Cappell-Shaneson homotopy 4-spheres are standard

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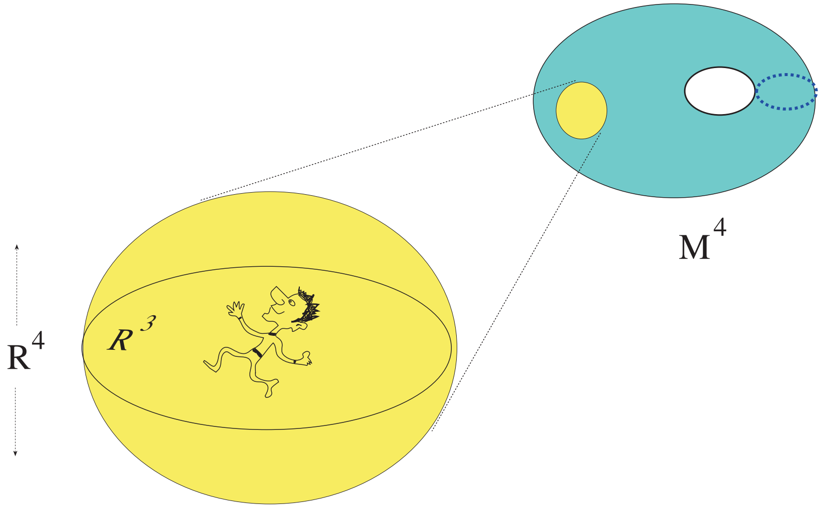
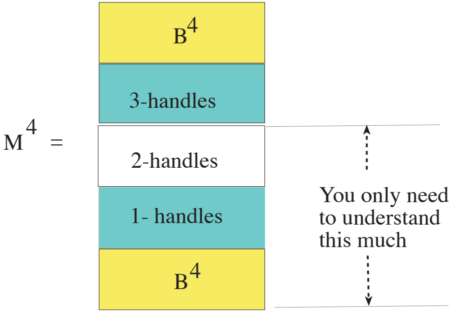
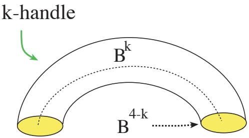
4-manifolds are spaces that locally look like R^4

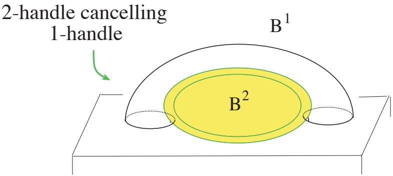


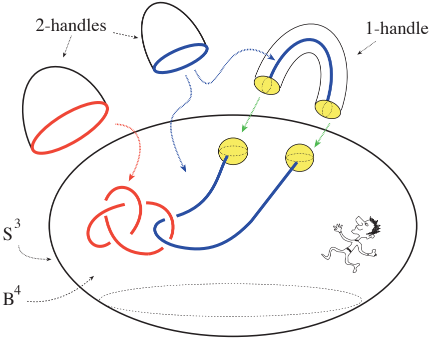
We visualize 4-manifolds by handles

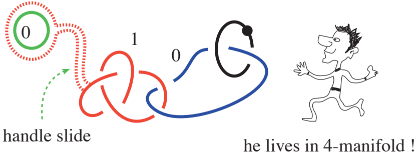


A k-handle is just a ball $B^4 = \mathbf{B}^k \times B^{4-k}$ $(k = 0, 1, 2, 3, 4)$ attached along $\partial \mathbf{B}^k \times B^{4-k} = \mathbf{S}^{k-1} \times B^{4-k}$.

