball $\mathbf{B}(P_j)$ into $\mathbf{N}(R_j)$. The intersection $L_i \cap \mathbf{B}(P_j)$ consists of a pair $B_1(P_j), B_2(P_j)$ of Lagrangian disks intersecting transversally at P_j as in Figure 10.

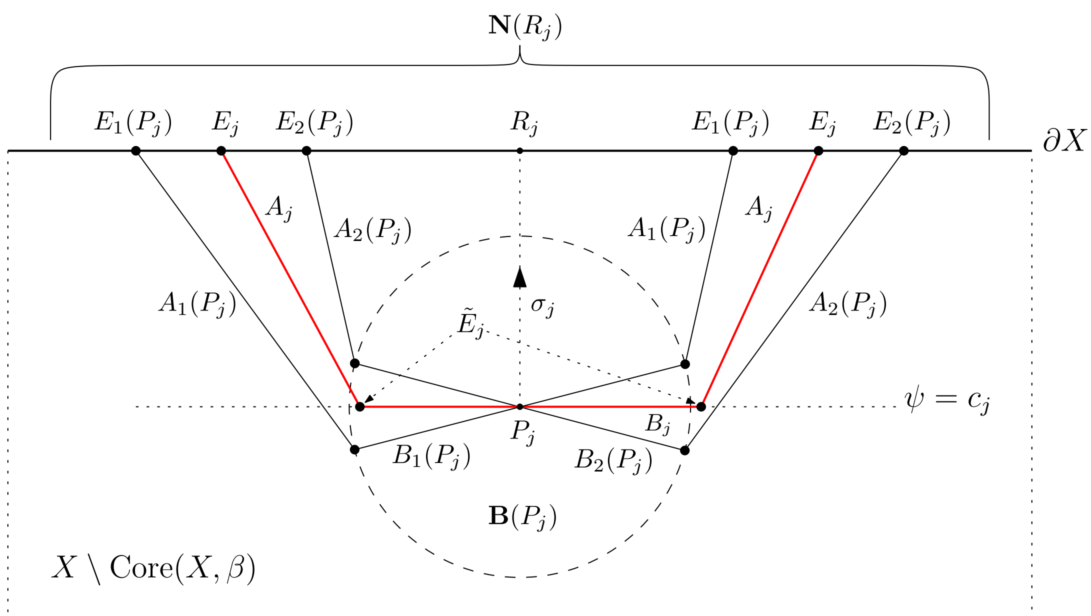

FIGURE 10. Forming the third Lagrangian disk D_j (a smoothing of $A_j \cup B_j$) at the double point P_j.

Observe that the $(n-1)$-spheres $\partial B_1(P_j)$, $\partial B_2(P_j) \subset \partial\mathbf{B}(P_j)$ link each other geometrically once. Denote by $A_1(P_j)$ (resp. $A_2(P_j)$ the cylinder ($\approx S^{n-1} \times [0,1]$) obtained by translating $\partial B_1(P_j)$ (resp. $\partial B_2(P_j)$) along χ to the boundary ∂X. Then we obtain a pair of $(n-1)$-spheres

$$E_1(P_j) := A_1(P_j) \cap \partial X, \quad E_2(P_j) := A_2(P_j) \cap \partial X$$

in $\mathbf{N}(R_j)$ which are also linked to each other geometrically once. Take a small trivial Legendrian sphere $E_j \subset \mathbf{N}(R_j)$ which links each of $E_1(P_j)$ and $E_2(P_j)$ geometrically once. Consider the Lagrangian cylinder A_j consisting of trajectories connecting E_j with its translate $\tilde{E}_j$ on the level $\psi = c_j$ of the point P_j. The Legendrian sphere $\tilde{E}_j$ bounds a Lagrangian disk B_j which also sits on the level $\psi = c_j$ as depicted in Figure 10. As a result, we obtain a (smooth) Lagrangian disk D_j which is a smoothing of the Lagrangian union $A_j \cup B_j$.

