

Casson's Invariant for
Oriented Homology 3-Spheres

An Exposition

by

Selman Akbulut and John D. McCarthy

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Preface

In the spring of 1985, A. Casson announced an interesting invariant of homology 3-spheres via beautiful constructions on representation spaces. This invariant generalizes the Rohlin invariant and gives surprising corollaries in low dimensional topology. Armed with the Casson lecture notes and many informal discussions with him, in the fall of 1985 we held a seminar on this invariant. These notes grew out of this seminar. We tried to remain close to Casson's original outline and proceeded by giving needed details including an exposition of Newstead's results. We have often chosen classical concrete approaches over general methods; this perhaps made this exposition at places longer and more elementary than necessary. For example, we did not attempt to give gauge theory explanations for the results of Newstead. Instead, we followed his original techniques. We would like to thank the seminar participants at M.S.U. and Tammy Hatfield and Cathy Friess for a speedy typing of these notes.

Introduction

Let M^3 be an oriented homology 3-sphere and K be a knot in M^3 . Let $N(K)$ be a closed regular neighborhood of K , so that $N(K) \cong S^1 \times D^2$. Let $M^3(K)$ be the complement of the interior of $N(K)$ in M^3 . $M^3(K)$ is a 3-manifold with boundary. The boundary is a torus T^2 . Since M^3 is a homology 3-sphere, we can choose the identification of $N(K)$ with $S^1 \times D^2$ so that $S^1 \times \{1\}$ is homologically trivial in $M^3(K)$. Under this assumption, let μ be the meridian $\{1\} \times \partial D^2$ and λ be the preferred longitude $S^1 \times \{1\}$. Since M^3 is oriented, T^2 inherits a natural orientation from the induced orientation on $M^3(K)$. We may orient the curves μ and λ so that (μ, λ) is a positively oriented basis for the first homology of T^2 $H_1(T^2)$.

Now for each integer n , the homology class $\mu + n\lambda$ can be represented by a simple closed curve γ in T^2 . We may obtain a new oriented homology 3-sphere by attaching $S^1 \times D^2$ to $M^3(K)$ along their respective boundaries in such a manner that $\{1\} \times \partial D^2$ is attached to γ . The resulting oriented homology 3-sphere K_n is well defined up to orientation preserving homeomorphism. We say that K_n is the manifold obtained from M by $1/n$ -surgery on K . In particular, K_0 is M .

As for knots in the 3-sphere S^3 , one can associate an Alexander-polynomial to K . This is a polynomial in the variables t and t^{-1} with integer coefficients and is well-defined up to multiplication by t^n for any integer n . Since this polynomial satisfies the standard identities for Alexander polynomials of knots in S^3 , one obtains a well defined polynomial, known as the symmetrized Alexander polynomial Δ_K by imposing the conditions:

(xii)

$$\Delta_K(t) = \Delta_K(t^{-1}) \quad \Delta_K(1) = 1.$$

It is easy to see that these conditions imply that the second derivative of the Alexander polynomial evaluated at 1 $\Delta_K''(1)$ is an even integer.

With the above notation, we may state Casson's main theorem concerning his new invariant for oriented homology spheres.

Theorem (Casson): There is a unique invariant, $\lambda(M)$, of oriented homology 3-spheres satisfying the following properties:

$$(1) \quad \lambda(K_{n+1}) - \lambda(K_n) = \frac{1}{2} \Delta_K''(1) \quad \text{for any knot } K \text{ in any oriented homology 3-sphere } M^3.$$

$$(2) \quad \lambda(S^3) = 0.$$

The idea of Casson's invariant comes from the observation that if M^3 is a 3-manifold with a Heegard splitting $M^3 = W_1 \cup W_2$, (where W_1 and W_2 are solid handlebodies of genus g meeting along their common boundary F), then the commutative diagram of inclusions:

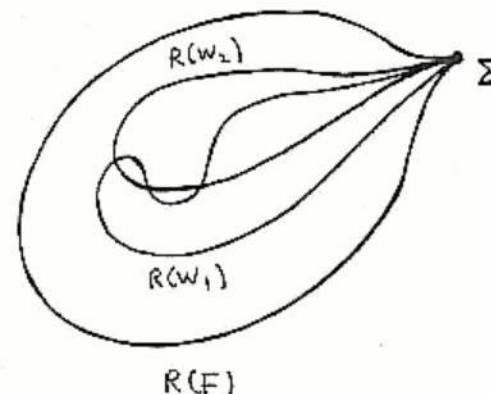
$$\begin{array}{ccccc} & & \pi_1(W_1) & & \\ & \nearrow & & \searrow & \\ \pi_1(F) & & & & \pi_1(M^3) \\ & \searrow & & \nearrow & \\ & & \pi_1(W_2) & & \end{array}$$

gives rise to imbeddings of the space of conjugacy classes of representations of these groups into $SU(2)$:

$$\begin{array}{ccccc} & & R(W_1) & & \\ & \nearrow & & \searrow & \\ R(F) & & & & R(M) \\ & \searrow & & \nearrow & \\ & & R(W_2) & & \end{array}$$

(xiii)

It turns out that all these spaces are real algebraic sets and $R(W_1)$ and $R(W_2)$ are complementary dimensional subspaces of $R(F)$. The singular set Σ of $R(W_1)$ is included in the singular set Σ of $R(F)$. Casson's invariant is roughly the "algebraic intersection number" of the smooth manifolds $R(W_i) \setminus \Sigma$ $i = 1, 2$ in $R(F) \setminus \Sigma$. The technical difficulties are to make sense out of the algebraic intersection number of these proper open submanifolds and to show that it is independent of the Heegard splitting of M .



The properties (1) and (2) actually allow us to compute $\lambda(M)$ for any homology sphere. This may be done as follows. Using a well known result ([L]), we may represent M by surgery on a framed link in S^3 (with integer framings). Moreover, we can assume that the linking matrix is unimodular, odd and indefinite. With this assumption, it is possible, by handle sliding, to diagonalize the matrix. Hence, we find that M is given by

surgery on a framed link in S^3 with a diagonal linking matrix whose diagonal entries are all ± 1 .

This implies that we can obtain M^3 by a sequence of ± 1 Dehn surgeries on knots in homology spheres beginning with a knot in S^3 . Hence, we can calculate $\lambda(M)$ inductively by applying property (1) beginning with property (2). In particular, $\lambda(M)$ is an integer.

This observation establishes the uniqueness of λ . Hence, it remains to establish existence. This task will occupy our attention throughout this exposition. In the course of proving existence, it is found that Casson's invariant satisfies the following additional properties.

Theorem (Casson): The invariant λ satisfies the following properties:

- (3) $\lambda(-M) = -\lambda(M)$.
- (4) If $\lambda(M) \neq 0$, then $\Pi_1(M)$ has a nontrivial representation in $SU(2, \mathbb{C})$.
- (5) $\lambda(M) \equiv \mu(M) \pmod{2}$, where $\mu(M)$ is the Rohlin invariant.
- (6) $\lambda(M_1 \# M_2) = \lambda(M_1) + \lambda(M_2)$.

From our description of calculating λ , it is easy to see that properties (1) and (2) imply properties (3) and (6). Property (5) also follows from properties (1) and (2) and known relations between the Rohlin invariant of 3-manifolds obtained by surgery to a knot K , the Arf invariant of K and the Alexander polynomial of K (see section V.3(c)).

Property (4), as we shall see, follows immediately from the definition of λ . Representations of $\Pi_1(M)$ in $SU(2, \mathbb{C})$ form the context for λ in Casson's formulation of this theory. As Casson suggested in his lectures in the summer

of 1985, and Taubes later verified, there is an alternative approach to this invariant via the gauge theory of flat connections on principal $SU(2, \mathbb{C})$ -bundles. This links Casson's work with the developments of Donaldson in 4-manifolds in a way that is yet to be fully understood.

We now list some immediate corollaries of the properties of λ , ($\lambda'(K)$ will denote $\lambda(K_{n+1}) - \lambda(K_n)$).

Corollary (Casson): Any homotopy 3-sphere has zero Rohlin invariant.

Corollary (Casson): Any amphicheiral homology 3-sphere has zero Rohlin invariant.

Corollary (Casson): If $\lambda'(K)$ is nonzero and K is in S^3 , then K has property P.

Proof: From property (1), we deduce that:

$$\lambda(K_n) - \lambda(K_0) = n \cdot \lambda'(K).$$

Since $K_0 = S^3$, by property (2), we have:

$$\lambda(K_n) = n \cdot \lambda'(K).$$

Hence, by property (2) and our assumption on $\lambda'(K)$, we conclude that $K_n = S^3$ if and only if n is zero. $\square$

Casson observed in his lectures that λ is not an invariant of homology cobordism. There exist homology 3-spheres which bound smooth contractible 4-manifolds but have nonzero λ .

Corollary (Casson): There exist non-triangulable 4-manifolds. For example, Freedman's E_8 manifold cannot be triangulated as a simplicial complex.

This is because, if M^4 were triangulated it would be P.L. in the complement of the open star of a vertex $x_0 \in M^4$. This would give the $\text{Link}(x_0)$ a P.L. structure $\mathbb{E}^3$. Since M^4 is a topological manifold, $\mathbb{E}^3$ has to be simply connected. So it must have zero Casson invariant, hence, zero Rohlin invariant $\mu(\mathbb{E})$. On the other hand, $\mathbb{E}^3$ is the boundary of the Spin 4-manifold $W^4 = M^4 - \text{open star}(x_0)$ and, therefore, $\mu(\mathbb{E}) = 1/8 \text{ signature}(W) \pmod{2} = 1$, giving us a contradiction. We must add that in dimension 3 and 4 P.L. manifolds have smooth structures.

We close this introduction by giving an outline of the subsequent exposition.

Chapters are labeled by Roman numerals. Sections of a chapter are labeled by standard numerals. Subsections of a section are labeled by lowercase letters. Significant statements are labeled consecutively within each section. Hence, section V.2(c) refers to the third subsection of the second section of the fifth chapter. Likewise, Proposition IV.2.1 refers to the first proposition of the second section of the fourth chapter. Within each chapter, the reference to the chapter is omitted. Hence, a reference to Proposition 5.1 in the text of chapter V refers to the first proposition in the fifth section of that chapter.

Chapters I and II are background material, concerning, respectively, the theory of representation spaces and the theory of Heegard decompositions, two essential tools to Casson's approach. The results here, especially those in

chapter II are well known propositions which the reader may wish to consult at various points in the subsequent chapters.

Chapter III is also background information concerning the particular representation spaces associated to an Heegard decomposition of an homology sphere.

Chapter IV concerns the definition of Casson's invariant and the proof of its invariance (independence of Heegard decomposition). Properties (2) and (4) are obvious from this definition.

Chapter V is a study of the associated invariant for knots in homology spheres $\lambda'(K)$. In this chapter, we establish the crucial axiom, property (1). This involves a careful study of a "characteristic" cycle associated to the difference $\lambda(K_{n+1}) - \lambda(K_n)$, the difference cycle δ . This cycle plays a central role in the exposition. (It may be of some interest to determine whether there are other "characteristic" cycles present in these representation spaces.)

In chapter VI, we study the topology of an associated representation space which arises in studying δ . The main purpose of this section is to prove that δ is invariant under the Torelli group, a technical fact needed in chapter V. The main content of this section is extracted from the articles of Newstead.

One final disclaimer. The purpose of this text is purely expository. We have not made a systematic attempt to cross-reference the various sources consulted. The major portion of the exposition is as elementary as possible. We hope that the reader will find the exposition helpful and that the brief list

of references given at the end of the text will provide a bridge to more sophisticated treatments of this important subject.

CHAPTER I: REPRESENTATION SPACES

1. The special unitary group - $SU(2, \mathbb{C})$.

(a) Identification with S^3

Let

$$M(2, \mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{C} \right\}$$

$$SU(2, \mathbb{C}) = \{ A \in M(2, \mathbb{C}) \mid A \bar{A}^t = I, \det(A) = 1 \}.$$

$$S^3 = \{ (a, b) \in \mathbb{C}^2 \mid a\bar{a} + b\bar{b} = 1 \}.$$

There is a natural identification of S^3 with $SU(2, \mathbb{C})$:

$$\begin{aligned} S^3 &\xrightarrow{\cong} SU(2, \mathbb{C}) \\ (a, b) &\longmapsto \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \end{aligned}$$

Conventions:

- (1) We shall identify S^3 and $SU(2, \mathbb{C})$ by this fixed diffeomorphism.
- (2) We shall consider S^3 to have a fixed orientation throughout our discussion.

(b) The tangent bundle of S^3 TS^3

By differentiating the defining equations for S^3 , we obtain the tangent bundle of S^3 :

$$TS^3 = \left\{ \left(\begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}, \begin{bmatrix} u & v \\ -\bar{v} & \bar{u} \end{bmatrix} \right) \mid \begin{aligned} a\bar{u} + \bar{a}u + b\bar{v} + \bar{b}v &= 0 \\ a\bar{a} + b\bar{b} &= 1 \end{aligned} \right\}.$$

In particular, the Lie algebra of S^3 is given as:

$$\mathfrak{S} = \mathfrak{SU}(2, \mathbb{C}) = T_1(S^3)$$

$$\mathfrak{S} = \left\{ \begin{bmatrix} is & v \\ -\bar{v} & -is \end{bmatrix} \mid s \in \mathbb{R}, v \in \mathbb{C} \right\}.$$

The action of S^3 on itself by left translations provides a trivialization of TS^3 :

$$\begin{aligned} S^3 \times S &\xrightarrow{\cong} TS^3 \\ (A, X) &\longmapsto (A, AX). \end{aligned}$$

Convention: We identify TS^3 with $S^3 \times S$ by the natural trivialization provided above.

A natural Riemannian metric arises via translation from the inner product on $\mathfrak{S}$:

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathfrak{S} \times \mathfrak{S} &\longrightarrow \mathbb{R} \\ (X, Y) &\longrightarrow \frac{1}{2} \operatorname{trace}(XY^t). \end{aligned}$$

Associated to this metric, we have a natural exponential map:

$$\exp : \mathfrak{S} \longrightarrow S^3.$$

The natural identification of S^3 with $SU(2, \mathbb{C})$ is an isometry of the standard metric on S^3 and the above metric on $SU(2, \mathbb{C})$.

(c) The conjugation actions on S^3 and TS^3

S^3 acts on S^3 by conjugation:

$$\begin{aligned} * : S^3 \times S^3 &\longrightarrow S^3 \\ (A, B) &\longmapsto ABA^{-1}. \end{aligned}$$

The derivative of this action is simply:

$$\begin{aligned} * : S^3 \times TS^3 &\longrightarrow TS^3 \\ (A, (B, X)) &\longmapsto (ABA^{-1}, AXA^{-1}). \end{aligned}$$

From the definition of the inner product on $\mathfrak{S}$:

$$\langle AXA^{-1}, AY A^{-1} \rangle = \langle X, Y \rangle.$$

Hence, these actions are actions by isometries and:

$$\exp(AXA^{-1}) = A \exp(X) A^{-1}.$$

Let:

$$\begin{aligned} S^0 &= \{\pm I\} \\ S^1 &= \left\{ \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \mid \theta \in \mathbb{R} \right\} = \text{diagonals}. \end{aligned}$$

Every element of S^3 is conjugate to a diagonal element:

Proposition 1.1: (1) $S^0 \subset S^1 \subset S^3$

(2) $S^3 = \cup \{CS^1C^{-1} \mid C \in S^3\}$

Proof: (1) This is, of course, obvious. It is stated in anticipation of an analogous statement.

(2) Suppose $A \in S^3$. Since $\mathbb{C}$ is algebraically closed, A has an eigenspace of eigenvalue λ . Choose a unit eigenvector in this eigenspace:

$$(x_0, y_0) \in S^3 \subset \mathbb{C}^2$$

$$V = \{(x, y) \in \mathbb{C}^2 \mid (x, y) = t(x_0, y_0); t \in \mathbb{C}\}$$

$$AV = V.$$

Let $C = \begin{bmatrix} x_0 & -\overline{y_0} \\ y_0 & \overline{x_0} \end{bmatrix}$, so that:

$$C^{-1}AC \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \end{pmatrix}, \text{ or } C^{-1}AC = \begin{pmatrix} \lambda & \alpha \\ 0 & \beta \end{pmatrix}.$$

Since $C \in \text{SU}(2, \mathbb{C})$,

$$C^{-1}AC = \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix} \quad \lambda \overline{\lambda} = 1.$$

That is to say, $A \in \text{CS}^1 C^{-1}$. $\square$

From the previous facts and the fact that:

$$\exp \begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

we deduce that the exponential map is a diffeomorphism on the ball of radius π about 0:

$$\exp : B_\pi(0) \xrightarrow{\cong} S^3 \setminus \{-I\}.$$

(Clearly, the injectivity radius of S^3 is π .)

Notation: We shall use the following abuses of notation:

$$i\theta = \begin{pmatrix} i\theta & 0 \\ 0 & -i\theta \end{pmatrix} \quad e^{i\theta} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

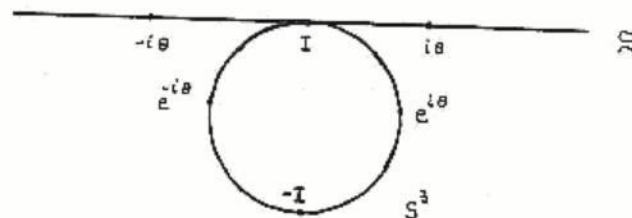


Figure 1

In particular, the natural contraction of $B_\pi(0)$:

$$G : B_\pi(0) \times [0, 1] \longrightarrow B_\pi(0)$$

$$(X, t) \longmapsto tX.$$

exponentiates to a natural contraction of $S^3 \setminus \{-I\}$:

$$G : (S^3 \setminus \{-I\}) \times [0, 1] \longrightarrow S^3 \setminus \{-I\}$$

$$(A, t) \longmapsto A^t.$$

Note: By the previous observations, these contractions are equivariant with respect to the conjugation actions.

We note that this contraction extends to continuous maps:

$$G : B_n(o) \times \mathbb{R} \longrightarrow S$$

$$(x, t) \longmapsto tX$$

$$G : (S^3 \setminus \{-I\}) \times \mathbb{R} \longrightarrow S^3$$

$$(A, t) \longmapsto A^t.$$

Note: G does not extend continuously over $S^3 \times [0, 1]$.

Note: The fact that $\exp(i\theta) = e^{i\theta}$ follows from the previous observation concerning the metrics on S^3 and $SU(2, \mathbb{C})$. We shall frequently take advantage of these two perspectives (as well as others to be introduced below).