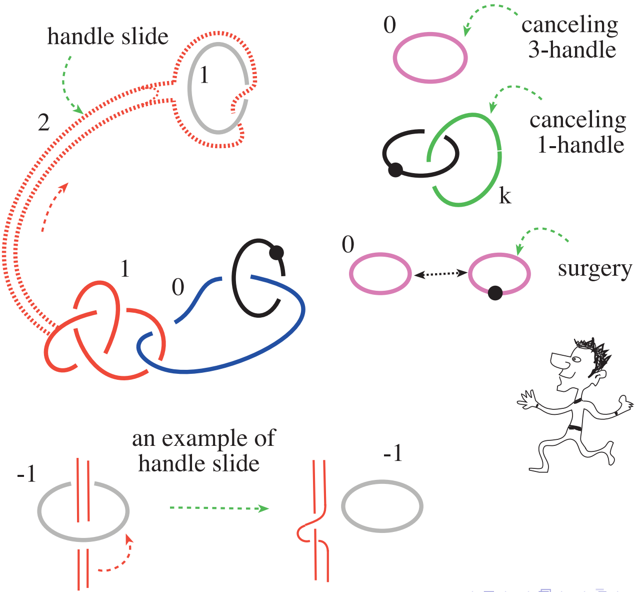





2-handles
1-handle
S
3
B
4


0
1
0
handle slide
he lives in 4-manifold !

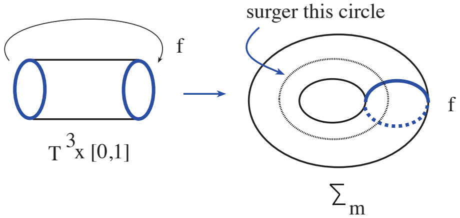
The rules of 4-dimensional world:



$M^4 \simeq S^4 \Rightarrow M^4 \approx S^4$? (Smooth Poincare Conjecture)

- 1976 Cappell and Shaneson gave a sequence Σ_m, $m = 1, 2, ..$ of homotopy S^4's and asked whether they are standard or exotic (some of them double cover exotic $\mathbf{RP}^4$'s which they constructed). Σ_m is obtained surgering the circle from the mapping torus of T^3 with the diffeomorphism $T^3 \to T^3$ induced by the following matrix

$$A_m = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & m+1 \end{pmatrix}$$



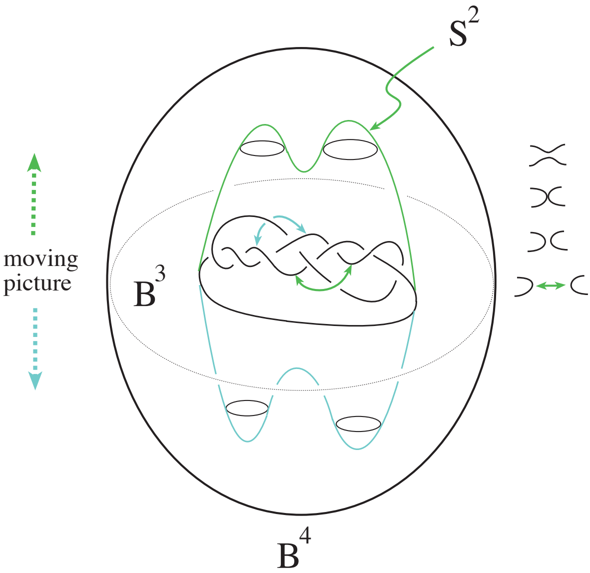
A Gluck construction

- 1977 (A and Kirby) Σ_0 is obtained from S^4 by a Gluck construction to an imbedded 2-sphere $\mathbf{S}^2 \subset S^4$ (a knotted copy of S^2) with tubular neighborhood $\mathbf{S}^2 \times D^2$ so that

$$\Sigma_0 = (S^4 - \mathbf{S}^2 \times D^2) \smile_f (S^2 \times D^2)$$

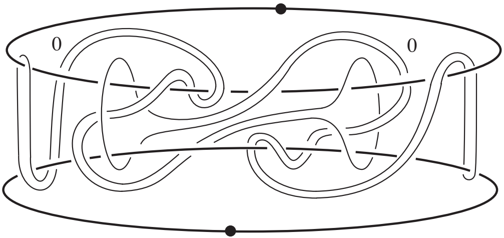
  where the gluing map $f : S^2 \times S^1 \to S^2 \times S^1$ is the nontrivial diffeomorphism given by $f(x, y) = (\alpha(y)x, y)$, where $\alpha \in \pi_1 SO_3 = \mathbf{Z}_2$ is the nontrivial generator $\alpha : S^1 \to SO_3$
- At the time we mistakenly claimed Σ_0 is S^4, since we overlooked checking if the gluing map is the identity or f. It turned out it was in fact f. This was first noticed by Aitchison and Rubinstein.