Next, by following [AA1] we positively stabilize the Liouville Lefschetz fibration

$$(\widetilde{\pi}, \widetilde{W}, \widetilde{\Omega}, \widetilde{\Lambda}, \widetilde{X}, \widetilde{h})$$

for each double point P_j along the Lagrangian disk D_j. Each such stabilization is performed in two stages which can be interprated in terms of attaching $(2n+2)$-dimensional handles H^j_n and H^j_{n+1} of indices n and $n+1$, respectively. The result is another Liouville Lefschetz fibration $(\pi_{\mathcal{V}}, W_{\mathcal{V}}, \Omega_{\mathcal{V}}, \Lambda_{\mathcal{V}}, X_{\mathcal{V}}, h_{\mathcal{V}})$ on a Liouville domain $(W_{\mathcal{V}}, \Omega_{\mathcal{V}}, Z_{\mathcal{V}})$ where

$$W_{\mathcal{V}} = \widetilde{W} \;\cup\; \bigcup_j H^j_n \;\cup\; \bigcup_j H^j_{n+1}$$

is diffeomorphic to $\widetilde{W}$ (as each $\{H^j_n, H^j_{n+1}\}$ is a canceling pair in smooth category) and $Z_{\mathcal{V}}$ is the $\Omega_{\mathcal{V}}$-dual of the Liouville form $\Lambda_{\mathcal{V}}$ [S2] (see also Proposition 6.5 of [AA1]). From the definition of such stabilizations, the new page $X_{\mathcal{V}}$ is obtained from X by attaching $2n$-dimensional Weinstein handles H_j (of indicies n) along the Legendrian spheres $\{E_j \mid j = 1, ..., d\}$ in the binding ∂X, and the new monodromy is given by

$$h_{\mathcal{V}} = \delta_1 \circ \delta_2 \circ \cdots \circ \delta_d$$

where each δ_j is the right-handed Dehn twist about the Lagrangian sphere V_j obtained by gluing D_j with the Lagrangian core disk C_j of the handle H_j along E_j. Note that each V_j in the set

$$\mathcal{V} := \{V_j \mid j = 1, ..., d\}$$

is the vanishing cycle of the corresponding Lefschetz handle H^j_{n+1}. The new boundary convex open book $\mathcal{OB}_{\pi_{\mathcal{V}}}$ (induced by $(\pi_{\mathcal{V}}, W_{\mathcal{V}}, \Omega_{\mathcal{V}}, \Lambda_{\mathcal{V}}, X_{\mathcal{V}}, h_{\mathcal{V}})$) supports the contact structure $(\partial W_{\mathcal{V}}, \mathrm{Ker}(\Lambda_{\mathcal{V}}|_{\partial W_{\mathcal{V}}}))$ by Theorem 4.9 of [AA1]. We also remark that $\partial W_{\mathcal{V}}$ is obtained from $\partial\widetilde{W}$ by the corresponding surgeries (canceling in pairs) along the spheres E_j and V_j which are away from S (as S misses the page $X \times \{z_0\}$ on which all V_j's are constructed and, in particular, it also misses the binding $B = \partial(X \times \{z_0\})$ where all E_j's live). Thus, S is still an embedded sphere in $\partial W_{\mathcal{V}}$.

The pair $\{H^j_n, H^j_{n+1}\}$ can be replaced with a symplectically canceling pair of $(2n+2)$-dimensional Weinstein handles $\{\mathcal{H}^j_n, \mathcal{H}^j_{n+1}\}$. From Lemma 6.2 of [AA1] we know how to realize H^j_n as a Weinstein handle $\mathcal{H}^j_n$. Replacing H^j_{n+1} with $\mathcal{H}^j_{n+1}$ can be done in two different ways: In the first one, by deforming the Liouville structure on $\widetilde{W}$ near V_j, one can realize V_j as a Legendrian sphere (Theorem 1.3 of [AA2]). In the second one, instead of altering the Liouville structure on $\widetilde{W}$, we lift V_j to a Legendrian sphere in $\partial\widetilde{W}$. In the present proof, we will only use the latter technique.