From Proposition 1.1 and the previous observations, we conclude that the extension of the contraction defined above is given by the following rule:

$$G : (S^3 \setminus \{-I\}) \times \mathbb{R} \longrightarrow S^3$$

$$(Ce^{i\theta}C^{-1}, t) \longmapsto Ce^{it\theta}C^{-1}$$

where $0 \neq \theta < \pi$.

Note: The restriction on θ is possible due to the fact that $e^{i\theta}$ is conjugate to $e^{-i\theta}$:

$$\begin{bmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The next proposition describes the stabilizers of various elements in the "stratification" of S^3 above. For this reason, we use the standard notation:

$$B \star A = BAB^{-1}$$

$$C \star S^1 = \{CAC^{-1} \mid A \in S^1\}$$

$$S_A^3 = \{B \in S^3 \mid B \star A = A\}.$$

Proposition 1.2:

- (1) If $A \in S^0$, then $S_A^3 = S^3$.
- (2) If $A \in (C \star S^1) \setminus S^0$, then $S_A^3 = C \star S^1$.

Proof: (1) This is obvious.

(2) Suppose $A \in (C \star S^1) \setminus S^0$. Since S^1 is an abelian group, it is clear that $C \star S^1 \subset S_A^3$. We need to establish the reverse inclusion.

By part (1), $C^{-1} \star A \in S^1 \setminus S^0$. That is:

$$C^{-1}AC = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}; \quad e^{i\theta} \neq e^{-i\theta}.$$

Suppose that $B \in S_A^3$, so that:

$$BAB^{-1} = A \quad \text{or} \quad (C^{-1}BC)(C^{-1}AC) = (C^{-1}AC)(C^{-1}BC).$$

If $C^{-1}BC = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$ with $aa + b\bar{b} = 1$, then:

$$\begin{bmatrix} ae^{i\theta} & be^{-i\theta} \\ * & * \end{bmatrix} = \begin{bmatrix} ae^{i\theta} & be^{i\theta} \\ * & * \end{bmatrix}.$$

Hence, $b = 0$, and $B \in C \star S^1$ as before. $\square$

(d) The trace and argument maps

We have the usual trace function:

$$\begin{aligned} \text{tr} : S^3 &\longrightarrow \mathbb{R} \\ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} &\longmapsto a + \bar{a} . \end{aligned}$$

Of course, this function is SO_3 invariant:

$$\text{tr}(CAC^{-1}) = \text{tr}(A) .$$

Hence, by the results of the previous section, we have a surjection:

$$\text{tr} : S^3 \longrightarrow [-2, 2] .$$

Hence, we may define the argument function via the inverse cosine:

$$\begin{aligned} \arg : S^3 &\longrightarrow [0, \pi] \\ A &\longmapsto \cos^{-1}(\text{tr}(A)/2) . \end{aligned}$$

Then:

$$\arg(A) = \theta \quad \text{iff} \quad A = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

$$\arg(A) = 0 \quad \text{iff} \quad A = I$$

$$\arg(A) = \pi \quad \text{iff} \quad A = -I .$$

The argument respects the natural contraction of the previous section:

$$\arg(A^t) = t \cdot \arg(A) \quad A \neq -I \quad 0 \leq t \leq 1 .$$

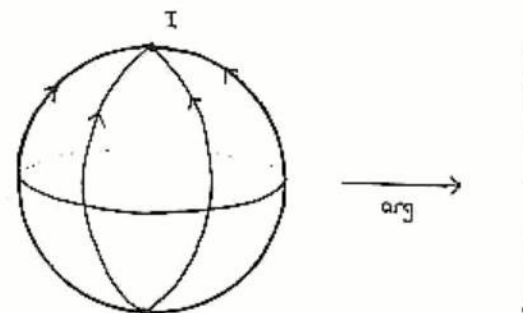


Figure 2

Let:

$$\begin{aligned} D_+^3 &= \{A \mid 0 \leq \arg(A) \leq \pi/2\} \\ &= \{A \mid 0 \leq \text{trace}(A) \leq 2\} \end{aligned}$$

$$\begin{aligned} S^2 &= \{A \mid \arg(A) = \pi/2\} \\ &= \{A \mid \text{trace}(A) = 0\} \end{aligned}$$

$$\begin{aligned} D_-^3 &= \{A \mid \pi/2 \leq \arg(A) \leq \pi\} \\ &= \{A \mid -2 \leq \text{trace}(A) \leq 0\} . \end{aligned}$$

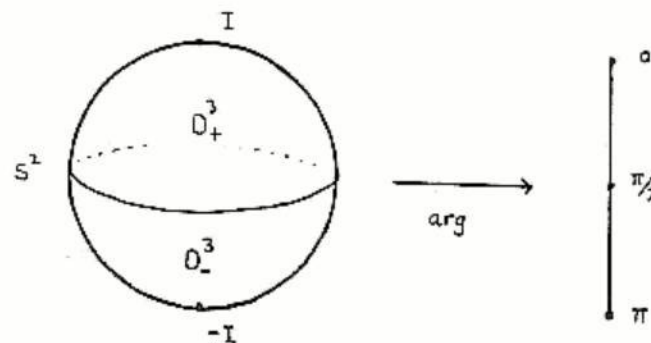


Figure 3

There is a natural involution of this decomposition:

$$i : (S^1, D_+^3, D_-^3, S^2) \xrightarrow{\quad} (S^1, D_+^3, D_-^3, S^2) \\ A \longmapsto -A^{-1}.$$

We observe that this involution is SO_3 equivariant:

$$i(CAC^{-1}) = Ci(A)C^{-1}.$$

In addition, it fixes S^2 pointwise:

$$i(A) = A \text{ iff } A \in S^2.$$

Moreover, it is respected by the argument:

$$\arg(-A^{-1}) = \pi - \arg(A).$$

(e) The unit quaternions $Sp(1)$ and SO_3

We may identify $\mathbb{C}^2$ with the quaternions $\mathbb{H}$:

$$\mathbb{C}^2 \longrightarrow \mathbb{H} \\ (a, b) \longmapsto a + bj.$$

This identifies S^3 with the unit quaternions:

$$S^3 = Sp(1) = \text{unit quaternions}.$$

Under this identification, the diffeomorphism of part (a) identifying S^3 with $SU(2, \mathbb{C})$ is an isomorphism of group structures and an isometry of metrics.

The action of S^3 on $\mathbb{H}$ by conjugation is a linear action by isometries of S^3 . Hence, we obtain a representation:

$$S^3 \xrightarrow{\quad r \quad} SO_4 \\ r(x)(y) = xyx^{-1}$$

Note: The representation is in SO_4 because S^3 is connected.

Now $\mathbb{R} \subset \mathbb{H}$ and $\mathbb{R}$ is fixed by the action. Hence, we can reduce r to a representation in SO_3 :

$$S^3 \xrightarrow{\quad r^1 \quad} SO_3.$$

From Proposition 1.2, we conclude that the kernel of r^1 is S^0 . Hence, r^1 is onto:

$$1 \longrightarrow S^0 \longrightarrow S^3 \longrightarrow SO_3 \longrightarrow 1 \quad \text{exact}$$

$$\frac{SU(2, \mathbb{C})}{\pm I} \cong \frac{S^3}{S^0} = SO_3.$$

By Proposition 1.2, it follows that the conjugation action descends to a well-defined faithful action of SO_3 on S^3 :

$$\ast : SO_3 \times S^3 \longrightarrow S^3 \\ ([A], B) \longmapsto ABA^{-1}.$$

Convention:

(1) We consider all of the above isomorphisms as identifications.

We observe that the natural involution of S^3 defined in the previous section, under the above identification with the unit quaternions, is given by the rule:

$$i : S^3 \xrightarrow{\cong} S^3$$

$$w \longmapsto \bar{w}.$$

Note: We shall primarily work with $SU(2, \mathbb{C})$. It is, however, convenient to have these alternative models. Frequently an argument is more transparent in one model than another.

(f) The pure unit quaternions S^2

Under the identification of S^3 with $Sp(1)$ given above, we see that S^2 , (as defined in part (d)), is the set of pure unit quaternions:

$$S^2 = \left\{ \begin{bmatrix} it & b \\ -\bar{b} & -it \end{bmatrix} \mid t \in \mathbb{R}, b \in \mathbb{C}, t^2 + b\bar{b} = 1 \right\}.$$

Clearly, S^3 acts on S^2 by conjugation:

$$S^3 \times S^2 \longrightarrow S^2$$

$$(A, B) \longmapsto ABA^{-1}.$$

This action yields an identification of S^2 with the right coset space of S^3 by S^1 :

Proposition 1.3: $S^3/S^1 \cong S^2$

Proof: Let:

$$i = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Then, from Proposition 1.1, we conclude that $S^2 = S^3 \cdot i$. On the other hand, from Proposition 1.2, $S^3_1 = S^1$. Hence, $S^2 = S^3 \cdot i \cong S^3/S^3_1 = S^3/S^1$. $\square$

2. The space of representations of G , $R(G)$.

(a) the functor R

Let G be a group. The space of representations of G in S^3 is defined as follows:

$$R(G) = \text{Hom}(G, S^3)$$

$$= \{ \rho : G \longrightarrow S^3 \mid \rho \text{ is a representation} \}.$$

We give $R(G)$ the compact open topology, where G has the discrete topology and S^3 the usual topology.

To each homomorphism $\lambda : G_1 \rightarrow G_2$, there is an associated continuous map:

$$R(\lambda) : R(G_2) \longrightarrow R(G_1) \quad \rho \longmapsto \rho \circ \lambda.$$

Clearly, we have the following identities:

$$R(\lambda \circ \mu) = R(\mu) \circ R(\lambda), \quad R(\text{Id}) = \text{Id}.$$

In other words, R is a contravariant functor from the category of groups and homomorphisms to the category of topological spaces and continuous maps.

(b) The action of SO_3 on $R(G)$

By our previous remarks, the conjugation action of S^3 on S^3 descends to a faithful action of SO_3 on S^3 :

$$* : SO_3 \times S^3 \longrightarrow S^3 \quad (\text{faithful}).$$

Given a group G , we may define an action of SO_3 on $R(G)$ (or an action of $SU_2(\mathbb{C})$):

$$\begin{aligned} \cdot : SO_3 \times R(G) &\longrightarrow R(G) \\ ([A] \cdot \rho)(g) &= A \rho(g) A^{-1} \end{aligned}$$

This is a natural action in the following sense. For any homomorphism $\lambda : G \rightarrow H$, the following diagram commutes:

$$\begin{array}{ccc} SO_3 \times R(H) & \xrightarrow{\cdot} & R(H) \\ \text{id} \times R(\lambda) \downarrow & & \downarrow R(\lambda) \\ SO_3 \times R(G) & \xrightarrow{\cdot} & R(G) \end{array}$$

Let:

$$\begin{aligned} S_0(G) &= \{\rho \mid \rho : G \longrightarrow S^0\} \\ S_1(G) &= \{\rho \mid \rho : G \longrightarrow S^1\} = \text{diag. reps} \\ S(G) &= \{\rho \mid \rho : G \longrightarrow S^3 \text{ is reducible}\} \end{aligned}$$

Proposition 2.1: (1) $S_0(G) \subset S_1(G) \subset S(G) \subset R(G)$

$$(2) \quad S(G) = \cup \{C \cdot S_1(G) \mid C \in SU_2(\mathbb{C})\}$$

Proof: (1) This is immediate from the definitions.

(2) Since the conjugate of a reducible representation is clearly reducible, part (1) implies that:

$$S(G) \supset \cup \{C \cdot S_1(G) \mid C \in SU_2(\mathbb{C})\}.$$

It remains to show that a reducible representation is conjugate to a diagonal representation. Suppose, therefore, that we have a reducible representation:

$$\rho : G \longrightarrow SU(2, \mathbb{C}) \text{ reducible.}$$

By definition, $\rho(G)$ leaves a proper subspace V of $\mathbb{C}^2$ invariant. The proof of Proposition 1.1 (2) yields an element C in $SU_2(\mathbb{C})$ such that:

$$C^{-1} \rho(g) C \in S^1 \text{ for all } g \in G.$$

Equivalently, $\rho \in C \cdot S_1$. $\square$

The next corollary follows immediately from the definitions, Proposition 2.1 and the naturality of the SO_3 action.

Corollary 2.2: If $\lambda : G \rightarrow H$ is a homomorphism, then we have a commutative diagram:

$$\begin{array}{ccccccc} S_0(H) & \subset & S_1(H) & \subset & S(H) & \subset & R(H) \\ \downarrow R(\lambda) & & \downarrow R(\lambda) & & \downarrow R(\lambda) & & \downarrow R(\lambda) \\ S_0(G) & \subset & S_1(G) & \subset & S(G) & \subset & R(G) \end{array}$$

Note: In fact, the decomposition of Proposition 1.1(2) is also natural.

The next proposition describes the stabilizers of representations in this "stratification" of $R(G)$. For that purpose, we use the standard notation:

$$S_\rho^3 = \{A \in S^3 \mid A \cdot \rho = \rho\}.$$

Clearly:

$$A \in S_\rho^3 \text{ iff } A \in S_{\rho(g)}^3 \text{ for all } g \in G$$

$$A \in S_\rho^3 \text{ iff } \rho(g) \in S_A^3 \text{ for all } g \in G.$$

Proposition 2.3:

- (1) If $\rho \in S_0(G)$, then $S_\rho^3 = S^3$.
- (2) If $\rho \in C \cdot S_1(G) \setminus S_0(G)$, then $S_\rho^3 = CS^1C^{-1}$.
- (3) If $\rho \in R(G) \setminus S(G)$, then $S_\rho^3 = S^0$.

Let:

$$\begin{aligned}\bar{R}(G) &= SO_3 \setminus R(G) \\ &= \text{space of conjugacy classes of reps.} \\ &\text{of } G \text{ in } S^3.\end{aligned}$$

By the previous observations, we conclude that if $\lambda : G \rightarrow H$ is a homomorphism, there is a continuous map:

$$\bar{R}(\lambda) : \bar{R}(H) \longrightarrow \bar{R}(G)$$

such that the following diagram commutes:

$$\begin{array}{ccc} R(H) & \xrightarrow{R(\lambda)} & R(G) \\ \downarrow \pi & & \downarrow \pi \\ \bar{R}(H) & \xrightarrow{\bar{R}(\lambda)} & \bar{R}(G) \end{array}$$

Let:

$$\begin{aligned}\bar{S}_0(G) &= \pi(S_0(G)) \\ \bar{S}_1(G) &= \pi(S_1(G)) \\ \bar{S}(G) &= \pi(S(G)) = SO_3 \setminus S(G).\end{aligned}$$

The diagram below commutes:

$$\begin{array}{ccccccc} \bar{S}_0(H) & \subset & \bar{S}_1(H) & \subset & \bar{S}(H) & \subset & \bar{R}(H) \\ \downarrow \bar{R}(\lambda) & & \downarrow \bar{R}(\lambda) & & \downarrow \bar{R}(\lambda) & & \downarrow \bar{R}(\lambda) \\ \bar{S}_0(G) & \subset & \bar{S}_1(G) & \subset & \bar{S}(G) & \subset & \bar{R}(G) \end{array}$$

(c) The algebraic set associated to a set of generators, S , of G

Suppose that G is a finitely generated group. Consider a finite set of generators for G :

$$S = \{g_1, \dots, g_n\}.$$

We have a natural map:

$$\begin{aligned}f = f_S : R(G) &\longrightarrow (S^3)^n \\ \rho &\longmapsto (\rho(g_1), \dots, \rho(g_n))\end{aligned}$$

Let:

$$V(S) = f_S(R(G)).$$

Now, if R is a set of relations for G , so that:

$$G = \langle g_1, \dots, g_n \mid r(g_1, \dots, g_n) = 1 \quad r \in R \rangle$$

is a presentation of G , then:

$$V(S) = \{(A_1, \dots, A_n) \in (S^3)^n \mid r(A_1, \dots, A_n) = 1 \quad r \in R\}.$$

Hence, we have a natural homeomorphism from $R(G)$ to a real algebraic set:

$$\begin{aligned} f_S : R(G) &\longrightarrow V(S) \\ \rho &\longmapsto (\rho(g_1), \dots, \rho(g_n)) . \end{aligned}$$

Note: $V(S)$ is actually the locus of finitely many polynomial equations:

$$r_j(A_1, \dots, A_m) = 1 \quad r_j \in R \quad 1 \leq j \leq m .$$

Hence, $R(G) = R(H)$, where H is a finitely presented group with quotient group G .

(d) Induced polynomial maps

If, in addition, we are given another group H , a finite generating set for H T , and a homomorphism $\lambda : G \rightarrow H$, then there is a unique polynomial map λ_S^T completing the following commutative diagram:

$$\begin{array}{ccc} R(H) & \xrightarrow{R(\lambda)} & R(G) \\ \cong \downarrow & & \downarrow \cong \\ V(T) & \xrightarrow{\lambda_S^T} & V(S) \end{array} .$$

Let:

$$T = \{h_1, \dots, h_m\} .$$

If we express the images of generators of G as words in the generators of H :

$$\lambda(g_j) = h_{\sigma_1 j}^{a_{1j}} \dots h_{\sigma_{pj} j}^{a_{pj} j} \quad 1 \leq j \leq n$$

then λ_S^T is given by the rule:

$$\lambda_S^T(A_1, \dots, A_m) = (A_{\sigma_{11}}^{a_{11}} \dots A_{\sigma_{p1}}^{a_{p1}}, \dots, A_{\sigma_{1n}}^{a_{1n}} \dots A_{\sigma_{pn}}^{a_{pn}}) .$$

Note: We may assume, as above, that p is independent of j by introducing trivial exponents if necessary.

These induced polynomial maps behave very much like matrices in the theory of linear transformations. In particular, if K is a third group with finite generating set U and γ is a homomorphism from H to K , then:

$$(\gamma \circ \lambda)_S^U = \lambda_S^T \cdot \gamma_T^U .$$

Likewise:

$$(\text{id})_S^S = \text{id} .$$

(e) The real algebraic structure of $R(G)$

From these observations, we observe that if $\lambda : G \rightarrow H$ is an isomorphism, then the induced polynomial map is an algebraic isomorphism of real algebraic sets:

$$\lambda_S^T : V(T) \xrightarrow{\cong} V(S) .$$

It follows that $R(G)$ has a well defined structure of a real algebraic set induced from the natural homeomorphisms associated to finite generating sets of G .