The knotted S^2 in S^4



Killing 3-handles of Σ_0 by turning it upside down

- 1981 (A) By turning the handlebody of Σ_0 upside down and then cancelling its 1-handles, you can get rid of all the 3-handles of Σ_0. Here is the picture:



Σ_0

$$\pi_1(\Sigma_0) = < x, y \mid xyx = yxy,\ x^5 = y^4 >$$

Checking $\pi_1(\Sigma_0) = 0$ (original check was lengthy, first done with the Rutgers group theory computer, then gradually simplified with some help from A.Casson)

$$\pi_1(\Sigma_0) = < x, y \mid yxy = xyx, \ x^5 = y^4 >$$

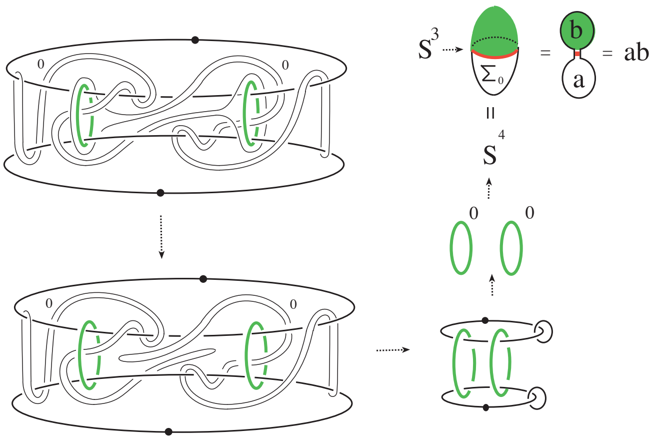
$$yxy = xyx$$

$$y = (yx)^{-1}x(yx)$$

$$y^5 = (yx)^{-1}x^5(yx) = (yx)^{-1}y^4(yx) = x^{-1}y^4x = x^{-1}x^5x = x^5 = y^4$$

$$\Rightarrow \quad y = 1 \text{ and } x = 1$$

Σ_0 is homemorphic to S^4 ! . Is it diffeomorphic to S^4?