Each vanishing cycle $V_j \subset X = X \times \{z_0\}$ can be lifted to a Legendrian sphere

$$S_j \subset (X \times \mathbb{R}, \mathrm{Ker}(\beta + K dz)$$

and by choosing z_0 with $|z_0|$ large enough we may assume that $S_j \cap S = \emptyset$ for each j. This can be seen as follows: Lifting each V_j to S_j is a standard process such that the real coordinate assigned to each point $p \in V_j$ is determined by the value $F_j(p)$ where $F_j : V_j \to \mathbb{R}$ is the function such that $dF_j = \beta|_{V_j}$ (note that $V_j \approx S^n, n > 1$, is exact Lagrangian). Observe that we are free to add any constant K_j to F_j while forming the lift S_j. Since V_j is compact, there exists a real number z_j with $|z_j| > K'$ such that the lift S_j of $V_j \subset X \times \{z_j\}$ lies in the region $X \times (\mathbb{R} \setminus [-K', K'])$ and, therefore, $S_j \cap S = \emptyset$ because $S \subset X \times (-K', K')$. Then, we simply choose z_0 as the number z_j with $|z_j|$ the largest. This will guarantee that $S_j \cap S = \emptyset$, $\forall j$. Set

$$\mathcal{S} := \{S_j \mid j = 1, ..., d\}.$$

Now we let $(W_\mathcal{S}, \Omega_\mathcal{S}, Z_\mathcal{S})$ be the Liouville domain described by the handle attachment

$$W_\mathcal{S} = \widetilde{W} \, \cup \, \bigcup_j \mathcal{H}_n^j \, \cup \, \bigcup_j \mathcal{H}_{n+1}^j$$

where $\mathcal{H}_n^j$ (resp. $\mathcal{H}_{n+1}^j$) is the Weinstein handle attached along the isotropic sphere E_j (resp. Legendrian sphere S_j). Then, first of all, $W_\mathcal{S}$ is diffomorphic to $W_\mathcal{V}$. Moreover, one can arrange the lifts S_j by adding appropriate constants K_j to each F_j so that the set $\mathcal{S}$ becomes a basis for Lefschetz type Legendrian surgery, and hence, by Proposition 8.1 of [BEE], the Liouville domains $(W_\mathcal{S}, \Omega_\mathcal{S}, Z_\mathcal{S})$, $(W_\mathcal{V}, \Omega_\mathcal{V}, Z_\mathcal{V})$ are Liouville isomorphic. In particular, note that the convex boundaries $(\partial W_\mathcal{V}, \mathrm{Ker}(\Lambda_\mathcal{V}|_{\partial W_\mathcal{V}})), (\partial W_\mathcal{S}, \mathrm{Ker}(\Lambda_\mathcal{S}|_{\partial W_\mathcal{S}}))$ are contactomorphic.

On the other hand, since $\{\mathcal{H}_n^j, \mathcal{H}_{n+1}^j\}$ is a symplectically canceling pair for each j, the Liouville domains $(\widetilde{W}, \widetilde{\Omega}, \widetilde{Z})$, $(W_\mathcal{S}, \Omega_\mathcal{S}, Z_\mathcal{S})$ are also Liouville isomorphic. In particular, there exists a contactomorphism

$$F : (\partial\widetilde{W}, \mathrm{Ker}(\widetilde{\Lambda}|_{\partial\widetilde{W}})) \to (\partial W_\mathcal{S}, \mathrm{Ker}(\Lambda_\mathcal{S}|_{\partial W_\mathcal{S}}))$$

defined by the corresponding contact surgeries along the isotropic spheres E_j, S_j (canceling in pairs) in $(\partial\widetilde{W}, \mathrm{Ker}(\widetilde{\Lambda}|_{\partial\widetilde{W}}))$. Similar to the above discussion, S is disjoint from the region where these surgeries are performed, and so F can be considered as the identity map near the Legendrian link S. As a result, we may regard S as a Legendrian sphere embedded in the image $(\partial W_\mathcal{S}, \mathrm{Ker}(\Lambda_\mathcal{S}|_{\partial W_\mathcal{S}})) \cong (\partial W_\mathcal{V}, \mathrm{Ker}(\Lambda_\mathcal{V}|_{\partial W_\mathcal{V}}))$. Hence, we obtain a Liouville domain $(\overline{W}, \overline{\Omega}, \overline{Z})$ (which is Liouville isomorphic to (W, Ω, Z)) described by the handle decomposition

$$\overline{W} = W_\mathcal{S} \cup \mathcal{H} \quad \left(= \widetilde{W} \, \cup \, \bigcup_j \mathcal{H}_n^j \, \cup \, \bigcup_j \mathcal{H}_{n+1}^j \cup \mathcal{H} \right)$$

where $\mathcal{H}$ is the $(2n+2)$-dimensional Weinstein handle (of index $n+1$) attached along the Legendrian sphere $S \in (\partial W_\mathcal{S}, \mathrm{Ker}(\Lambda_\mathcal{S}|_{\partial W_\mathcal{S}}))$.