Conventions: Given a group G with a finite generating set S , we make the identification:

$$R(G) = V(S) .$$

Moreover, given a homomorphism $\lambda : G \rightarrow H$ and finite generating set T for H :

$$R(\lambda) = \lambda_S^T.$$

When we do not need to emphasize the generating sets we shall write:

$$\lambda^\# = \lambda_S^T.$$

$$\begin{array}{lll} \text{(eg)} & G = \langle x \mid \rangle \cong \mathbb{Z} & H = \langle y \mid y^2 = 1 \rangle \cong \mathbb{Z}_2 \\ & \lambda : G \longrightarrow H & S = \{x\} \quad T = \{y\} \\ & x^n \longmapsto y^n & \end{array}$$

$$V(S) = S^3$$

$$V(T) = \{A \in S^3 \mid A^2 = I\} = \{\pm I\} \cong S^0$$

$$\begin{array}{lll} \lambda^\# : V(T) & \longrightarrow & V(S) \quad \text{inclusion} \\ \pm I & \longmapsto & \pm I \end{array}$$

3. Representations of a free group.

(a) Standard model with respect to a basis

Let:

$$G = \langle g_1, \dots, g_n \rangle = \text{free group of rank } n.$$

Then:

$$\begin{array}{ll} R(G) & \xrightarrow{\cong} (S^3)^n \\ \rho & \longmapsto (\rho(g_1), \dots, \rho(g_n)) \end{array}$$

If $\lambda : G \rightarrow G$ is an endomorphism, then:

$$R(\lambda) : (S^3)^n \longrightarrow (S^3)^n$$

$$(\rho(g_1), \dots, \rho(g_n)) \longmapsto (\rho(\lambda(g_1)), \dots, \rho(\lambda(g_n)))$$

Note: As in the previous example, if H is a quotient of G , then the quotient map:

$$\lambda : G \longrightarrow H$$

induces an inclusion of representation spaces:

$$R(\lambda) : R(H) \longrightarrow R(G).$$

This is a general property of Hom functors.

(b) The homology of $R(G)$.

Since $R(G)$ is 2-connected, we have the natural isomorphism:

$$\alpha : H^3(R(G), \mathbb{Z}) \xrightarrow{\cong} \text{Hom}(H_3(R(G), \mathbb{Z}), \mathbb{Z}).$$

In addition, we have the commutative diagram:

$$\begin{array}{ccc} H^3(R(G), \mathbb{Z}) & \xrightarrow{(\lambda^\#)^*} & H^3(R(G), \mathbb{Z}) \\ \alpha \downarrow \cong & & \cong \downarrow \alpha \\ \text{Hom}(H_3(R(G), \mathbb{Z}), \mathbb{Z}) & \xrightarrow{\text{Hom}(\lambda^\#)} & \text{Hom}(H_3(R(G), \mathbb{Z}), \mathbb{Z}). \end{array}$$

For each element g of G , we have a natural evaluation map:

$$\begin{array}{ccc} \text{ev}_g : R(G) & \longrightarrow & S^3 \\ p & \longmapsto & p(g) \end{array} .$$

We have the following identity:

$$\text{ev}_{gh}(\rho) = \text{ev}_g(\rho) \cdot \text{ev}_h(\rho) .$$

Likewise, since S^3 is 2-connected, we have the natural isomorphism:

$$\beta : H^3(S^3, \mathbb{Z}) \xrightarrow{\cong} \text{Hom}(H_3(S^3), \mathbb{Z}) .$$

Let μ be the "dual" to the top class z of S^3 . (Recall that we fixed an orientation on S^3 .)

$$\begin{array}{ccc} \mu : \Pi_3(S^3, \mathbb{Z}) & \longrightarrow & \mathbb{Z} \\ z & \longmapsto & 1 \end{array}$$

$$z = \text{fund. class of } S^3 .$$

Using the above observations, we obtain a natural homomorphism:

$$\begin{array}{ccc} \varepsilon^* : H_1(G, \mathbb{Z}) & \longrightarrow & H^3(R(G), \mathbb{Z}) \\ g & \longmapsto & \text{ev}_g^*(\mu) \end{array} .$$

If λ is an endomorphism of G , then we have the following identity:

$$\text{ev}_g^* \circ R(\lambda) = \text{ev}_{\lambda(g)} .$$

The next proposition follows immediately.

Proposition 3.1: Suppose λ is an endomorphism of G . Then the following diagram commutes:

$$\begin{array}{ccc} H_1(G, \mathbb{Z}) & \xrightarrow{\lambda^*} & H_1(G, \mathbb{Z}) \\ \varepsilon^* \downarrow & & \downarrow \varepsilon^* \\ H^3(R(G), \mathbb{Z}) & \xrightarrow{R(\lambda)^*} & H^3(R(G), \mathbb{Z}) \end{array} .$$

We consider the naturally oriented 3-cycles in $R(G)$:

$$\begin{array}{ccc} x_j : S^3 & \longrightarrow & R(G) \\ A & \longmapsto & (A_{1,j}, \dots, A_{n,j}) \end{array}$$

where $A_{i,j} = I$ if $i \neq j$
 $= A$ if $i = j$.

Since G is a free group, we obtain an homomorphism:

$$\begin{array}{ccc} \varepsilon : G & \longrightarrow & \pi_3(R(G)) \\ x_j & \longmapsto & x_j \end{array} .$$

Note: Given the identification of $R(G)$ with $(S^3)^n$, $R(G)$ is a topological group. Hence, the homomorphism ε is given by taking products with respect to the group structure of $R(G)$. This agrees with the usual structure on π_3 .

Since $\pi_3(R(G))$ is an abelian group, ε factors through $G/[G, G]$:

$$\begin{array}{ccc} \varepsilon : H_1(G, \mathbb{Z}) & \longrightarrow & \pi_3(R(G)) \\ g_j & \longmapsto & x_j \end{array} .$$

Finally, we can compose with the Kurewicz homomorphism:

$$\begin{array}{ccc} \varepsilon : H_1(G, \mathbb{Z}) & \xrightarrow{\cong} & H_3(R(G), \mathbb{Z}) \\ g_j & \longmapsto & x_j \end{array}$$

This last map is an isomorphism, since $\{g_1, \dots, g_n\}$ is a basis for $H_1(G, \mathbb{Z})$ and $\{x_1, \dots, x_n\}$ is, clearly, the standard basis for $H_3(\langle S^3 \rangle)$.

$$H_1(G, \mathbb{Z}) = \langle g_1, \dots, g_n \rangle \cong \mathbb{Z}^n$$

$$H_3(R(G), \mathbb{Z}) = \langle x_1, \dots, x_n \rangle \cong \mathbb{Z}^n$$

Now, the Burewicz homomorphism yields an isomorphism:

$$\gamma : \pi_3(S^3) \xrightarrow{\cong} H_3(S^3, \mathbb{Z}) .$$

Under this identification, the top class z of S^3 is given by the identity map:

$$\begin{array}{ccc} z = [\text{id}] & \text{id} : S^3 & \longrightarrow S^3 \\ & A & \longmapsto A \end{array}$$

We compute that $\{\varepsilon^*(g_1), \dots, \varepsilon^*(g_n)\}$ is a dual basis for $H^3(R(G), \mathbb{Z})$:

$$\begin{aligned} \varepsilon^*(g_i)(x_j) &= \delta_{i,j} = 1 & i = j \\ &= 0 & i \neq j \end{aligned}$$

Hence, ε^* is a natural isomorphism:

$$\begin{array}{ccc} \varepsilon^* : H_1(G, \mathbb{Z}) & \xrightarrow{\cong} & H^3(R(G), \mathbb{Z}) \\ g_j & \longmapsto & x_j^* \end{array}$$

The next proposition follows immediately.

Proposition 3.2: Suppose λ is an endomorphism of G . Then the following diagram commutes:

$$\begin{array}{ccc} H_1(G, \mathbb{Z}) & \xrightarrow{\lambda_*} & H_1(G, \mathbb{Z}) \\ \varepsilon \downarrow \cong & & \cong \downarrow \varepsilon \\ H_3(R(G), \mathbb{Z}) & \xrightarrow{(\lambda_*)^t} & H_3(R(G), \mathbb{Z}) \end{array}$$

where

λ_* = induced homomorphism of $H_1(G, \mathbb{Z})$

$\lambda_*^{\#} = (R(\lambda))_* = \text{ind. hom. of } H_3(R(G), \mathbb{Z})$

and

t = transpose w.r.t. the ordered basis $(x_1, \dots, x_n)$.

(c) The tangent bundle of $R(G)$

We fix a standard model for $R(G)$ as in section (a):

$$G = \langle g_1, \dots, g_n \rangle$$

$$R(G) = (S^3)^n \quad \rho = (A_1, \dots, A_n)$$

We see that:

$$T(R(G)) = \{ (A_1, \dots, A_n, X_1, \dots, X_n) \mid A_i \in S^3, X_i \in T_{A_i}(S^3) \} .$$

Let:

$$1 = (I, \dots, I) = \text{trivial representation} .$$

Then the "Lie Algebra" of $R(G)$ is given as:

$$T_1(R(G)) = S^n = S_1 \oplus \dots \oplus S_n$$

where S is the Lie algebra of S^3 described in section 1(b). As for S^3 , we have a "natural" trivialization:

$$R(G) \times S^n \xrightarrow{\alpha} TR(G)$$

$$(A_1, \dots, A_n, X_1, \dots, X_n) \longmapsto (A_1, \dots, A_n, A_1 X_1, \dots, A_n X_n)$$

Likewise, we have a "natural" Riemannian metric arising via translation from the inner product on S^n :

$$\langle \cdot, \cdot \rangle : S^n \times S^n \longrightarrow \mathbb{R}$$

$$((X_1, \dots, X_n), (Y_1, \dots, Y_n)) \longmapsto \sum_i \text{tr}(X_i \bar{Y}_i^t)$$

With respect to these models, the conjugation actions of SO_3 (or S^3) are given by:

$$\cdot : SO_3 \times R(G) \longrightarrow R(G)$$

$$([A], (A_1, \dots, A_n)) \longmapsto (AA_1 A^{-1}, \dots, AA_n A^{-1})$$

$$\cdot : SO_3 \times TR(G) \longrightarrow TR(G)$$

$$([A], (A_i)_i, (X_i)_i) \longmapsto ((AA_i A^{-1})_i, (AX_i A^{-1})_i)$$

Again, these actions are actions by isometries.

Note: We are inclined to consider $R(G)$ as a Lie group via the identification with $(S^3)^n$. Although this is possible, it is not natural; the induced group structure depends strongly on the choice of basis. Nevertheless, this auxiliary structure is useful for technical purposes.

We now wish to compare the action of $\text{End}(G)$ on $H_1(G, \mathbb{Z})$ with the action on the "Lie Algebra" of $R(G)$ $T_1(R(G))$. As in Proposition 3.1, we construct an identification.

Note: We may also consider cotangent spaces. This is most natural, as we saw in Proposition 2.3. We shall not develop this approach.

Using the given basis for $H_1(G, \mathbb{Z})$ we can identify $T_1(R(G))$ with $H_1(G, S)$:

$$H_1(G, S) = H_1(G, \mathbb{Z}) \otimes S$$

$$= \mathbb{Z}^n \otimes S$$

$$= S_1 \oplus \dots \oplus S_n = T_1(R(G))$$

We obtain the following analog of Proposition 2.3:

Proposition 3.3: Suppose λ is an endomorphism of G . Then the following diagram commutes:

$$\begin{array}{ccc} H_1(G, \mathbb{Z}) \otimes S & \xrightarrow{\lambda_*^t \otimes \text{id}} & H_1(G, \mathbb{Z}) \otimes S \\ \epsilon \downarrow & & \downarrow \epsilon \\ T_1(R(G)) & \xrightarrow{D_1 R(\lambda)} & T_1(R(G)) \end{array}$$

Proof: Observe that the tangent space at 1 to $R(G)$ is spanned by curves of cyclic representations, curves lying in the 3-cycles in the proof of Proposition 3.2. A typical curve has the form:

$$a_j(t) = (A_{1j}(t), \dots, A_{nj}(t)) \quad A_{ij}(t) = I \quad \text{if } i \neq j$$

$$A_{jj}(0) = I$$

Write the images of generators of G under λ as words in the generators of G :

$$\lambda(g_j) = w_j(g_1, \dots, g_n)$$

Then the action of λ on $H_1(G, \mathbb{Z})$ is given by the following:

$$(i) \quad \lambda_*(g_j) = \sum_{i=1}^n \exp_{g_i}(w_j) g_i$$

where $\exp_{g_i}(w_j)$ is the sum of the exponents of occurrences of g_i in w_j .

On the other hand, we compute the action on the tangent space as follows:

$$\begin{aligned} R(\lambda)(e_i(t)) &= R(\lambda)(A(t), I, \dots, I) \\ &= ((A(t))^{\exp_{g_1}(w_i)}, \dots, (A(t))^{\exp_{g_n}(w_i)}) \end{aligned}$$

$$\begin{aligned} &\left[\frac{d(R(\lambda)(e_i(t)))}{dt} \right]_{t=0} \\ &= \left[\exp_{g_1}(w_i) \left(\frac{dA}{dt} \right)_{t=0}, \dots, \exp_{g_n}(w_i) \left(\frac{dA}{dt} \right)_{t=0} \right]. \end{aligned}$$

Under the identification of $T_1(R(G))$ with $H_1(G, \mathbb{Z}) \otimes \mathbb{R}$ described above, the tangent vector to the curve $e_i(t)$ at the trivial representation is clearly $g_i \otimes X$, where

$$X = \left(\frac{dA}{dt} \right)_{t=0} \in \mathbb{R}.$$

Generalizing, we conclude that:

$$(ii) \quad D_*(R(\lambda))(x_j \otimes X) = \sum_{i=1}^n \exp_{g_i}(w_j) x_i \otimes X.$$

Comparing (i) and (ii), we are done. $\square$

(d) Induced orientations of $R(G)$ via a basis of G

Given a basis for G , we use the identification of part (a) to induce an orientation on $R(G)$. (The fixed orientation of S^3 induces the obvious

orientation on $(S^3)^n$.) That is, by the Kunneth formula:

$$\begin{aligned} H_3(S^3) \otimes \dots \otimes H_3(S^3) &\xrightarrow{\cong} H_{3n}((S^3)^n) \\ x_1 \otimes \dots \otimes x_n &\longmapsto x_1 \times \dots \times x_n \end{aligned}$$

and the fundamental class w is the image:

$$z \otimes \dots \otimes z \longmapsto w = z \times \dots \times z.$$

In a similar manner, as we have seen, we obtain an identification:

$$T_1(S^3) \otimes \dots \otimes T_1(S^3) \xrightarrow{\cong} T_1(R(G))$$

or more generally:

$$\begin{aligned} T_{A_1}(S^3) \otimes \dots \otimes T_{A_n}(S^3) &\xrightarrow{\cong} T_{\rho}(R(G)) \\ \text{where } (A_1, \dots, A_n) &= (\rho(g_1), \dots, \rho(g_n)). \end{aligned}$$

If we denote by $\tilde{z}$ and $\tilde{w}$ the orientations on the tangent spaces induced by z and w , we see that:

$$\tilde{z} \otimes \dots \otimes \tilde{z} \longmapsto \tilde{w}.$$

We consider the following bilinear forms. First, we have the standard pairing of homology and cohomology:

$$\langle , \rangle : H^p(R(G), \mathbb{Z}) \times H_p(R(G), \mathbb{Z}) \longrightarrow \mathbb{Z}.$$

Secondly, we have the cup product:

$$\cup : H^p(R(G), \mathbb{Z}) \times H^q(R(G), \mathbb{Z}) \longrightarrow H^{p+q}(R(G), \mathbb{Z}).$$

Likewise, we have the cap product:

$$\cap : H^p(R(G), \mathbb{Z}) \times H_q(R(G), \mathbb{Z}) \longrightarrow H_{q-p}(R(G), \mathbb{Z}).$$

recall that there is a natural identification:

$$H_0(R(G), \mathbb{Z}) \longrightarrow \mathbb{Z}.$$

Under this identification, we have:

$$\langle \alpha, \beta \rangle = n \text{ in case } p = q.$$

Furthermore, $\cup$ and $\cap$ are dual:

$$\langle \alpha \cup \beta, \gamma \rangle = \langle \alpha, \beta \cap \gamma \rangle.$$

Given an orientation of $R(G)$ as above, we obtain the Poincaré duality:

$$\begin{aligned} D_w : H^p(R(G), \mathbb{Z}) &\longrightarrow H_{3n-p}(R(G), \mathbb{Z}) \\ \alpha &\longmapsto \alpha \cap w \end{aligned}$$

Furthermore, by "counting" signed intersections with respect to this orientation, we obtain the intersection product:

$$\langle \cdot, \cdot \rangle_w : H_p(R(G), \mathbb{Z}) \times H_q(R(G), \mathbb{Z}) \longrightarrow H_{p+q-3n}(R(G), \mathbb{Z}).$$

Again, by using the identification above, we obtain as a special case:

$$\langle \cdot, \cdot \rangle_w : H_p(R(G), \mathbb{Z}) \times H_{3n-p}(R(G), \mathbb{Z}) \longrightarrow \mathbb{Z}.$$

The fundamental identity relating these various forms is:

$$\langle D(\alpha), D(\beta) \rangle_w = D(\alpha \cup \beta).$$

In particular, in the special case, duality arises via the formula:

$$\langle D(\alpha), x \rangle_w = \langle \alpha, x \rangle, \dim(\alpha) + \dim(x) = 3n.$$

Considering the natural map ϵ^* of part (b), we easily compute that:

$$\langle \epsilon^*(g_1) \cup \dots \cup \epsilon^*(g_n), w \rangle = \prod_{i=1}^n \epsilon^*(g_i)(x_i) = 1.$$

Hence, the "dual" of w w^* is given by:

$$w_i^* = x_1^* \cup \dots \cup x_n^* \quad x_i^* = \epsilon^*(g_i) \quad 1 \leq i \leq n.$$

Note: As in section 3(b), "dual" refers to the dual in the functional sense. We reserve the term dual for Poincaré dual.

From Propositions 3.2 and 3.3, and these observations, we deduce the following proposition.

Proposition 3.4: If λ is an endomorphism of G , then

- (1) $R(\lambda)^*(w^*) = \det(\lambda_*) \cdot w^*$,
- (2) $R(\lambda)_*(w) = \det(\lambda_*) \cdot w$.

Suppose g is a generator of G (an element of a basis of G).

Let:

$$y(g) = \{ \rho \mid \rho(g) = I \}.$$

Choosing a basis for G which contains g , we can order the basis so that:

$$G = \langle g'_1, \dots, g'_n \rangle \quad g = g'_1.$$

With respect to the associated identification, we obtain an homeomorphism:

$$\begin{aligned} y(g) &\xrightarrow{\cong} (S^3)^{n-1} \\ &\longmapsto (\rho(g'_2), \dots, \rho(g'_n)). \end{aligned}$$

Hence, $y(g)$ determines a $3n - 3$ cycle in $R(G)$ which is well defined up to sign.

Given an orientation on $R(G)$, we obtain a well defined cycle from the following observations:

$$\begin{aligned} \langle y(g), x_j' \rangle_W &= \pm 1 & j &= 1 \\ &= 0 & j &\neq 1 \end{aligned}$$

Hence, we conclude:

$$y(g) = \pm D(\varepsilon^*(g)) \quad .$$

(We choose the orientation of $y(g)$ to make the sign equal to +.)