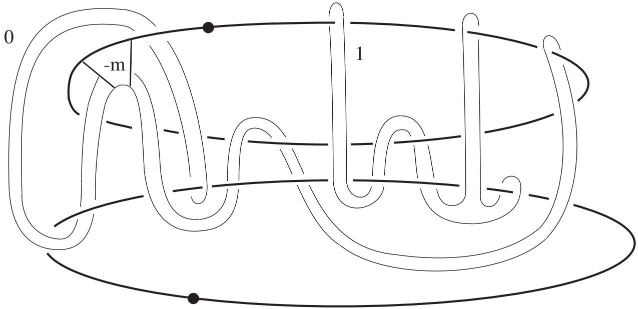


Mazur's swindle: ab=1, ba=1, so a = a(ba)(ba).. = (ab)(ab).. = 1

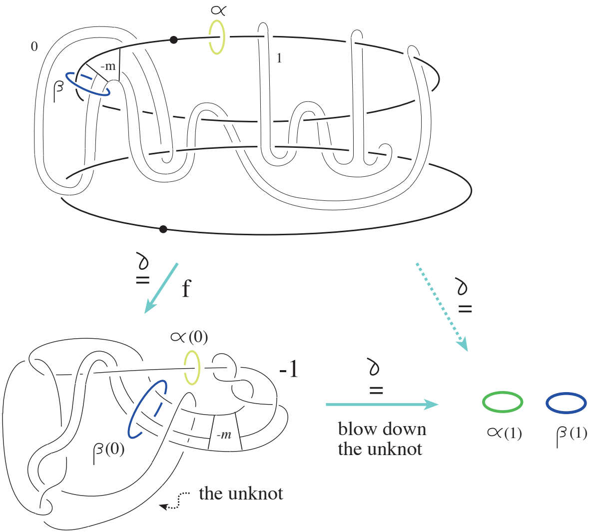
Hard times

- 1985 (A, Kirby) Conjecture Σ_0 is possibly exotic.
- 1987 (Gompf) $\Sigma_0 \approx S^4$! (using the 3-handle free handlebody Σ_0, and the trivialization of $\pi_1(\Sigma_0) =< x, y \mid xyx = yxy,\ x^4 = y^5 >$
- 1991 (Gompf) A similar handlebody picture for Σ_m discussed.
- 06/28/2009 (Gompf, Freedman, Morrison and Walker) Conjecture Σ_m are possibly exotic when $m \neq 0$ by using modern tools from : Khovanov homology and Microsoft computers. "Man and machine thinking about the smooth 4-dimensional Poincare conjecture"
- 07/01/2009 (A) All $\Sigma_m \approx S^4$ (by locating cancelling 2/3-handle pairs from upsidedown view, and then identifying Σ_m with Σ_{m-1}).
- 08/13/2009 (Gompf) Some more CS-spheres are standard (corresponding to some matrices other then A_m's) (by using first A-Kirby paper, plus an “undoing a log-transform by fishtail” trick).

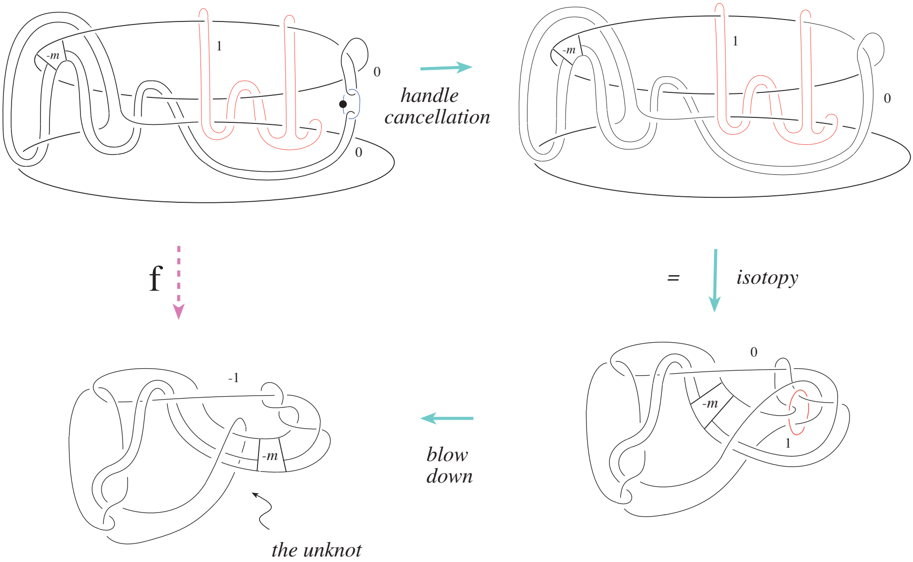
Σ_m



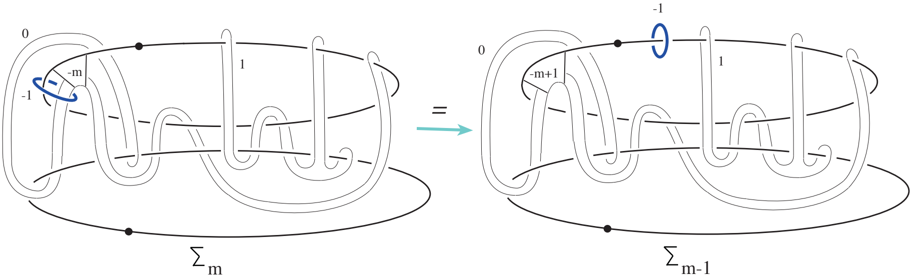
The loops α and β are the unknots on the boundary



Describing the diffeomorphism f



The proof:



$$\Sigma_m \approx \Sigma_m + \beta^{-1} \approx \Sigma_{m-1} + \alpha^{-1} \approx \Sigma_{m-1}$$

$$\Sigma_m \approx \Sigma_{m-1} \ldots \approx \Sigma_0 \approx S^4$$