Next, we slide the Weinstein handle $\mathcal{H}$ over each $\mathcal{H}_{n+1}^1, \mathcal{H}_{n+1}^2, ..., \mathcal{H}_{n+1}^d$ to obtain a new Weinstein handle which we will denote by $\mathcal{H}'$. More precisely, after d Legendrian handle slide operations we obtain a Liouville domain $(\overline{\overline{W}}, \overline{\overline{\Omega}}, \overline{\overline{Z}})$ (which is Liouville isomorphic to $(\overline{W}, \overline{\Omega}, \overline{Z})$) described by the handle decomposition

$$\overline{\overline{W}} = W_\mathcal{S} \cup \mathcal{H}' \quad \left(= \widetilde{W} \, \cup \, \bigcup_j \mathcal{H}_n^j \, \cup \, \bigcup_j \mathcal{H}_{n+1}^j \cup \mathcal{H}' \right)$$

where $\mathcal{H}'$ is the $(2n+2)$-dimensional Weinstein handle (of index $n+1$) attached along the Legendrian sphere $S' := S\#S_1\#S_2\#\cdots\#S_d$. Here "#" sign refers to "Legendrian connected-sum operation" which can be performed as follows (also see [DG], [EES3], [Et]): Since both S and S_j are disjoint from a page, we may consider them as embedded Legendrian spheres in the contact manifold $(X_{\mathcal{V}} \times \mathbb{R}, \mathrm{Ker}(\alpha + dz)$ where α is the Liouville form on $X_{\mathcal{V}}$ obtained by gluing β (on X) with the primitives of the standard symplectic structures on the Weinstein handles H_j (which were attached to X in the formation of $X_{\mathcal{V}}$). By construction the projections L, V_j on the page $X_{\mathcal{V}} \times \{z_0\}$ (z_0 as above) intersect transversally once at the double point P_j. Denote by Q_1^j, Q_2^j the points in S projecting onto P_j, and let R_j be the (unique) point in S_j projecting onto P_j. WLOG assume that Q_2^j lies between Q_1^j and R_j. Consider the Reeb chord ρ_j with endpoints Q_2^j, R_j. By translating S_j along the Reeb direction, ρ_j can be taken arbitrarly small. We first Legendrian isotope a small neighbourhood of Q_2^j along ρ_j until the pushed S intersects S_j at R_j. Note that one can achieve this isotopy using a contact vector field whose contact Hamiltonian is obtained by multiplying the contact Hamiltonian (constant 1 function) of the Reeb vector field ∂z by a smooth cut-off function which is equal to 1 near ρ_j, and identically vanishes in the complement of a slightly larger set. Then we take the connected-sum of pushed S with S_j at R_j (i.e., we slide the pushed S over S_j) to form a new Legendrian sphere $S\#S_j$ (see Figure 11). Repeating this for each j, we obtain $S' = S\#S_1\#S_2\#\cdots\#S_d$