Summarizing, we may write:

$$D(\varepsilon^*(g)) = y(g) = \pm \{ \rho \mid \rho(g) = I \} \quad .$$

CHAPTER II: HEGBARD DECOMPOSITIONS AND STABLE EQUIVALENCE

1. Heegard decompositions and models

(a) The standard handlebody W

Let:

W = standard (model) handlebody of genus g ($g \geq 1$)

$F = \partial W$ = boundary of W

D = embedded 2-disk in F

O = basepoint of F on ∂D

$F^* = F \setminus \text{interior}(D)$, $S^1 = \partial D$.

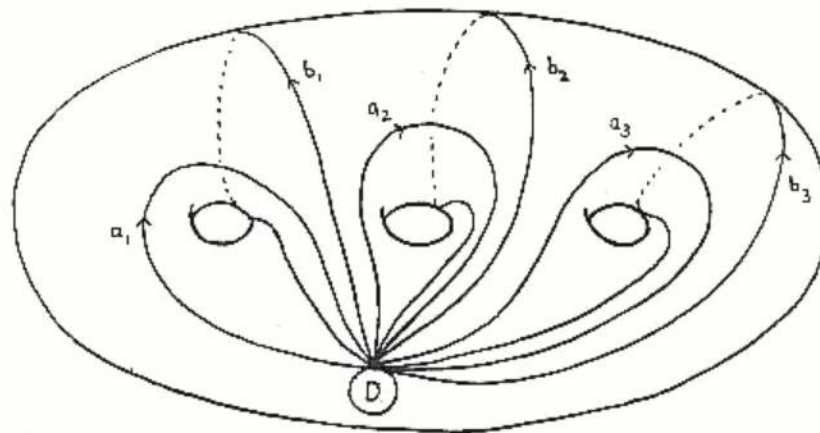


Figure 4

We may choose a family of loops on (F^*, O) , as in Figure 4, such that:

$$H_1(F^*, 0) = \langle a_1, \dots, a_g, b_1, \dots, b_g \rangle$$

$$H_1(F, 0) = \langle a_1, \dots, a_g, b_1, \dots, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$

$$\text{where } [a, b] = aba^{-1}b^{-1}$$

$$H_1(W, 0) = \langle a_1, \dots, a_g, b_1, \dots, b_g \mid b_1 = \cdots = b_g = 1 \rangle$$

$$= \langle a_1, \dots, a_g \rangle.$$

Let c_i and d_i denote the homology classes of a_i and b_i respectively.

Then we have abelian presentations:

$$H_1(F, \mathbb{Z}) = H_1(F^*, \mathbb{Z}) = \langle c_1, d_1, \dots, c_g, d_g \rangle \cong \mathbb{Z}^{2g}$$

$$H_1(W, \mathbb{Z}) = \langle c_1, \dots, c_g \rangle \cong \mathbb{Z}^g.$$

(b) The Heegard model (W, h)

Given an orientation preserving homeomorphism:

$$h : (F, D, 0) \xrightarrow{\cong} (-F, -D, 0) \text{ or. pres.}$$

where $(-F, -D)$ is (F, D) with the opposite orientation, we define a closed orientable 3-manifold:

$$|(W, h)| = \frac{(W \times \{1\}) \cup (-W \times \{2\})}{\langle (x, 1) \sim (h(x), 2) \mid x \in F \rangle}.$$

(c) Heegard decomposition of a three-manifold M^3

An Heegard decomposition of a three-manifold M^3 is a pair of subsets (W_1, W_2) of M^3 with the following properties:

$$(1) \quad M^3 = W_1 \cup W_2$$

$$(2) \quad \partial W_1 = W_1 \cap W_2 = \partial W_2$$

$$(3) \quad W_1 \neq W \neq W_2.$$

Given an Heegard decomposition of M^3 , we may construct an homeomorphism to an Heegard model as follows:

$$(4) \text{ choose } f_i : W_i \xrightarrow{\cong} W \quad i = 1, 2$$

$$(5) \text{ let } h = f_2 \circ f_1^{-1}.$$

Note: We choose f_2 and f_1 so that h is an automorphism of the triple $(F, D, 0)$ as above.

$$(6) \quad f : M^3 \xrightarrow{\cong} |(W, h)|$$

$$x \longmapsto [(f_1(x), 1)] \quad x \in W_1$$

$$\longmapsto [(f_2(x), 2)] \quad x \in W_2.$$

It is well known that every closed 3-manifold has an Heegard decomposition. Hence, every closed 3-manifold is homeomorphic to an Heegard model. We shall work with Heegard models for the most part. This is primarily done for purely computational and technical reasons.

(d) Associated presentations of π_1 and H_1

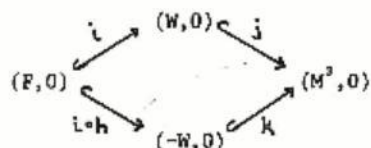
Identify M^3 with an Heegard model (W, h) . By the Van Kampen Theorem, we may obtain a presentation for $H_1(M^3, 0)$ as follows:

$$\begin{aligned}\Pi_1(W,0) &= \langle a_1^1, \dots, a_g^1, b_1^1, \dots, b_g^1 \mid b_1^1 = \dots = b_g^1 = 1 \rangle \\ &= \langle a_1^1, \dots, a_g^1 \rangle\end{aligned}$$

$$\begin{aligned}\Pi_1(-W,0) &= \langle a_1^2, \dots, a_g^2, b_1^2, \dots, b_g^2 \mid b_1^2 = \dots = b_g^2 = 1 \rangle \\ &= \langle a_1^2, \dots, a_g^2 \rangle\end{aligned}$$

$$\Pi_1(F,0) = \langle a_1, \dots, a_g, b_1, \dots, b_g \mid [a_1, b_1] \cdots [a_g, b_g] = 1 \rangle$$

Here we have identified $(W,0)$ with $W \times \{1\} \subset |(W,h)|$ via the map j and $(-W,0)$ with $-W \times \{2\}$ via k :



Then we conclude that:

$$\Pi_1(M^3,0) = \left\langle a_1^1, \dots, a_g^1, a_1^2, \dots, a_g^2 \mid \begin{array}{l} (1) \ i_g(a_1) = (i \cdot h)_\#(a_1) \\ (2) \ i_g(b_1) = (i \cdot h)_\#(b_1) \\ 1 \leq i \leq g \end{array} \right\rangle$$

$$\Pi_1(M^3,0) = \langle a_1^1, \dots, a_g^1, a_1^2, \dots, a_g^2 \mid \begin{array}{l} (1) \ a_1^1 = (i \cdot h)_\#(a_1) \\ (2) \ 1 = (i \cdot h)_\#(b_1) \\ 1 \leq i \leq g \end{array} \rangle$$

$$\Pi_1(M^3,0) = \langle a_1^2, \dots, a_g^2 \mid (i \cdot h)_\#(b_i) \quad 1 \leq i \leq g \rangle$$

Alternatively, we can obtain a presentation by attaching discs to F and capping off holes with 3 discs. We obtain M^3 by attaching discs to $b_1, \dots, b_g$ and capping off the resulting 2 sphere to get W_1 and then attaching discs to $h^{-1}(b_1), \dots, h^{-1}(b_g)$ and capping to attach W_2 . This yields the following presentation:

$$\Pi_1(M^3,0) = \left\langle a_1, b_1, \dots, a_g, b_g \mid \begin{array}{l} (1) \ b_1 = 1 \\ (2) \ h_*^{-1}(b_1) = 1 \end{array} \right\rangle$$

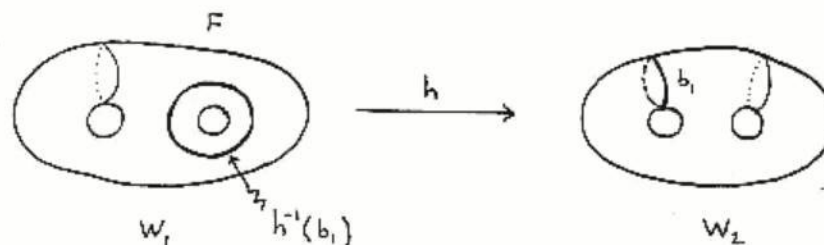


Figure 5

These two presentations, of course, are of the same type. The first considers F as identified with the target of h , the latter with the source. We shall prefer the latter perspective.

Abelianizing, we obtain $H_1(M^3, \mathbb{Z})$:

$$\Pi_1(M^3, \mathbb{Z}) = \left\langle c_1, d_1, \dots, c_g, d_g \mid \begin{array}{l} (1) \ d_i = 1 \\ (2) \ h_*^{-1}(d_i) = 1 \end{array} \right\rangle$$

Since F is oriented, we have a well defined intersection pairing on $H_1(F, \mathbb{Z})$:

$$\langle \cdot, \cdot \rangle_F : H_1(F, \mathbb{Z}) \times H_1(F, \mathbb{Z}) \longrightarrow \mathbb{Z}.$$

We assume that the basis for $H_1(F, \mathbb{Z})$ described above is chosen so that this pairing has the following matrix with respect to the ordered basis $(c_1, \dots, c_g, d_1, \dots, d_g)$:

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}, \quad \begin{array}{c} \uparrow c_i \\ \hline \longrightarrow d_i \end{array}, \quad \langle c_i, d_i \rangle = 1.$$

That is, if we consider the pairing:

$$\langle \cdot, \cdot \rangle : H_1(F, \mathbb{Z}) \times H_1(F, \mathbb{Z}) \longrightarrow \mathbb{Z}$$

- (1) $\langle c_i, c_j \rangle = \delta_{ij}$
- (2) $\langle c_i, d_j \rangle = \langle d_j, c_i \rangle = 0$
- (3) $\langle d_i, d_j \rangle = \delta_{ij}$

we have the identity:

$$(x, y)_F = \langle x, Jy \rangle.$$

Recall that h is an orientation preserving homeomorphism from F to $-F$. It follows immediately from the definition of the intersection product that:

$$(h_*(x), h_*(y))_F = -(x, y)_F.$$

Combining these identities, we compute:

$$\langle x, Jy \rangle = (x, y)_F = -(h_*(x), h_*(y))_F$$

$$\langle x, Jy \rangle = -\langle h_*(x), Jh_*(y) \rangle$$

$$\langle x, Jy \rangle = \langle x, -h_*^t J h_* y \rangle.$$

Hence:

$$J = -h_*^t J h_*$$

$$h_*^{-1} = -J^{-1} h_*^t J \quad \text{where} \quad J^{-1} = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

Writing h_* and h_*^{-1} with respect to this basis, we obtain the equivalence:

$$h_* = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \Leftrightarrow h_*^{-1} = - \begin{bmatrix} D^t & -B^t \\ -C^t & A^t \end{bmatrix}.$$

Hence, we obtain a presentation matrix for $H_1(M^3, \mathbb{Z})$ P :

$$P = \begin{bmatrix} 0 & B^t \\ I & -A^t \end{bmatrix} = (b_1, \dots, b_g, h_*^{-1}(b_1), \dots, h_*^{-1}(b_g)).$$

That is to say, we have a short exact sequence:

$$1 \longrightarrow \mathbb{Z}^{2g} \xrightarrow{P} \mathbb{Z}^{2g} \longrightarrow H_1(M^3, \mathbb{Z}) \longrightarrow 1.$$

In particular, we conclude that:

$$\beta_1(M^3) \neq 0 \quad \text{iff} \quad \det(P) = 0$$

$$\beta_1(M^3) = 0 \quad \text{iff} \quad \det(P) = |H_1(M^3, \mathbb{Z})|$$

where $|H_1(M^3, \mathbb{Z})|$ is the order of $H_1(M^3, \mathbb{Z})$.

2. Stable equivalence of Heegard decompositions

(a) Stable equivalence of Heegard decompositions

Suppose M^3 is a closed 3-manifold with Heegard decompositions (W_1, W_2) and (W'_1, W'_2) . We say that (W_1, W_2) is isotopic to (W'_1, W'_2) if there is an ambient isotopy of M^3 taking W_1 to W'_1 . We write $(W_1, W_2) \approx (W'_1, W'_2)$.

An unknotted one handle H in a handlebody W is an embedding of $D^2 \times I$ such that:

- (1) $H \cap F = D^2 \times \{0, 1\}$ ($F = \partial W$)
- (2) if α is the arc $\{0\} \times I$ there is an arc β on F such that $\alpha \cup \beta$ is the boundary of an embedded disc B^2 with $B^2 \cap F = \beta$.

Note: Up to ambient isotopy of W there is exactly one unknotted one handle H in W .

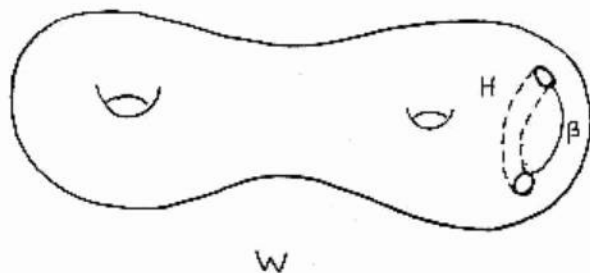


Figure 6

Given an unknotted one handle H in W_1 , we obtain a new Heegard decomposition of M^3 :

$$\sigma(W_1, W_2) = (W_1 \cup H, \overline{W_2 \setminus H})$$

well defined up to isotopy.

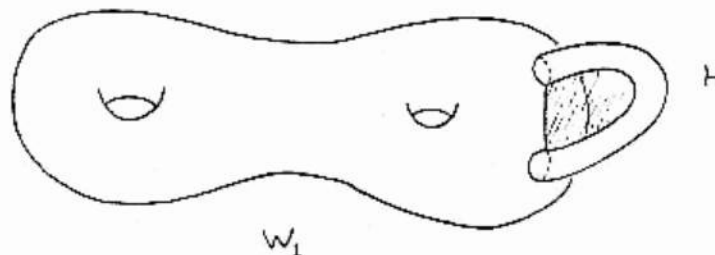


Figure 7

We say that (W_1, W_2) is stably equivalent to (W'_1, W'_2) if there exists a pair of nonnegative integers, $j, k \geq 0$, such that

$$\sigma^j(W_1, W_2) \approx \sigma^k(W'_1, W'_2)$$

Note: This move of adding an unknotted one handle to increase the genus of an Heegard decomposition has an alternative interpretation in terms of connected sums of Heegard decompositions. (See sections (c) and (d) below).

(b) Singer's Theorem on Stable Equivalence of Heegard Decompositions of Closed Orientable Three Manifolds [SIN]

Stable equivalence is the basic equivalence relating the Heegard decompositions of a fixed three-manifold. This is the theorem of Singer:

Theorem 2.1 (Singer): Any two Heegard decompositions of an orientable closed 3 manifold are stably equivalent.

(c) Connected sums of Heegard decompositions

Suppose that (W_1, W_2) is an Heegard decomposition of M^3 and (V_1, V_2) is an Heegard decomposition of N^3 . Choose a 3-ball B in M^3 and a 3-ball C in N^3 such that:

$$B \cap \partial W_1 = D = 2\text{-disc} \quad D \cap \partial B = \partial D$$

$$C \cap \partial V_1 = E = 2\text{-disc} \quad E \cap \partial C = \partial E$$

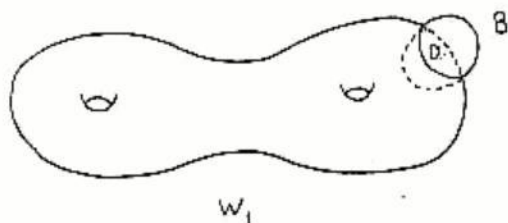


Figure 8

If we remove the interiors of B and C from M^3 and N^3 respectively and glue ∂B to ∂C by a homeomorphism respecting the decomposition:

$$\varphi : (\partial B, \partial B \cap W_1) \xrightarrow{\cong} (\partial C, \partial C \cap V_1)$$

we evidently obtain an Heegard decomposition for the connected sum of M^3 and N^3 $M^3 \# N^3$:

$$(W_1, W_2) \# (V_1, V_2) = ((\overline{W_1 \setminus B}) \cup_{\varphi} (\overline{V_1 \setminus C}), (\overline{W_2 \setminus B}) \cup_{\varphi} (\overline{V_2 \setminus C})) .$$

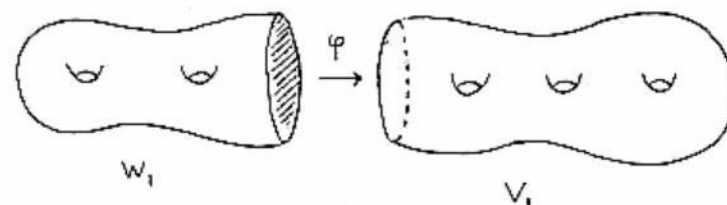


Figure 9

We may think of this operation as taking boundary connected sums of the component handle bodies:

$$(W_1, W_2) \# (V_1, V_2) = (W_1 \# V_1, W_2 \# V_2) .$$

Note: We adopt the usual convention that φ is orientation reversing with respect to the induced orientations on ∂B and ∂C .

The operation of connected sum of Heegard decompositions has a particularly simple description in terms of the Heegard models of section 1(b). If M^3 is identified with $|(W, h)|$ and N^3 with $|(V, g)|$ so that:

$$h : (\partial W, D, 0) \xrightarrow{\cong} (-\partial W, -D, 0) \quad \text{or. pres}$$

$$g : (\partial V, E, p) \xrightarrow{\cong} (-\partial V, -E, p) \quad \text{or. pres}$$

and if, by abuse of notation, we denote the corresponding Heegard decompositions by (W, h) and (V, g) , then:

$$(W, h) \# (V, g) = (W \# V, h \# g)$$

where the boundary connect sum identifies $(D, 0)$ and (E, p) and $h \# g$ is well defined by assumption.

Note: We choose the new disc and base point as in Figure 10.

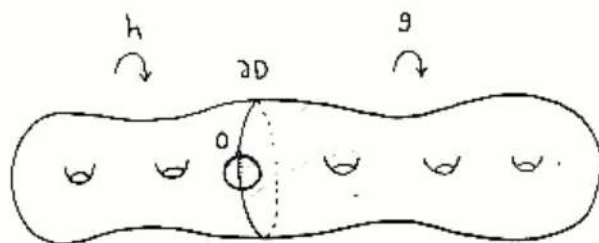


Figure 10

(d) The genus one decomposition of S^3 (T_1, T_2)

Of particular interest to us is the genus one Heegard decomposition of S^3 (T_1, T_2) where T_1 is the standard torus embedded in R^3 and S^3 is $R^3 \cup \{\infty\}$. (Actually there is only one genus one Heegard decomposition of S^3 [W].)

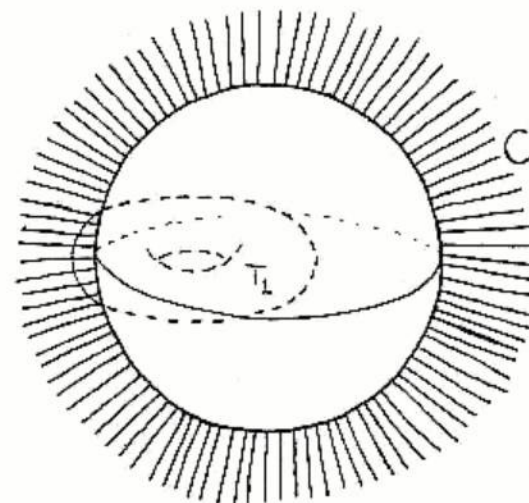


Figure 11

It is evident from Figure 7 that Singer's move of adding an unknotted one-handle is equivalent to forming the connect sum with the genus-one Heegard decomposition of S^3 :

$$\sigma(W_1, W_2) = (W_1, W_2) \# (T_1, T_2)$$

The genus one Heegard decomposition of S^3 has the following Heegard model.

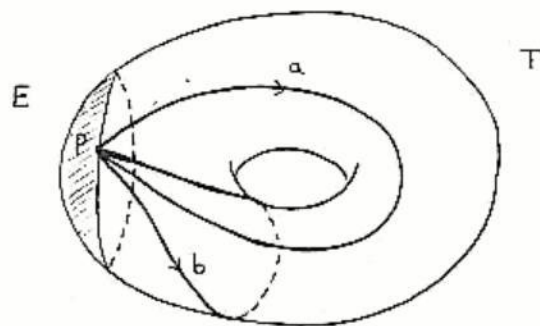


Figure 12

Choose a homeomorphism of (T, E, p) which fixes E pointwise and interchanges the simple closed curves a and b :

$$\begin{aligned} \tau : (T, E, p) &\xrightarrow{\cong} (-T, -E, p) \\ \tau_* : \pi_1(T^*, p) &\longrightarrow \pi_1(T^*, p) \\ a &\longmapsto b \\ b &\longmapsto a \end{aligned}$$

Clearly, this yields the desired Heegaard model:

$$S^3 = |(T, \tau)|$$

(T, τ) = the genus one Heeg. dec. of S^3 .

CHAPTER III: REPRESENTATION SPACES ASSOCIATED TO HEEGARD DECOMPOSITIONS

1. The diagram of representation spaces

(a) The diagram of representation spaces

Given an Heegaard decomposition (W_1, W_2) of a closed oriented 3-manifold M^3 , we have an associated commutative diagram of inclusions:

$$\begin{array}{ccccc} & & & \langle W_1, 0 \rangle & \\ & \swarrow & & \searrow & \\ \langle \partial F^*, 0 \rangle & \xleftarrow{i} & \langle F^*, 0 \rangle & \longleftrightarrow & \langle F, 0 \rangle \\ & \searrow & & \swarrow & \\ & & & \langle W_2, 0 \rangle & \end{array} \quad \begin{array}{c} \hookrightarrow \\ \hookrightarrow \end{array} \langle M^3, 0 \rangle$$

where:

$$F = \partial W_1 = \partial W_2 = W_1 \cap W_2$$

F^* is as in section II.1(a).

Note that each inclusion except for i is obtained by attaching 2 and 3 cells. Hence, by applying π_1 , we obtain a commutative diagram of groups where all homomorphisms except i_* are surjective. "All the fundamental group is in the punctured surfaces":

$$\begin{array}{ccccc} & & & \pi_1(W_1, 0) & \\ & \swarrow & & \searrow & \\ \pi_1(\partial F^*, 0) & \xleftarrow{i_*} & \pi_1(F^*, 0) & \longrightarrow & \pi_1(F, 0) \\ & \searrow & & \swarrow & \\ & & & \pi_1(W_2, 0) & \end{array} \quad \begin{array}{c} \twoheadrightarrow \\ \twoheadrightarrow \end{array} \pi_1(M^3, 0)$$

Clearly, i_* is an inclusion.

Finally, by applying the representation functor R , we obtain a commutative diagram of spaces:

$$R_3 \hookleftarrow R^* \hookleftarrow R \begin{matrix} \nearrow Q_1 \\ \searrow Q_2 \end{matrix} R(\Pi_1(M^3, 0))$$

where:

$$R_3 = R(\Pi_1(\partial F^*, 0))$$

$$R^* = R(\Pi_1(F^*, 0))$$

$$R = R(\Pi_1(F, 0))$$

$$Q_1 = R(\Pi_1(W_1, 0))$$

$$Q_2 = R(\Pi_1(W_2, 0))$$

$$j = R(i_*)$$

As usual for Hom functors, all the surjections become injections. (It can be shown that j is surjective. We shall show this later.)

(b) Intersection properties of Q_1 and Q_2

From our previous discussion, we observe:

$$R^* \cong (S^3)^{2g}, \quad Q_1 \cong (S^3)^g, \quad Q_2 \cong (S^3)^g.$$

Since all the injections are natural, we consider them as inclusions. Under this convention, we conclude that:

$$R(\Pi_1(M^3, 0)) = Q_1 \cap Q_2$$

Q_1 and Q_2 are half-dimensional cycles in R^* .

Hence, we can consider their intersection product in R^* . We shall prove the following proposition.

Proposition 1.1:

- (a) $\langle Q_1, Q_2 \rangle_{R^*} = 0$ iff $\beta_1(M^3) \neq 0$
- (b) $|\langle Q_1, Q_2 \rangle_{R^*}| = |H_1(M^3, \mathbb{Z})|$ iff $\beta_1(M^3) = 0$
- (c) $Q_1 \neq Q_2$ at 1 if $\beta_1(M^3) = 0$

Note: We have denoted the top class of R^* by R^* .

Propositions 1.3.1, 1.3.2 and 1.3.3 thru 9 are the key tools in the proof. We need to develop some background before beginning the proof.

Identify M^3 with an Heegard model (W, h) . The diagram of inclusions in part (a) may be written as follows:

$$\begin{array}{ccccccc} (S^1, 0) & \xrightarrow{i} & (F^*, 0) & \xrightarrow{i} & (F, 0) & \xrightarrow{i} & (W, 0) \\ \cong \downarrow h & & \cong \downarrow h & & \cong \downarrow h & & \searrow \\ (-S^1, 0) & \xrightarrow{i} & (-F^*, 0) & \xrightarrow{i} & (-F, 0) & \xrightarrow{i} & (-W, 0) \end{array} \begin{matrix} \nearrow \\ \searrow \end{matrix} \begin{matrix} \\ \end{matrix} (M^3, 0)$$

where:

$$j : (W, 0) \longrightarrow | (W, h) |$$

$$x \longmapsto [(x, 1)]$$

$$k : (-W, 0) \longrightarrow | (W, h) |$$

$$x \longmapsto [(x, 2)]$$

From this, we obtain the corresponding diagram of representation spaces:

$$\begin{array}{ccccc}
R(\Pi_1(F^*, 0)) & \longleftrightarrow & R(\Pi_1(F, 0)) & \longleftrightarrow & R(\Pi_1(W, 0)) \\
\uparrow R(h_*) & & \uparrow R(h_*) & & \searrow \\
R(\Pi_1(F^*, 0)) & \longleftrightarrow & R(\Pi_1(F, 0)) & \longleftrightarrow & R(\Pi_1(-W, 0)) \\
& & & & \nearrow \\
& & & & R(\Pi_1(M^3, 0))
\end{array}$$

considering all the injections as inclusions, we may make the identifications below:

$$(1) R^* = R(\Pi_1(F^*, 0)) = (S^3)^{2g}$$

$$\rho \longmapsto (\rho(a_1), \dots, \rho(a_g), \rho(b_1), \dots, \rho(b_g))$$

$$(2) R = R(\Pi_1(F)) = \{(A_1, \dots, A_g, B_1, \dots, B_g) \mid [A_1, B_1] \cdots [A_g, B_g] = 1\}$$

$$(3) Q_1 = R(\Pi_1(W_1)) = \{(A_1, \dots, A_g, I, \dots, I)\} \cong (S^3)^g$$

$$(4) Q_2 = R(\Pi_1(W_2)) = h^*(Q_1) \cong (S^3)^g$$

$$(5) R(\Pi_1(M^3)) = Q_1 \cap Q_2 \quad (\text{under identifications})$$

In particular, we have fixed orientations on R^* and Q_1 .

Proof of Proposition 1.1:

Consider the natural cocycles in $H^3(R^*, \mathbb{Z})$:

$$\alpha_i = z^*(c_i) \quad \beta_i = z^*(d_i) \quad i = 1, \dots, g$$

If we consider R^* as having the induced orientation via the ordered basis $(c_1, \dots, c_g, d_1, \dots, d_g)$, the dual $3n - 3$ cycles are given by:

$$D(\alpha_i) = (-1)^1 \{\rho \mid A_i = 1\} \quad D(\beta_i) = (-1)^{g-i} \{\rho \mid B_i = 1\}$$

Notes:

(1) $\{\rho \mid A_i = 1\} = \{(A_1, \dots, A_{i-1}, 1, A_{i+1}, \dots, A_g, B_1, \dots, B_g)\}$ is oriented by $z \times \dots \times z \times 1 \times z \times \dots \times z \times z \times \dots \times z$.

(2) Similarly, $\{\rho \mid B_i = 1\}$ has the obvious orientation.

Clearly:

$$Q_1 = D(\beta_1 \cup \dots \cup \beta_g)$$

Then:

$$\begin{aligned}
\langle Q_1, Q_2 \rangle_{R^*} &= \langle Q_1, (h^*)^*(Q_1) \rangle_{R^*} \\
&= \langle \beta_1 \cup \dots \cup \beta_g \cup (h^*)^*(\beta_1) \cup \dots \cup (h^*)^*(\beta_g), w \rangle
\end{aligned}$$

where w is the top class of R^* . By Proposition I.3.1, the matrix for $(h^*)^*$ with respect to the ordered basis $(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g)$ is the same as the matrix for h_* with respect to $(a_1, \dots, a_g, b_1, \dots, b_g)$:

$$(h^*)^* = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = h_* \quad , \quad B = (b_{ij})$$

Using the fact that $\beta_i \cup \beta_i = 0$:

$$\begin{aligned}
\langle Q_1, Q_2 \rangle_{R^*} &= \langle \beta_1 \cup \dots \cup \beta_g \cup \sum_{\sigma_1} b_{1\sigma_1} \alpha_{\sigma_1} \cup \dots \cup \sum_{\sigma_g} b_{g\sigma_g} \alpha_{\sigma_g}, w \rangle \\
&= \sum_{\sigma} b_{1\sigma_1} \dots b_{g\sigma_g} \langle \beta_1 \cup \dots \cup \beta_g \cup \alpha_{\sigma_1} \cup \dots \cup \alpha_{\sigma_g}, w \rangle \\
\langle Q_1, Q_2 \rangle_{R^*} &= (-1)^{2g} \sum_g b_{1\sigma_1} \dots b_{g\sigma_g} \langle \alpha_{\sigma_1} \cup \dots \cup \alpha_{\sigma_g} \cup \beta_1 \cup \dots \cup \beta_g, w \rangle \\
&= (-1)^g \sum_g b_{1\sigma_1} \dots b_{g\sigma_g} (-1)^{\sigma} \langle \alpha_1 \cup \dots \cup \alpha_g \cup \beta_1 \cup \dots \cup \beta_g, w \rangle
\end{aligned}$$

But, as established in section III.3(d), $\alpha_1 \cup \dots \cup \alpha_g \cup \beta_1 \cup \dots \cup \beta_g$ is "dual" to w :

$$\langle Q_1, Q_2 \rangle_{R^*} = (-1)^{k \det(B)} .$$

Hence, by comparison with section II.1(c):

$$\beta_1(M) \neq 0 \text{ iff } \langle Q_1, Q_2 \rangle_{R^*} = 0 .$$

For the same reason:

$$|\langle Q_1, Q_2 \rangle_{R^*}| = |H_1(M^3, \mathbb{Z})| \text{ iff } \beta_1(M^3) = 0 .$$

We turn now to the proof of part (c). Using the same models as above, we may make the identifications:

$$(6) \quad H_1(F^*, \mathbb{Z}) = \langle c_1, \dots, c_g, d_1, \dots, d_g \rangle \cong \mathbb{Z}^{2g}$$

$$(7) \quad T_1(R^*) = H_1(F^*, \mathbb{Z}) \otimes S$$

$$(8) \quad T_1(Q_1) = \langle c_1, \dots, c_g \rangle \otimes S$$

$$(9) \quad T_1(Q_2) = D_1 R(h_*) (T_1(Q_1))$$

Transversality of Q_1 and Q_2 at 1 is equivalent to the equation:

$$T_1(Q_1) + T_1(Q_2) = T_1(R^*) .$$

(10) But $T_1(Q_1) + T_1(Q_2)$ is the image of the map:

$$T_1(Q_1) \oplus T_1(Q_1) \xrightarrow{i \oplus D_1(R(h_*))} T_1(R^*) .$$

(11) The direct product here is "abstract" and we identify the second factor as:

$$T_1(Q_1) = \langle a'_1, \dots, a'_g \rangle \otimes S .$$

(12) By Proposition I.3.3, the map is given by the matrix:

$$i \oplus D_1(R(h_*)) = \begin{bmatrix} I & A^t \\ 0 & B^t \end{bmatrix} \otimes \text{id} .$$

(13) By the above remarks $Q_1 \neq Q_2$ at 1 iff $i \oplus D_1(R(h_*))$ is non-singular. Clearly, this is equivalent to the nonsingularity of B . The result follows as before. $\square$

Note: Apparently, the fact that h_* is symplectic is important here.

2. The boundary map ∂ and induced orientations

(a) The boundary map ∂

Consider:

$$(\partial F^*, 0) \xrightarrow{i} (F^*, 0) .$$

This induces a map of representation spaces, the restriction map:

$$R(\pi_1(F^*, 0)) \xrightarrow{R(i_*)} R(\pi_1(\partial F^*, 0)) .$$

We denote this map as the boundary map ∂ :

$$\partial : R^* \longrightarrow R(\pi_1(\partial F^*, 0)) .$$

We have the following properties of ∂ ([Sh], [I]):

Proposition 2.1: (1) ∂ is surjective

(2) {critical pts of ∂ } = {reducible reps}

Proof of (1):

As in section 1(a):

$$R^* = (S^3)^{2g} = \{(A_1, B_1, \dots, A_g, B_g) \mid A_i, B_i \in S^3\}$$

and $R(\Pi_1(\partial F^*, 0))$ is identified with S^3 .

This latter identification may be chosen so that the map ∂ may be written as:

$$(1) \quad \begin{aligned} \partial : (S^3)^{2g} &\longrightarrow S^3 \\ (A_1, B_1, \dots, A_g, B_g) &\longmapsto [A_1, B_1] \cdots [A_g, B_g] \end{aligned}$$

(2) Let:

$$R_\theta = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

(3) Then:

$$\partial(R_\theta \sigma, I, \dots, I) = R_{2\theta}.$$

(4) If $A \in S^3$, then $A = CR_\theta C^{-1}$ for some $\theta, C \in S^3$.

(5) Hence:

$$A = \partial(CR_\theta C^{-1}, C\sigma C^{-1}, I, \dots, I).$$

Proof of (2):

Let:

$$\rho = (A_1, B_1, \dots, A_g, B_g) \in R^*$$

$$\partial\rho = [A_1, B_1] \cdots [A_g, B_g] \in R(\Pi_1(\partial F^*, 0)).$$

(6) Of course:

$$T_\rho(R^*) = T_{A_1}(S^3) \oplus T_{B_1}(S^3) \oplus \cdots \oplus T_{A_g}(S^3) \oplus T_{B_g}(S^3)$$

$$T_{\partial\rho}(R(\Pi_1(\partial F^*, 0))) = T_{\partial\rho}(S^3).$$

(7) By computing derivatives as before and applying appropriate left and right translations to each factor of the given splitting of $T_\rho(R^*)$ and to $T_{\partial\rho}(S^3)$, we can construct a commutative diagram:

$$\begin{array}{ccc} T_\rho(R^*) = T_{A_1}(S^3) \oplus \cdots \oplus T_{B_g}(S^3) & \xrightarrow{D_\rho(\partial)} & T_{\partial\rho}(S^3) \\ \cong \downarrow & & \downarrow \cong \\ T_1(R^*) = T_1(S^3) \oplus \cdots \oplus T_1(S^3) & \xrightarrow{F} & T_1(S^3) \end{array}$$

where:

$$\begin{aligned} F(X_1, Y_1, \dots, X_g, Y_g) &= \sum_{j=1}^g (X_j - \partial_{j-1} A_j B_j A_j^{-1} \partial_{j-1}^{-1} X_j \partial_{j-1} A_j B_j A_j^{-1} \partial_{j-1}^{-1}) \\ &\quad + \sum_{j=1}^g (Y_j - \partial_{j-1} A_j B_j A_j^{-1} \partial_{j-1}^{-1} Y_j \partial_{j-1} A_j B_j A_j^{-1} \partial_{j-1}^{-1}). \end{aligned}$$

(8) Since the following elements of $\Pi_1(F^*, 0)$ are a free basis:

$$\{\partial_{j-1} a_j b_j a_j^{-1} b_j^{-1} \partial_{j-1}^{-1}, \partial_{j-1} a_j b_j a_j^{-1} \partial_{j-1}^{-1}\}_{j=1}^g$$

we see that Proposition 2.1(2) is equivalent to the following claim.

Claim: Let $(C_1, D_1, \dots, C_g, D_g) \in (S^3)^{2g} = R^*$. Let:

$$F : T_1(S^3) \oplus \cdots \oplus T_1(S^3) \longrightarrow T_1(S^3)$$

where:

$$F(X_1, Y_1, \dots, X_g, Y_g) = \sum_{j=1}^g (X_j - D_j X_j D_j^{-1}) + \sum_{j=1}^g (Y_j - C_j Y_j C_j^{-1})$$

Then F is surjective if and only if $\langle C_1, \dots, D_g \rangle$ is irreducible.