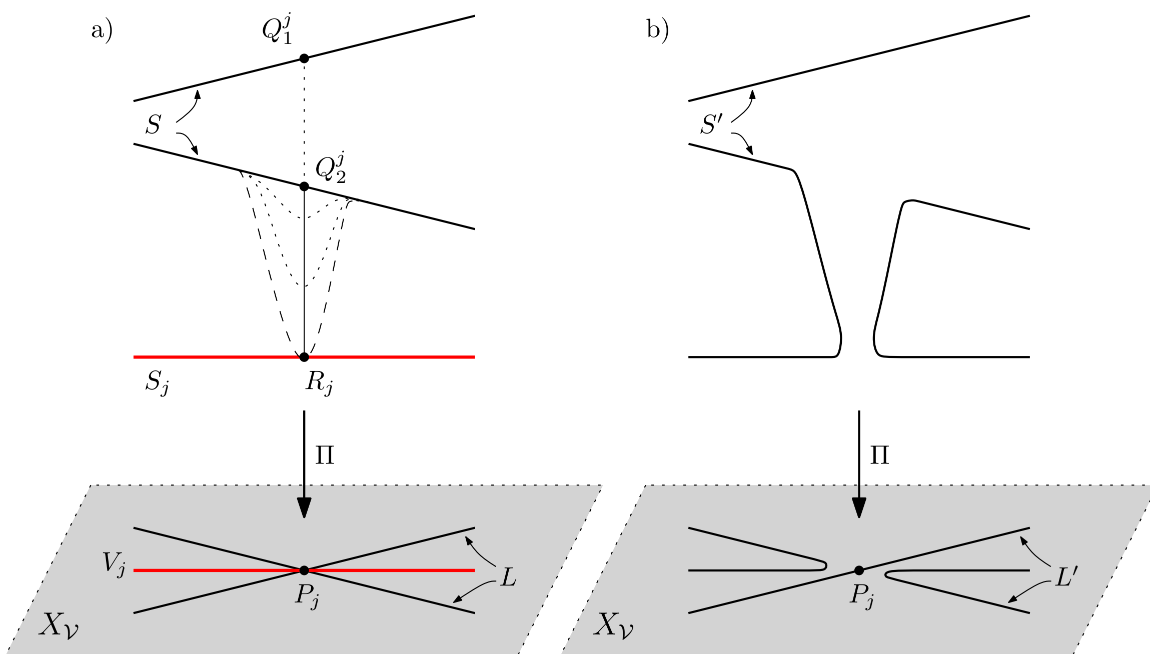

FIGURE 11. Performing Legendrian connected-sum of S with S_j to build up $S' = S\#S_1\#S_2\#\cdots\#S_d$ (This corresponds to sliding $\mathcal{H}$ over each $\mathcal{H}_{n+1}^j$), and the embedded Lagrangian sphere $L' \subset X_{\mathcal{V}} = X_{\mathcal{V}} \times \{z_0\}$ (the projection of S').

Observe that the projection L' of S' onto the page $X_{\mathcal{V}} = X_{\mathcal{V}} \times \{z_0\}$ is an embedded Lagrangian sphere (Figure 11-b). Therefore, by attaching a Lefschetz handle H along each L', we obtain a Liouville Lefschetz fibration $(\pi_{\mathcal{V}}(L'), W_{\mathcal{V}}(L'), \Omega_{\mathcal{V}}(L'), \Lambda_{\mathcal{V}}(L'), X_{\mathcal{V}}, h_{\mathcal{V}}(L'))$ on a Liouville domain $(W_{\mathcal{V}}(L'), \Omega_{\mathcal{V}}(L'), Z_{\mathcal{V}}(L'))$ where

$$h_{\mathcal{V}}(L') = h_{\mathcal{V}} \circ \delta$$

is the new monodromy obtained by composing $h_{\mathcal{V}}$ with the right-handed Dehn twist δ about the Lagrangian sphere L'. Observe that the boundary convex open book $(X_{\mathcal{V}}, h_{\mathcal{V}}(L'))$ supports the induced contact structure $\mathrm{Ker}(\Lambda_{\mathcal{V}}(L')|_{\partial W_{\mathcal{V}}}(L'))$ (Theorem 4.9 of [AA1]), and that the handle decomposition of $W_{\mathcal{V}}(L')$ is given by

$$W_{\mathcal{V}}(L') = W_{\mathcal{V}} \,\cup\, H \quad \left(= \widetilde{W} \,\cup\, \bigcup_j H_n^j \,\cup\, \bigcup_j H_{n+1}^j \,\cup\, H \right).$$