(9) Hence, we begin by computing the image of the map:

$$L: T_I(S^1) \longrightarrow T_I(S^1) \\ X \longmapsto X - DXD^{-1}$$

(10) Suppose:

$$D = R_\theta, \quad X = \begin{bmatrix} it & -\bar{u} \\ u & -it \end{bmatrix}$$

Then:

$$L(X) = \begin{bmatrix} 0 & (e^{2i\theta} - 1)\bar{u} \\ (1 - e^{-2i\theta})u & 0 \end{bmatrix}$$

Hence:

$$\text{Image}(L) = \{0\} \quad \text{if } D = \pm I \\ = S_0 = \left\{ \begin{bmatrix} 0 & -\bar{w} \\ w & 0 \end{bmatrix} \mid w \in \mathbb{C} \right\} \quad \text{if } D \neq \pm I$$

(11) In general, $D = ER_\theta E^{-1}$ and:

$$\text{Image}(L) = \{0\} \quad D = \pm I \\ = ES_0 E^{-1} \quad D \neq \pm I$$

(12) Hence, let $C_j = E_{2j-1} R_{\theta_{2j-1}} E_{2j-1}^{-1}$, $D_j = E_{2j} R_{\theta_{2j}} E_{2j}^{-1}$, so:

$$\text{Image}(F) = \sum \{ E_j S_0 E_j^{-1} \mid E_j R_{\theta_j} E_j^{-1} \neq \pm I \}$$

(13) Since $\dim(S_0) = 2$, we conclude that:

F is surjective if and only if there exists j, k with $1 \leq j < k \leq 2g$ such that:

$$(a) \quad E_j R_{\theta_j} E_j^{-1} \neq \pm I$$

$$(b) \quad E_k R_{\theta_k} E_k^{-1} \neq \pm I$$

$$\text{and } (c) \quad E_j S_0 E_j^{-1} \neq E_k S_0 E_k^{-1}$$

(14) Let $F_j = E_j R_{\theta_j} E_j^{-1}$, $F_k = E_k R_{\theta_k} E_k^{-1}$, $F_j \neq \pm I \neq F_k$.

Suppose $\langle F_j, F_k \rangle$ is reducible. Then there exists $E \in S^1$ such that:

$$EF_j E^{-1} = R_{\theta_j}$$

$$EF_k E^{-1} = R_{\theta_k}$$

So:

$$E^{-1} R_{\theta_j} E = R_{\theta_j}$$

$$E^{-1} R_{\theta_k} E = R_{\theta_k}$$

So:

$$E^{-1} R_{\theta_j} E \in \text{Normalizer}(\langle R_{\theta_j} \rangle) = \text{Normalizer}(S^1)$$

$$E^{-1} R_{\theta_k} E \in \text{Normalizer}(\langle R_{\theta_k} \rangle) = \text{Normalizer}(S^1)$$

So:

$$E_j^{-1} E_k \in \text{Normalizer}(S^1)$$

Claim: $\langle F_j, F_k \rangle$ is reducible iff $E_j^{-1} E_k \in \text{Norm}(S^1)$, (assuming $F_j \neq \pm I \neq F_k$).

Subproof: We have just shown the necessity. Now suppose $E = E_j^{-1} E_k \in S^1$.

Then:

$$E_j F_j E_j^{-1} = R_{\theta_j}$$

$$E_j^{-1} F_k E_j = E_j^{-1} E_k R_{\theta_k} E_k^{-1} E_j = R_{\theta_k}.$$

In other words, $\langle F_j, F_k \rangle$ is diagonalizable. $\square$

(15) Finally, it is a straightforward exercise to verify that:

$$E S_a E^{-1} = S_a \text{ iff } E \in \text{Normalizer}(S^1).$$

The result follows immediately from (13) and (15). $\square$

(b) The singular set S

Let:

$$S = \{ \rho \mid \rho : \pi_1(F^*, 0) \longrightarrow S^3 \text{ reducible} \}.$$

By Proposition 1.2.1, every reducible representation is conjugate to a diagonal representation. Hence, every reducible representation satisfies the generating relation for $\pi_1(F, 0)$ and we conclude that:

$$S \subset R \quad S = S(\pi_1(F, 0)) = S(\pi_1(F^*, 0)).$$

As an immediate corollary to Proposition 2.1 (and the definition of R and ∂) we conclude:

Corollary 2.2: (a) $R = \partial^{-1}(1)$

(b) $R \setminus S$ is an open smooth manifold of dimension $6g - 3$.

Appealing to Proposition 1.2.1 and the naturality of the SO_3 action, we deduce the existence of the following free actions:

$$SO_3 \times (R^* \setminus S) \longrightarrow (R^* \setminus S) \text{ freely}$$

$$SO_3 \times (R \setminus S) \longrightarrow (R \setminus S) \text{ freely}$$

$$SO_3 \times (Q_i \setminus S) \longrightarrow (Q_i \setminus S) \text{ freely}$$

$$SO_3 \times (Q_j \setminus S) \longrightarrow (Q_j \setminus S) \text{ freely}.$$

Let:

$$\hat{R} = (R \setminus S) / SO_3$$

$$\hat{Q}_i = (Q_i \setminus S) / SO_3$$

$$\hat{Q}_j = (Q_j \setminus S) / SO_3.$$

From the above observation, we derive:

Proposition 2.3:

(a) $\hat{R}$ is a smooth open manifold of dimension $6g - 6$

(b) $\hat{Q}_i$ is a smooth open manifold of dimension $3g - 3$ $i = 1, 2$.

(c) Induced orientations

Consider the following choices of orientation:

W_i - induced orientation from M^3 $i = 1, 2$

F - induced orientation from W_i

F^* - induced orientation from F

∂F^* - induced orientation from F^* .

The orientation on ∂F^* determines a unique generator of $\pi_1(\partial F^*, 0)$. We adopt the following convention:

R_j - induced orientation from the basis of $\Pi_1(\partial F^*, 0)$
corresponding to the orientation on ∂F^* .

Recall that a short exact sequence of oriented vector spaces:

$$0 \longrightarrow U \xrightarrow{\alpha} V \xrightarrow{\beta} W \longrightarrow 0$$

$[(u_1, \dots, u_p)] = \text{orientation on } U$

$[(v_1, \dots, v_q, v_{q+1}, \dots, v_{q+p})] = \text{orientation of } V$

$[(w_1, \dots, w_q)] = \text{orientation on } W$

is said to be compatibly oriented if we can assume:

$$v_{q+j} = \alpha(u_j) \quad 1 \leq j \leq p$$

$$w_j = \beta(v_j) \quad 1 \leq j \leq q$$

Given a short exact sequence of vector spaces with two of U, V, W oriented, there is a unique orientation on the third making the sequence compatibly oriented. We shall refer to this orientation as the induced orientation.

Next, we orient $R \setminus S$ by the following convention. Consider the sequence:

$$R \setminus S \longrightarrow R^* \xrightarrow{\partial} R_j$$

By Corollary 2.2 and Proposition 2.1, we conclude that if $\rho \in R \setminus S$, then:

$$0 \longrightarrow T_\rho(R \setminus S) \longrightarrow T_\rho(R^*) \xrightarrow{D_\rho \partial} T_1(R_j) \longrightarrow 0$$

is a short exact sequence.

Now, given an arbitrary orientation of R^* , we orient these vector spaces as follows:

$T_1(R_j)$ - induced orientation from orientation on R_j

$T_\rho(R^*)$ - induced orientation from R^*

$T_\rho(R \setminus S)$ - induced orientation from $T_\rho(R^*)$ and $T_1(R_j)$.

It is easy to see that this gives a well-defined continuous family of orientations on the tangent spaces of $R \setminus S$ and, hence, a well-defined orientation on $R \setminus S$.

In a similar manner, we may orient $\hat{R}$. More precisely, we consider the sequence:

$$SO_3 \cdot \rho \longrightarrow R \setminus S \xrightarrow{\Pi} \hat{R} \quad \rho \in R \setminus S$$

and the associated short exact sequence:

$$0 \longrightarrow T_\rho(SO_3 \cdot \rho) \longrightarrow T_\rho(R \setminus S) \xrightarrow{D_\rho \Pi} T_\rho(\hat{R}) \longrightarrow 0$$

Since SO_3 acts freely on $R^* \setminus S$, we have a natural homeomorphism:

$$SO_3 \longrightarrow SO_3 \cdot \rho$$

$$A \longmapsto A \cdot \rho$$

which induces an orientation on $SO_3 \cdot \rho$. (Recall that S^3 has a fixed orientation. SO_3 is oriented as the quotient S^3/S^1 .)

We orient the vector spaces above as follows:

$T_\rho(SO_3 \cdot \rho)$ - induced orientation from $SO_3 \cdot \rho$

$T_\rho(R \setminus S)$ - induced orientation from $R \setminus S$

$T_\rho(\hat{R})$ - induced orientation from $T_\rho(SO_3 \cdot \rho)$ and $T_\rho(R \setminus S)$.

again, it is easy to see that we obtain in this manner a well defined orientation on $\hat{R}$.

Finally, in the same manner, given an arbitrary orientation on Q_i , we obtain an orientation on $\hat{Q}_i$:

$T_{\rho}(SO_3 \cdot \rho)$ - induced orientation from $SO_3 \cdot \rho$

$T_{\rho}(Q_i \setminus S)$ - induced orientation from Q_i

$T_{\rho}(\hat{Q}_i)$ - induced orientation from $T_{\rho}(SO_3 \cdot \rho)$, $T_{\rho}(Q_i \setminus S)$.

Note: Under our conventions, the only fixed orientations are those of R_2 and the orbits of SO_3 . $R \setminus S$ and R^* , $R \setminus S$ and $\hat{R}$, $Q_i \setminus S$ and $\hat{Q}_i$ need only be compatibly oriented with respect to these. The motivation for these conventions will be seen shortly.

CHAPTER IV: CASSON'S INVARIANT FOR ORIENTED HOMOLOGY 3-SPHERES

1. Casson's invariant for Heegard decompositions

(a) The intersection products

Let M^3 be an homology 3-sphere:

$$H_0(M^3, \mathbb{Z}) \cong H_3(M^3, \mathbb{Z}) \cong \mathbb{Z}$$

$$H_1(M^3, \mathbb{Z}) \cong H_2(M^3, \mathbb{Z}) = \{0\}.$$

Let (W_1, W_2) be a Heegard decomposition of M^3 . As an immediate consequence of Proposition III.1.1, we obtain:

Proposition 1.1:

$$(a) \langle Q_1, Q_2 \rangle_{R^*} = \pm 1$$

$$(b) Q_1 \uparrow Q_2 \text{ at } 1$$

By Proposition 1.2.1, every reducible representation is conjugate to a diagonal representation. Hence, every reducible representation of $\Pi_1(M^3, 0)$ factors through $H_1(M^3, \mathbb{Z})$. We deduce the following corollary:

Corollary 1.2:

$$(a) S(\Pi_1(M^3, 0)) = R(\Pi_1(M^3, 0)) \cap S = Q_1 \cap Q_2 \cap S = \{1\}.$$

$$(b) (Q_1 \setminus S) \cap (Q_2 \setminus S) \text{ is compact.}$$

$$(c) \hat{Q}_1 \text{ and } \hat{Q}_2 \text{ are properly embedded open submanifolds of } \hat{R}.$$

$$(c) \hat{Q}_1 \cap \hat{Q}_2 \text{ is compact.}$$

Note: If the genus of the Heegard decomposition is one, then $\Pi_1(F, 0)$ is abelian. It follows from Proposition 1.2.3 that:

$$R = R(\Pi_1(F, 0)) \subset S \quad \text{if} \quad \text{genus} = 1$$

$$\hat{R} = \hat{Q}_1 = \hat{Q}_2 = \emptyset \quad \text{if} \quad \text{genus} = 1.$$

Choose an isotopy of $\hat{R}$ with compact support which carries $\hat{Q}_2$ to $\tilde{Q}_2$ where $\tilde{Q}_2$ is transverse to $\hat{Q}_1$. Then, by Proposition III.2.3:

- $(\tilde{Q}_2)$ is isotopic to $\hat{Q}_2$,
- $\hat{Q}_1 \pitchfork (\tilde{Q}_2)$
- $\hat{Q}_1 \cap \tilde{Q}_2 = \text{finite set of points.}$

Given orientations on $\hat{Q}_1$ and $\hat{Q}_2$, we can define the algebraic intersection of $\hat{Q}_1$ and $\hat{Q}_2$:

$$\begin{aligned} \langle \hat{Q}_1, \hat{Q}_2 \rangle_{\hat{R}} &= \sum (\text{sign}(\hat{\rho}; \hat{Q}_1, \tilde{Q}_2) | \hat{\rho} \in \hat{Q}_1 \cap \tilde{Q}_2) \\ [(u_1, \dots, u_{2g-3})] &= \text{ind. or. of } T_{\hat{\rho}} \hat{Q}_1 \\ [(v_1, \dots, v_{2g-3})] &= \text{ind. or. of } T_{\hat{\rho}} \tilde{Q}_2 \\ [(w_1, \dots, w_{2g-4})] &= \text{ind. or. of } T_{\hat{\rho}} \hat{R} \\ [(w_1, \dots, w_{2g-4})] &= \text{sign}(\hat{\rho}; \hat{Q}_1, \tilde{Q}_2) [(u_1, \dots, u_{2g-3}, v_1, \dots, v_{2g-3})]. \end{aligned}$$

By standard arguments, this is well-defined.

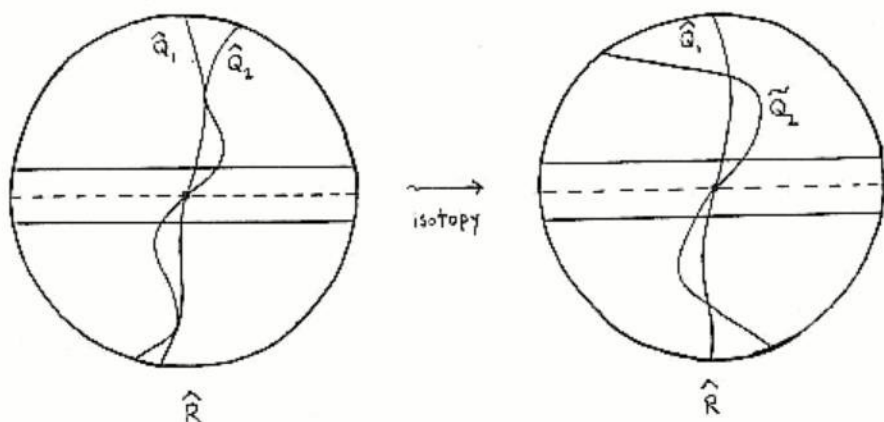


Figure 13

(b) Cusson's invariant for Heegard decompositions

If M^3 is an oriented homology 3-sphere with an Heegard decomposition (W_1, W_2) , we define:

$$\lambda(M^3; W_1, W_2) = \frac{(-1)^{\langle \hat{Q}_1, \hat{Q}_2 \rangle_{\hat{R}}} \langle \hat{Q}_1, \hat{Q}_2 \rangle_{\hat{R}}}{2 \langle Q_1, Q_2 \rangle_{R^*}} \cdot S$$

where all the orientations are chosen by the conventions of section III.2(c), and S is either -1 or 1 . S will be determined in section V.5.

Notes: 1) $\lambda(M^3; W_1, W_2)$ is well-defined by the compatibility conventions of section III.2(c). For instance, if we change the orientation of R^* , we must change that of $R \setminus S$ and, hence, that of $\hat{R}$. Hence, by Proposition 1.1, we can always choose the orientations subject to the convention:

$$\langle Q_1, Q_2 \rangle_{R^*} = 1.$$

2) If $g = 1$, we let $\lambda(M^3; W_1, W_2) = 0$.

If we change the orientation of M^3 , this changes the orientation of ∂F^* . This changes the generator α of $\pi_1(\partial F^*)$ to α^{-1} . The induced orientations on R_2 come, respectively, from the identifications:

$$\begin{array}{ccc} R_2 & \xrightarrow{\alpha} & S^3 \\ \rho & \longrightarrow & \rho(\alpha) \end{array} \quad \begin{array}{ccc} R_2 & \xrightarrow{\alpha^{-1}} & S^3 \\ \rho & \longrightarrow & \rho(\alpha^{-1}) = (\rho(\alpha))^{-1}. \end{array}$$

Since inversion changes the orientation of S^3 , we see that the induced orientation of R_2 must change. We may maintain the given orientation on R^* by 1). In order to maintain compatibility, the orientation on $R \setminus S$ and, hence, on $\hat{R}$ must change. (SO_3 has a fixed orientation). Therefore:

$$\lambda(-M^3; -W_1, -W_2) = -\lambda(M^3; W_1, W_2).$$

c) Independence of Heegard decomposition

We wish to show that $\lambda(M^3; W_1, W_2)$ is independent of the Heegard decomposition (W_1, W_2) . It is evident that this invariant is an invariant of ambient isotopy of Heegard decompositions. Hence, from Singer's Theorem, we conclude that it suffices to show that it is also invariant under stabilization, i.e., under addition of an unknotted one-handle).

If (W, h) is an Heegard model, let:

$$\lambda(W, h) = \lambda(\{(W, h)\}; W \times \{1\}, W \times \{2\}).$$

Proposition 1.3: Given an Heegard model (W, h) , let:

$$W' = W \# T \quad h' = h \# \tau.$$

Then:

$$\lambda(W', h') = \lambda(W, h).$$

Proof: Choose a basis $\{a_1, \dots, a_g, b_1, \dots, b_g\}$ for $H_1(F^*, 0)$ as in section II.1(a). Let $a_{g+1} = a$, $b_{g+1} = b$ as in section II.2(d):

$$H_1(F^*, 0) = \langle a_1, \dots, a_g, b_1, \dots, b_g \rangle$$

$$H_1(F', 0) = \langle a'_1, \dots, a'_g, b'_1, \dots, b'_g, a'_{g+1}, b'_{g+1} \rangle$$

$$H_1(F, 0) = \langle a_1, \dots, a_g, b_1, \dots, b_g \mid \prod_{j=1}^g [a_j, b_j] = 1 \rangle$$

$$H_1(F', 0) = \langle a'_1, \dots, a'_g, b'_1, \dots, b'_g, a'_{g+1}, b'_{g+1} \mid \prod_{j=1}^{g+1} [a'_j, b'_j] = 1 \rangle$$

$$H_1(Q_1, 0) = \langle a_1, \dots, a_g \rangle$$

$$H_1(Q'_1, 0) = \langle a'_1, \dots, a'_g, a'_{g+1} \rangle.$$

We may make the following identifications:

$$(R')^* = (S^3)^{2g+2} = \{(A_1, \dots, A_g, B_1, \dots, B_g, A_{g+1}, B_{g+1})\}$$

$$R^* = (S^3)^{2g} = \{(A_1, \dots, A_g, B_1, \dots, B_g)\}$$

$$R' = \{(A_1, \dots, A_g, B_1, \dots, B_g, A_{g+1}, B_{g+1}) \mid \prod_{j=1}^{g+1} [A_j, B_j] = I\}$$

$$R = \{(A_1, \dots, A_g, B_1, \dots, B_g) \mid \prod_{j=1}^g [A_j, B_j] = I\}$$

$$Q'_1 = \{(A_1, \dots, A_g, I, \dots, I, A_{g+1}, I)\}$$

$$Q_1 = \{(A_1, \dots, A_g, I, \dots, I)\}$$

$$Q'_2 = (h')^* (Q'_1)$$

$$Q_2 = h^* (Q_1).$$

Then:

$$1.) \quad \lambda(W', h') = \frac{(-1)^{g+1} \langle \hat{Q}'_1, \hat{Q}'_2 \rangle_{\hat{R}'}}{2 \langle Q'_1, Q'_2 \rangle_{(R')^*}} \cdot S$$

$$\lambda(W, h) = \frac{(-1)^g \langle \hat{Q}_1, \hat{Q}_2 \rangle_{\hat{R}}}{2 \langle Q_1, Q_2 \rangle_{R^*}} \cdot S$$

We wish to compare the various terms.

(i) Comparison of denominators

From the above identifications, we observe that:

$$(R')^* = R^* \times S^3 \times S^3$$

$$Q'_1 = Q_1 \times S^3 \times \{I\}$$

$$Q'_2 = Q_2 \times \{I\} \times S^3$$

This last identity follows from the definition of τ given in section II.2(d).
Consider the Kunneth formulas:

$$H_*(S^3) \otimes \dots \otimes H_*(S^3) \otimes H_*(S^3) \otimes \dots \otimes H_*(S^3) \xrightarrow{u} H_*(R^*)$$

$$(x_1 \otimes \dots \otimes x_g) \otimes (y_1 \otimes \dots \otimes y_g) \mapsto (x_1 \otimes \dots \otimes x_g) \times (y_1 \otimes \dots \otimes y_g)$$

$$H_*(R^*) \otimes H_*(S^3) \otimes H_*(S^3) \xrightarrow{u} H_*((R')^*)$$

$$u \otimes (x \otimes y) \longrightarrow u \times (x \times y)$$

$$H^*(S^3) \otimes \dots \otimes H^*(S^3) \otimes H^*(S^3) \otimes \dots \otimes H^*(S^3) \xrightarrow{u} H^*(R^*)$$

$$H^*(R^*) \otimes H^*(S^3) \otimes H^*(S^3) \longrightarrow H^*((R')^*).$$

As in the proof of Proposition III.1.1, we consider the natural 3-cocycles in H^* and $(R')^*$:

$$\alpha_i = \tau^*(c_i) \quad \beta_i = \tau^*(d_i) \quad i = 1, \dots, g$$

$$\alpha'_i = \tau^*(c'_i) \quad \beta'_i = \tau^*(d'_i) \quad i = 1, \dots, g+1.$$

then:

$$\alpha'_i = \alpha_i \times 1 \quad \beta'_i = \beta_i \times 1 \quad 1 \leq i \leq g$$

$$\alpha'_{g+1} = 1 \times \alpha \quad \beta'_{g+1} = 1 \times \beta.$$

Let τ and μ be the top-class and coclass of S^3 as in section I.3(b).

then:

$$\beta_i = (1 \otimes \dots \otimes 1) \times (\mu_1 \otimes \dots \otimes \mu_{i-1} \otimes \mu \otimes 1 \otimes \dots \otimes 1) \quad \mu_j = 1$$

$$\alpha = \mu \times 1 \quad \beta = 1 \times \mu.$$

Our conventions on orientations allow us to choose the following orientations:

$$R^* : \quad w = (z \times \dots \times z) \times (z \times \dots \times z)$$

$$(R')^* : \quad w' = w \times (z \times z)$$

$$Q_1 : \quad w_1 = (z \times \dots \times z) \times (1 \times \dots \times 1)$$

$$Q'_1 : \quad w'_1 = w_1 \times (z \times 1)$$

$$Q_2 : \quad w_2 = (h^*)_*(w_1)$$

$$Q'_2 : \quad w'_2 = w_2 \times (1 \times z).$$

Let:

$$P_1 = (1 \times \dots \times 1) \times (z \times \dots \times z) \quad P_2 = (h^*)_*(P_1)$$

$$P'_1 = P_1 \times (1 \times z) \quad P'_2 = P_2 \times (z \times 1).$$

From the above observations, we compute that:

$$\langle Q_1, P_1 \rangle_{R^*} = 1 = \langle \beta_1 \cup \dots \cup \beta_g, P_1 \rangle$$

$$\langle Q'_1, P'_1 \rangle_{(R')^*} = (-1)^g = (-1)^g \langle \beta'_1 \cup \dots \cup \beta'_g \cup \beta'_{g+1}, P'_1 \rangle$$

$$\langle Q_2, P_2 \rangle_{R^*} = 1 = \langle h^*(\beta_1 \cup \dots \cup \beta_g), P_2 \rangle$$

$$\langle Q'_2, P'_2 \rangle_{(R')^*} = (-1)^{g+1} = (-1)^{g+1} \langle (h')^*(\beta'_1 \cup \dots \cup \beta'_{g+1}), P'_2 \rangle.$$

As in the proof of Proposition III.1.1, we conclude:

$$w_1 = D(\beta_1 \cup \dots \cup \beta_g)$$

$$w'_1 = (-1)^g D(\beta'_1 \cup \dots \cup \beta'_g \cup \beta'_{g+1})$$

$$w_2 = D((h^*)^*(\beta_1 \cup \dots \cup \beta_g))$$

$$w'_2 = (-1)^{g+1} D(((h')^*)^*(\beta'_1 \cup \dots \cup \beta'_g \cup \beta'_{g+1})).$$

Let:

$$\delta = \beta_1 \cup \dots \cup \beta_g.$$

Then:

$$\begin{aligned} \langle Q_1', Q_2' \rangle_{(R')}^* &= (-1)^{2g+1} \langle (\delta \times \beta) \cup (h^*(\delta) \times \alpha), w' \rangle \\ &= (-1)^{2g+1} \langle (\delta \cup h^*(\delta)) \times (\beta \cup \alpha), w \times z \times z \rangle. \end{aligned}$$

Hence:

$$\begin{aligned} \langle Q_1', Q_2' \rangle_{(R')}^* &= (-1)^{2g+2} \langle \delta \cup h^*(\delta), w \rangle \cdot \langle \alpha \cup \beta, z \times z \rangle \\ &= (-1)^K \langle \delta \cup h^*(\delta), w \rangle. \end{aligned}$$

$$2.) \quad \langle Q_1', Q_2' \rangle_{(R')}^* = (-1)^K \langle Q_1, Q_2 \rangle_{R^*}$$

(ii) Comparison of numerators

We wish to carry out a similar comparison of numerators. Although $\hat{R}$ is not a factor of $\hat{R}'$, we shall see that $\hat{R}$ is a submanifold of $\hat{R}'$ and, hence, a "local factor". This will be sufficient, since the following considerations imply that all the intersections occurring in the numerators will take place in a regular neighborhood of $\hat{R}$.

By an argument similar to section III.2(b):

$$\begin{aligned} R \times \{I\} \times \{I\} &\subset R \times S^3 \times \{I\} \subset R' \\ R \times \{I\} \times \{I\} &\subset R \times \{I\} \times S^3 \subset R' \\ S \times \{I\} \times \{I\} &= S' \cap (R \times \{I\} \times \{I\}) \subset S'. \end{aligned}$$

By an obvious abuse of notation:

$$\begin{aligned} R \setminus S &\subset R' \setminus S' & Q_i \setminus S &\subset Q_i' \setminus S' \\ Q_1 \cap Q_2 &= Q_1' \cap Q_2' & & \subset R. \end{aligned}$$

Since SO_3 acts by conjugation on each component, we obtain a well-defined commutative diagram:

$$\begin{array}{ccc} R \setminus S & \xrightarrow{\quad} & R' \setminus S' \\ \downarrow & & \downarrow \\ \hat{R} & \xrightarrow{\quad} & \hat{R}' \end{array}$$

from which we conclude that:

$$\begin{aligned} \hat{R} &= \hat{R}' & \hat{Q}_1 &= \hat{Q}_1' \\ \hat{Q}_1 \cap \hat{Q}_2 &= \hat{Q}_1' \cap \hat{Q}_2' & & \subset \hat{R}. \end{aligned}$$

In order to compute the numerators, we need to isotope $\hat{Q}_2$ and $\hat{Q}_2'$ to be transverse to $\hat{Q}_1$ and $\hat{Q}_1'$ and then count intersections. We would like to construct a single isotopy of $\hat{R}'$ to accomplish this. Furthermore, we want this isotopy to be as simple as possible so that we can compare intersections and orientations. To see that this is possible, we investigate the regular neighborhood of $\hat{R}$ in $\hat{R}'$.

$R \setminus S$ is a properly embedded submanifold of $R' \setminus S'$

In order to see this, consider the map:

$$\begin{aligned} f: (R')^* &\xrightarrow{\quad} S^3 \times S^3 \times S^3 \\ \rho' &\xrightarrow{\quad} (A', A, B). \end{aligned}$$

where $\rho' = (\rho, A, B)$.

Then:

$$R = f^{-1}(I, I, I)$$

$$R' = f^{-1}(\{I\} \times S^3 \times S^3).$$

Clearly:

$$D_{\rho}(f) = \begin{bmatrix} D_{\rho}(j) \cdot [A, B] & * & * \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

It follows from Proposition III.2.1 that $R \setminus S \subset R' \setminus S'$ is a differentiable embedding. Moreover, from the observations above, this embedding is proper. Hence, we conclude that:

$R \setminus S \subset R' \setminus S'$ is a proper embedding

$Q_1 \setminus S \subset Q'_1 \setminus S'$ is a proper embedding.

In addition, we have the following commutative diagram of proper embeddings:

$$\begin{array}{ccc} R \setminus S & \begin{array}{c} \nearrow \\ \searrow \end{array} & (R \setminus S) \times S^3 \times \{I\} \\ & & \nearrow \searrow \\ & & R' \setminus S' \\ & \begin{array}{c} \nearrow \\ \searrow \end{array} & (R \setminus S) \times \{I\} \times S^3 \end{array}$$

(The latter inclusions follow from the definition of R' and S' .)

$\hat{R}$ is a properly embedded submanifold of $\hat{R}'$

Since all the proper embeddings above are invariant under the SO_3 action, we conclude immediately that:

$\hat{R} \subset \hat{R}'$ is a proper embedding

$\hat{Q}_1 \subset \hat{Q}'_1$ is a proper embedding

and that we have a commutative diagram of proper embeddings:

$$\begin{array}{ccc} & \hat{R}_1 & \\ \hat{R} & \begin{array}{c} \nearrow \\ \searrow \end{array} & \hat{R}' \\ & \hat{R}_2 & \end{array}$$

where:

$$\hat{R}_1 = \wedge ((R \setminus S) \times S^3 \times \{I\})$$

$$\hat{R}_2 = \wedge ((R \setminus S) \times \{I\} \times S^3).$$

In addition, we observe that:

$$\hat{Q}'_1 \subset \hat{R}_1 \quad \hat{R} = \hat{R}_1 \cap \hat{R}_2.$$

Identification of oriented tangent spaces

From the map f , we conclude that if $\rho \in R$, then:

$$T_{\rho}(R' \setminus S') = T_{\rho}(R \setminus S) \oplus S \oplus S.$$

Note: Considering the given trivialization of $T((R')^*)$ as $T((S^3)^2 \otimes \mathbb{R}^2)$ and the associated Riemannian metric, this splitting can be considered as a natural orthogonal splitting.

Recall that our conventions on orientations stipulate that the following exact sequences of oriented vector spaces be compatible:

$$\begin{array}{ccccccc} 1 & \longrightarrow & T_{\rho}(R \setminus S) & \longrightarrow & T_{\rho}(R^*) & \longrightarrow & T_{\partial(\rho)}(R_{\partial}) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & T_{\rho}(R' \setminus S') & \longrightarrow & T_{\rho}((R')^*) & \longrightarrow & T_{\partial(\rho)}(R_{\partial}') \longrightarrow 1. \end{array}$$

With respect to the given basis, this diagram may be identified with:

$$\begin{array}{ccccccc} 1 & \longrightarrow & T_{\rho}(R \setminus S) & \longrightarrow & T_{\rho}((S^3)^{2g}) & \longrightarrow & T_c(S^3) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \text{id} \\ 1 & \longrightarrow & T_{\rho}(R' \setminus S') & \longrightarrow & T_{\rho}((S^3)^{2g+2}) & \longrightarrow & T_c(S^3) \longrightarrow 1 \end{array}$$

$$\text{where } c = \partial(\rho) = \partial'(\rho).$$

By the choice of orientations on R^* and $(R')^*$ given in part (i), we have the following equality of oriented vector spaces:

$$T_{\rho}((R')^*) = T_{\rho}(R^*) \oplus S \oplus S \quad (\text{oriented}).$$

From the definition of compatibility, we deduce:

$$T_{\rho}(R' \setminus S') = T_{\rho}(R \setminus S) \oplus S \oplus S \quad (\text{oriented}).$$

Using the same metric, we may split the tangent space to $R \setminus S$:

$$T_{\rho}(R \setminus S) = T_{\rho}(SO_3 \cdot \rho) \oplus \nu_{\rho}(SO_3 \cdot \rho)$$

$$\nu_{\rho}(SO_3 \cdot \rho) = (T_{\rho}(SO_3 \cdot \rho))^{\perp} \cap T_{\rho}(R \setminus S).$$

Considering, the short exact sequences:

$$\begin{array}{ccccccc} 1 & \longrightarrow & T_{\rho}(SO_3 \cdot \rho) & \longrightarrow & T_{\rho}(R \setminus S) & \longrightarrow & T_{\rho}^*(\hat{R}) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & T_{\rho}(SO_3 \cdot \rho) & \longrightarrow & T_{\rho}(R' \setminus S') & \longrightarrow & T_{\rho}^*(\hat{R}') \longrightarrow 1 \end{array}$$

we may make the following identifications:

$$T_{\rho}^*(\hat{R}) = \nu_{\rho}(SO_3 \cdot \rho)$$

$$T_{\rho}^*(\hat{R}') = \nu_{\rho}(SO_3 \cdot \rho) \oplus S \oplus S$$

$$T_{\rho}^*(\hat{R}') = T_{\rho}^*(\hat{R}) \oplus S \oplus S.$$

Recall again that our conventions stipulate that the above sequences be compatibly oriented. That is, considering the above identifications:

$$T_{\rho}(R \setminus S) = T_{\rho}^*(\hat{R}) \oplus T_{\rho}(SO_3 \cdot \rho) \quad (\text{oriented})$$

$$T_{\rho}(R' \setminus S') = T_{\rho}^*(\hat{R}') \oplus T_{\rho}(SO_3 \cdot \rho) \quad (\text{oriented}).$$

From the previous observations, if we write:

$$T_{\rho}^*(\hat{R}') = \varepsilon(T_{\rho}^*(\hat{R}) \oplus S \oplus S) \quad (\text{oriented})$$

then we obtain the equation:

$$\begin{aligned} T_{\rho}^*(\hat{R}) \oplus T_{\rho}(SO_3 \cdot \rho) \oplus S \oplus S & \quad (\text{oriented}) \\ = \varepsilon(T_{\rho}^*(\hat{R}) \oplus S \oplus S \oplus T_{\rho}(SO_3 \cdot \rho)). \end{aligned}$$

Counting dimensions we see that $\varepsilon = 1$:

$$3.) \quad T_{\rho}^*(\hat{R}') = T_{\rho}^*(\hat{R}) \oplus S \oplus S \quad (\text{oriented}).$$

By the choice of orientations on Q_i and Q_i' given in part (i), we have the equations:

$$T_{h^{\#}(\rho)}(Q_1 \setminus S) = (h^{\#})_*(T_{\rho}(Q_1 \setminus S)) \quad (\text{oriented})$$

$$T_{\rho}(Q_1' \setminus S') = T_{\rho}(Q_1 \setminus S) \oplus S \oplus \{0\} \quad (\text{oriented})$$

$$T_{\rho}(Q_2' \setminus S') = T_{\rho}(Q_2 \setminus S) \oplus \{0\} \oplus S \quad (\text{oriented}).$$

Our conventions stipulate the compatibility of:

$$\begin{array}{ccccccc} 1 & \longrightarrow & T_{\rho}(SO_3 \cdot \rho) & \longrightarrow & T_{\rho}(Q_1 \setminus S) & \longrightarrow & T_{\rho}^*(\hat{Q}_1) \longrightarrow 1 \\ & & \downarrow (h^{\#})_* & & \downarrow (h^{\#})_* & & \downarrow (\hat{h})_* \\ 1 & \longrightarrow & T_{\delta}(SO_3 \cdot \delta) & \longrightarrow & T_{\delta}(Q_2 \setminus S) & \longrightarrow & T_{\delta}^*(\hat{Q}_2) \longrightarrow 1 \end{array}$$

where $\delta = h^{\#}(\rho)$.

We observe that the first column is orientation preserving:

$$h^{\#} : SO_3 \cdot \rho \longrightarrow SO_3 \cdot \delta$$

$$c\rho c^{-1} \longrightarrow c\delta c^{-1}.$$

Having already observed the same for the second column, we conclude that:

$$T_{\delta}(\hat{Q}_2) = (\hat{h})_{\#}(T_{\rho}(\hat{Q}_1)) \quad (\text{oriented}).$$

Likewise, we must have compatibility of:

$$\begin{array}{ccccccc} 1 & \longrightarrow & T_{\rho}(SO_3 \cdot \rho) & \longrightarrow & T_{\rho}(Q_1 \setminus S) & \longrightarrow & T_{\rho}(\hat{Q}_1) \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & T_{\rho}(SO_3 \cdot \rho) & \longrightarrow & T_{\rho}(Q'_1 \setminus S') & \longrightarrow & T_{\rho}(\hat{Q}'_1) \longrightarrow 1. \end{array}$$

Following the argument for equation 3., write:

$$T_{\rho}(\hat{Q}'_1) = \varepsilon(T_{\rho}(\hat{Q}_1) \oplus S \oplus \{0\}) \quad (\text{oriented})$$

to obtain the equation:

$$\begin{aligned} \varepsilon(T_{\rho}(\hat{Q}_1) \oplus S \oplus \{0\}) \oplus T_{\rho}(SO_3 \cdot \rho) & \quad (\text{oriented}) \\ = T_{\rho}(\hat{Q}_1) \oplus T_{\rho}(SO_3 \cdot \rho) \oplus S \oplus \{0\}. \end{aligned}$$

Counting dimensions we see that $\varepsilon = -1$:

$$4.) \quad T_{\rho}(\hat{Q}'_1) = -T_{\rho}(\hat{Q}_1) \oplus S \oplus \{0\} \quad (\text{oriented}).$$

The same argument establishes the equation:

$$\begin{aligned} 5.) \quad T_{\rho}(\hat{Q}'_2) &= -T_{\rho}(\hat{Q}_2) \oplus \{0\} \oplus S \quad (\text{oriented}) \\ T_{\delta}(\hat{Q}'_2) &= (\hat{h}')_{\#}(T_{\rho}(\hat{Q}'_1)). \end{aligned}$$

Construction of the isotopy

Choose a compactly supported ambient isotopy of $\hat{R}$ which carries $\hat{Q}_2$ to $\tilde{Q}_2$, where $\tilde{Q}_2$ is transverse to $\hat{Q}_1$. By the isotopy extension theorem [Hi], extend this isotopy H to an ambient isotopy of $\hat{R}_2$. We assume that this isotopy respects the orthogonal splitting of the normal bundle of $\hat{R}$:

$$\begin{aligned} (H_t)_{\#} : T_{\rho}(R \setminus S) \oplus \{0\} \oplus S & \longrightarrow T_{\sigma}(R \setminus S) \oplus \{0\} \oplus S \\ (x, 0, y) & \longrightarrow (\alpha(x), 0, \beta(y)). \end{aligned}$$

Note: The existence of such an isotopy follows readily from the proof of the isotopy extension theorem. Alternatively, one can construct the isotopy equivariantly in $R' \setminus S'$ using the product structure on the normal bundle of $R \setminus S$.