We will now check that the Liouville domains

$$(\overline{\overline{W}}, \overline{\overline{\Omega}}, \overline{\overline{Z}}), \quad (W_{\mathcal{V}}(L'), \Omega_{\mathcal{V}}(L'), Z_{\mathcal{V}}(L'))$$

are Liouville isomorphic. To this end, observe that S and each S_j (and so S') are also embedded Legendrian spheres in $(\partial W_{\mathcal{V}}, \mathrm{Ker}(\Lambda_{\mathcal{V}}|_{\partial W_{\mathcal{V}}}))$, and so we can also attach the Weinstein handle $\mathcal{H}'$ to the Liouville domain $(W_{\mathcal{V}}, \Omega_{\mathcal{V}}, Z_{\mathcal{V}})$. Let $(W_{\mathcal{V}}(S'), \Omega_{\mathcal{V}}(S'), Z_{\mathcal{V}}(S'))$ be the Liouville domain described by

$$W_{\mathcal{V}}(S') = W_{\mathcal{V}} \cup \mathcal{H}' \quad \left(= \widetilde{W} \,\cup\, \bigcup_j H_n^j \,\cup\, \bigcup_j H_{n+1}^j \,\cup\, \mathcal{H} \right)$$

where $\mathcal{H}'$ is the $(2n+2)$-dimensional Weinstein handle (of index $n+1$) attached along the Legendrian sphere $S' \in (\partial W_{\mathcal{V}}, \mathrm{Ker}(\Lambda_{\mathcal{V}}|_{\partial W_{\mathcal{V}}}))$. Then Proposition 3.13 implies that the domains

$$(W_{\mathcal{V}}(S'), \Omega_{\mathcal{V}}(S'), Z_{\mathcal{V}}(S')), \quad (W_{\mathcal{V}}(L'), \Omega_{\mathcal{V}}(L'), Z_{\mathcal{V}}(L'))$$

are Liouville isomorphic. On the other hand, the domains

$$(\overline{\overline{W}}, \overline{\overline{\Omega}}, \overline{\overline{Z}}) \quad \text{and} \quad (W_{\mathcal{V}}(S'), \Omega_{\mathcal{V}}(S'), Z_{\mathcal{V}}(S'))$$

are also Liouville isomorphic because they are, respectively, obtained from attaching a Weinstein handle $\mathcal{H}'$ to Liouville isomorphic domains $W_{\mathcal{S}}$ and $W_{\mathcal{V}}$ along the same attaching sphere S'.

Finally, in order to finish the proof, we just set

$$(W', \Omega', Z') := (W_{\mathcal{V}}(L'), \Omega_{\mathcal{V}}(L'), Z_{\mathcal{V}}(L')).$$

□

As we metioned earlier, Claim 3.14 now immediately follows from Proposition 3.15 by taking $\widetilde{W} = W'_{i-1}$. This finishes the proof of Theorem 3.1. □

Having Theorem 3.1 in hand, it is now easy to prove our very first claim:

Proof of Theorem 1.1. Let (W, J, ψ) be a Stein domain of dimension $2n+2$ with $n > 1$. Consider the Weinstein structure $(\Omega, Z, \Psi) := (-d(d\psi \circ J), \nabla_\psi \psi, \psi)$ on W (Remark 2.9). We know, by Theorem 3.1, that there exists a Liouville domain (W', Ω', Z'), with W' diffeomorphic to W, which is Liouville isomorphic to (W, Ω, Z), such that (W', Ω', Z') admits a Liouville Lefschetz fibration which has Weinstein generic fiber $X_{\mathcal{V}}$, and satisfies the required properties. The Weinstein structure on a generic fiber $X_{\mathcal{V}}$ defines a Stein structure on $X_{\mathcal{V}}$ by a result of Eliashberg, and so the theorem follows. □

As mentioned at the beginning of the paper there is a slight strengthening of Theorem 3.1 in dimension 6:

Remark 3.16. Let W^6 be a compact smooth manifold which admits a Lefschetz fibration $\pi : W \to D^2$ with a generic fiber X^4 such that $\partial X \neq \emptyset$. Therefore, there exists a handle decomposition of X^4 consisting of handles with indices 0,1,2, and 3 only. By the arguments of [K] summarized in Section 2.2 , we know that W^6 is obtained from $X^4 \times D^2$ by attaching a finite number of "Lefschetz" handles of index 3. By thickening the handles of X^4, we get a handle decomposition of $X^4 \times D^2$ consisting of handles with indices 0,1,2, and 3. This shows that W^6 admits a handle decomposition whose handles have index 0,1,2, and 3. Hence, there exists a Stein structure on W^6 by [E1]. The converse follows from Theorem 1.1 above. As a result, a compact manifold W^6 is Stein if and only if it admits a Lefschetz fibration over D^2 with fibers having nonempty boundary.

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