Finally, extend to an isotopy of $\hat{R}'$ H . Let $\tilde{Q}'_2 = H_1(\hat{Q}'_2)$. The next observations follow immediately from the construction of H :

$$\begin{aligned} \tilde{Q}_2 &= \tilde{Q}'_2 \cap \hat{R} & \hat{Q}_1 \cap \tilde{Q}_2 &= \hat{Q}'_1 \cap \tilde{Q}'_2 \subset \hat{R} \\ T_{\rho}(\hat{Q}'_1) &= T_{\rho}(\hat{Q}_1) \oplus S \oplus \{0\} & \hat{\rho} &\in \hat{R} \\ T_{\rho}(\tilde{Q}'_2) &= T_{\rho}(\tilde{Q}_2) \oplus \{0\} \oplus S & \hat{\rho} &\in \hat{R}. \end{aligned}$$

Since $\hat{Q}_1$ and $\tilde{Q}_2$ are transverse, we conclude that:

$$\hat{Q}'_1 \quad \text{and} \quad \tilde{Q}'_2 \quad \text{are transverse.}$$

Moreover, by the previous results and the construction of the isotopy:

$$T_{\hat{\rho}}(\hat{Q}_1') = -T_{\hat{\rho}}(\hat{Q}_1) \oplus S_1 \oplus \{0\} \quad (\text{oriented})$$

$$T_{\hat{\rho}}(\hat{Q}_2') = -T_{\hat{\rho}}(\hat{Q}_2) \oplus \{0\} \oplus S_2 \quad (\text{oriented}).$$

Let:

$$s = \text{sign}(\hat{\rho}; \hat{Q}_1, \hat{Q}_2)$$

$$s' = \text{sign}(\hat{\rho}; \hat{Q}_1', \hat{Q}_2').$$

Then we have the equations:

$$T_{\hat{\rho}}(\hat{R}') = T_{\hat{\rho}}(\hat{R}) \oplus S_1 \oplus S_2 \quad (\text{oriented})$$

$$T_{\hat{\rho}}(\hat{R}) = s(T_{\hat{\rho}}(\hat{Q}_1) \oplus T_{\hat{\rho}}(\hat{Q}_2)) \quad (\text{oriented})$$

$$T_{\hat{\rho}}(\hat{R}') = s'(T_{\hat{\rho}}(\hat{Q}_1') \oplus T_{\hat{\rho}}(\hat{Q}_2')) \quad (\text{oriented}).$$

From this, we derive the equations:

$$\begin{aligned} s(T_{\hat{\rho}}(\hat{Q}_1) \oplus T_{\hat{\rho}}(\hat{Q}_2)) \oplus S_1 \oplus S_2 & \quad (\text{oriented}) \\ = s'((-T_{\hat{\rho}}(\hat{Q}_1) \oplus S_1) \oplus (-T_{\hat{\rho}}(\hat{Q}_2) \oplus S_2)). \end{aligned}$$

Counting dimensions, we conclude that:

$$s = s'(-1)^{3(g-1)}.$$

Since this holds for all $\hat{\rho} \in \hat{Q}_1 \cap \hat{Q}_2$, we conclude from the previous observations that:

$$6.) \quad \langle \hat{Q}_1', \hat{Q}_2' \rangle_{\hat{R}'} = (-1)^{g-1} \langle \hat{Q}_1, \hat{Q}_2 \rangle_{\hat{R}}.$$

From equations 1., 2. and 6., we conclude:

$$\lambda(W', h') = \lambda(W, h).$$

2. Casson's invariant for oriented homology 3-spheres

(a) Casson's invariant for oriented homology 3-spheres

If M^3 is an oriented homology 3-sphere, we define:

$$\lambda(M^3) = \lambda(M^3; W_1, W_2)$$

where (W_1, W_2) is an Heegard dec. of M^3 .

From the results of section 1(c), this is well-defined.

If M^3 and N^3 are oriented homology 3-spheres and $h: M^3 \xrightarrow{\alpha} N^3$ is an orientation preserving homeomorphism, then h carries any given Heegard decomposition of M^3 (W_1, W_2) to an Heegard decomposition of N^3 $(h(W_1), h(W_2))$. It is evident from our conventions that:

$$\lambda(M^3; W_1, W_2) = \lambda(N^3; h(W_1), h(W_2)).$$

Hence, we obtain a topological invariant:

Proposition 2.1: Casson's invariant for oriented homology 3-spheres λ is an homeomorphism invariant.

The obvious question at this point is whether Casson's invariant is computable. It is well-known that every homology 3-sphere M^3 is obtained from S^3 by a sequence of ± 1 surgeries on knots in homology 3-spheres. Hence, one approach to this question is to study the change in Casson's invariant when we perform ± 1 surgery on a knot in M^3 . It turns out that there is a very simple formula for this change (as we shall see). In the next chapter we begin the study of this "differential".

1. Dehn surgery on knots in homology spheres

(a) Dehn surgery on knots in homology spheres

Let M^3 be an homology 3-sphere.

Let K be a knot in M^3 .

Let $N(K)$ be a closed regular neighborhood of K .

Let T^2 be the boundary of $N(K)$, so that T^2 is a torus.

Let $M^3(K)$ be $M^3 \setminus \text{interior}(N(K))$.

Now:

$$H_1(M^3(K)) \cong H_1(N(K)) \cong \mathbb{Z}$$

$$H_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Consider the inclusions:

$$i : T^2 \longrightarrow M^3(K)$$

$$j : T^2 \longrightarrow N(K)$$

and the associated surjections:

$$i_* : H_1(T^2) \longrightarrow H_1(M^3(K))$$

$$j_* : H_1(T^2) \longrightarrow H_1(N(K)).$$

Let μ be a generator of the kernel of j_* . Let λ be a generator of the kernel of i_* . Since μ and λ are primitive elements, we may represent them by nontrivial simple closed curves on T^2 which we also denote as μ and λ . Since μ and λ form a basis for $H_1(T^2)$, we may assume that

they intersect transversally in precisely one point. Each of these curves is well defined up to isotopy and change in orientation.

Let M^3 be oriented. The orientation on M^3 restricts to an orientation on $M^3(K)$. This orientation on $M^3(K)$ induces a well defined orientation on T^2 by the convention that "the orientation on $M^3(K)$ is the orientation on T^2 plus the inward pointing normal vector to T^2 ". (We assume that all orientations are chosen subject to this convention.) We choose the orientations on μ and λ so that "the orientation on T^2 is equal to the orientation on μ plus the orientation on λ ". Subject to these assumptions, the pair of curves (μ, λ) is well defined up to ambient isotopy of T^2 and simultaneous change of orientation of μ and λ . The pair is called the standard meridian-longitude pair for K .

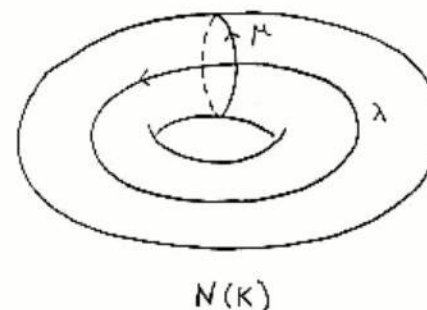


Figure 14

Any simple closed curve γ on T^2 is uniquely determined by its homology class (which is well defined up to a change in sign):

$$\gamma = p\mu + q\lambda \quad \langle p, q \rangle = 1.$$

Moreover, the relative primeness of p and q ensure that $p\mu + q\lambda$ is

represented by a simple closed curve. Hence, for each fraction p/q we obtain a well defined (up to isotopy) homotopically nontrivial simple closed curve on T^2 , and vice-versa. (We allow that $(p,q) = (\pm 1, 0)$ or $p/q = \infty$.)

$M^3(K, p/q)$ is p/q surgery on (M^3, K) , (or just p/q surgery on K).

This is defined as follows:

$$M^3(K, p/q) = M^3(K) \cup_{\sigma} (S^1 \times D^2)$$

$$\begin{aligned} \sigma : S^1 \times \partial D^2 &\xrightarrow{\cong} T^2 \\ (1) \times \partial D^2 &\longrightarrow \gamma \end{aligned}$$

($M^3(K, p/q)$ does not depend on the choice of σ .)

Now $H_1(M^3(K, p/q)) = \mathbb{Z}_p$ (assuming $p \neq 0$). Hence $M^3(K, p/q)$ is an homology 3-sphere precisely when $p = 1$.

$$K_n = M^3(K, 1/n) \text{ is } 1/n \text{ Dehn surgery on } K.$$

Note: $K_0 = M^3(K, \infty) = M^3$.

(b) Preferred Heegard decompositions for K_n

In this section, we construct Heegard decompositions which are compatible with surgery. These will be useful in comparing the Casson invariants for K_n and K_{n+1} .

Lemma 1.1: There exists an Heegard decomposition for $M^3 \setminus (W_1, W_2)$ such that K is a separating curve on ∂W_1 .

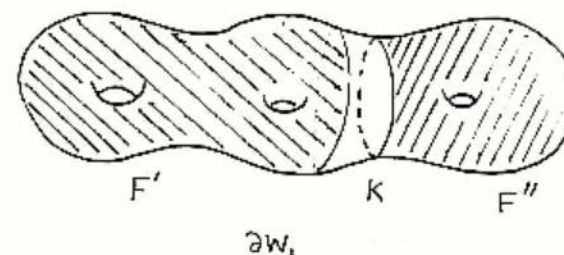


Figure 15

Proof: Choose a Seifert surface F^* for K . Thicken F^* to $F^* \times I \subset M^3$. (Here, $I = [0, 1]$ and F^* is identified with $F^* \times 1$.)

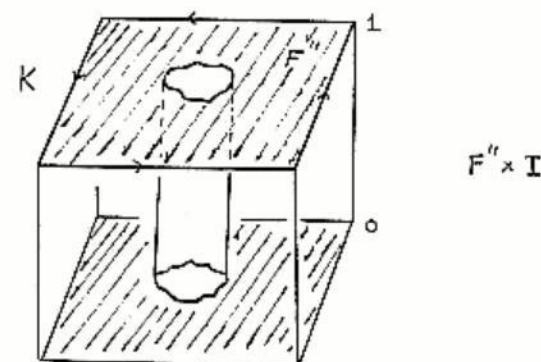


Figure 16

Attach 1-handles (from $M^3 \setminus (F^* \times I)$ to $F^* \times \{0\}$ to obtain a new handlebody W_1 such that $\overline{M^3 \setminus W_1}$ is an handlebody W_2 . $\square$



If we so desire, we may assume that $F' \equiv F''$:

Lemma 1.2: There exists an Heegaard decomposition for $M^3(W_1, W_2)$ such that

$$W_1 = F' \times I \quad K = \partial F' \times \{1/2\}.$$

roof: Repeat the proof of Lemma 1.1 to construct an Heegard decomposition (V_1, W_1') as in Figure 17. Then drill from $F' \times \{1\}$ to $F' \times \{0\}$ and through the cores of the 1-handles which were attached to $F' \times \{0\}$, effectively adding 1-handles to W_1' to obtain a new handlebody W_1 . Finally, observe that $\overline{M^3 \setminus W_1}$ is homeomorphic to $F^1 \times I$ and, therefore, is a handlebody W_1 .



By an ambient isotopy, we may assume $K = \partial F \times \{1/2\}$. $\square$

Now choose an Heegard decomposition of M^3 as in Lemma 1.1 (or Lemma 1.2):

$$\partial W_1 = F' \cup (\partial F'' \times I) \cup F''.$$

Let:

$$\Lambda = \partial F' \times I, \quad \lambda = \partial F' \cup \{0\}.$$

By an ambient isotopy, we assume that $K = \partial F^* \times \{1/2\}$.



note: K is homologically trivial in ∂W_1 . It is not, however, homotopically trivial in W_1 , (unless F^* is a disc, and, hence, K is a trivial knot).

Let W denote W_1 and identify the given Heegard decomposition of M^3 with an Heegard model (W, h) . Let τ denote the Dehn twist about K which we take to be supported on A .

lemma 1.3: $(W, h \circ \tau^n)$ is an Heegard model for $M^3(K, 1/n)$.

proof: Let B be $h(A)$ in ∂W . Then $M^3(K)$ is homeomorphic to the quotient space:

$$\frac{(W \times \{1\}) \cup (-W \times \{2\})}{h'}$$

where:

$$h' = h|_{\partial W \setminus A} : \partial W \setminus A \longrightarrow \partial W \setminus B.$$

Identifying $M^3(K)$ with this quotient, we observe that:

$$T^2 = A \cup B$$

$$\lambda = \partial F \times \{0\} = \text{longitude}.$$

Let α be an arc in A from $\partial F \times \{0\}$ to $\partial F \times \{1\}$. Let $\beta = h(\alpha)$.

Likewise, we observe that:

$$\mu = \alpha \cup \beta = \text{meridian}.$$

Since τ is supported on A we may consider τ as a Dehn twist on T^2 .

Under our conventions on orientations, we have:

$$\tau^{-n}(\mu) = \mu + n\lambda = \tau^{-n}(\alpha) \cup \beta.$$

(Recall, in general, $\tau_c(d) = d + (c, d)_{FC}$. Since $(\lambda, \mu)_F = 1$, we obtain a change of sign.)

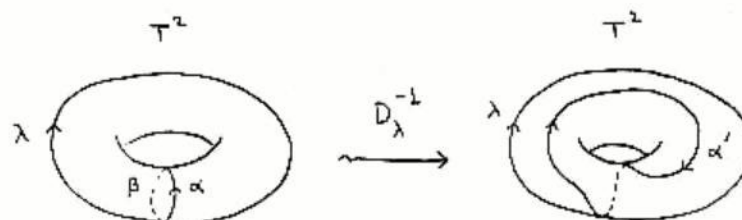


Figure 20

Let $\alpha' = \tau^{-n}(\alpha)$, so that:

$$\beta = h \circ \tau^n(\alpha')$$

$$\mu' = \alpha' \cup \beta = \mu + n\lambda.$$

In order to obtain $M^3(K, 1/n)$ we wish to attach a disc to μ' and cap off. From the preceding observation and the same considerations which allow us to identify $M^3(K)$ as a "prequotient" of M^3 , we conclude, as desired, that:

$$M^3(K, 1/n) \cong |(W, h \circ \tau^n)|$$

□

2. Casson's invariant for framed knots

(a) Casson's invariant for framed knots

A framed knot is a pair (K, n) where K is a knot and n is an integer. If K is a knot in an homology 3-sphere, as above, we define:

$$\lambda'(K, n) = \lambda(K_{n+1}) - \lambda(K_n).$$

2) The canonical isotopy

If we choose Heegard models for the Dehn surgeries on K as in the previous section, we compute, as in section IV.1(c), that:

$$\lambda(K_n) = \frac{(-1)^{2g} \langle \hat{Q}_1, (h\tau^n) \hat{Q}_1 \rangle_{\hat{R}}}{2 \langle Q_1, (h\tau^n)^\# Q_1 \rangle_{R^*}} \cdot S$$

where g is genus(F) and $2g$ is genus(∂W). That is:

$$\lambda(K_n) = \frac{\langle \hat{Q}_1, \tau^n \hat{Q}_2 \rangle_{\hat{R}}}{2 \langle Q_1, (\tau^n)^\# Q_2 \rangle_{R^*}} \cdot S$$

where Q_2 is $h_1^\#(Q)$.

Since K is a separating curve, τ acts trivially on the homology of ∂W . Following the proof of Proposition III.1.1, part (a), we conclude that:

$$\langle Q_1, (\tau^n)^\# Q_2 \rangle_{R^*} = \langle Q_1, Q_2 \rangle_{R^*}.$$

We assume, as in section IV.1(b), that $\langle Q_1, Q_2 \rangle_{R^*}$ is 1. With this assumption, we have:

$$\lambda(K_n) = \left[\langle \hat{Q}_1, \tau^n \hat{Q}_2 \rangle_{\hat{R}} / 2 \right]^* \cdot S.$$

Following thru our conventions on orientations, we see that $\tau(\hat{R})$ is $\hat{R}$ (as an oriented manifold) and, hence, that:

$$\begin{aligned} \lambda(K_{n+1}) - \lambda(K_n) &= \frac{1}{2} [\langle \hat{Q}_1, \tau^{n+1} \hat{Q}_2 \rangle_{\hat{R}} - \langle \hat{Q}_1, \tau^n \hat{Q}_2 \rangle_{\hat{R}}] \cdot S \\ &= \frac{1}{2} [\langle \hat{Q}_1, \tau^{n+1} \hat{Q}_2 \rangle_{\hat{R}} - \langle \tau \hat{Q}_1, \tau^{n+1} \hat{Q}_2 \rangle_{\hat{R}}] \cdot S \end{aligned}$$

$$\lambda(K_{n+1}) - \lambda(K_n) = -\frac{1}{2} \langle \tau \hat{Q}_1 - \hat{Q}_1, \tau^{n+1} \hat{Q}_2 \rangle_{\hat{R}} \cdot S$$

$$\lambda(K_{n+1}) - \lambda(K_n) = -\frac{1}{2} \langle \tau \hat{Q}_1 - \hat{Q}_1, h\tau^{n+1} \hat{Q}_1 \rangle_{\hat{R}} \cdot S.$$

We wish to study the difference $\tau \hat{Q}_1 - \hat{Q}_1$.

Note: It should be emphasized here that $\hat{Q}_1$ is not a cycle in the classical sense, since it is not compact.

We may identify R in the following manner:

$$(R')^* = R(\pi_1(F')) \cong (S^3)^{2g_1} \quad g_1 = \text{genus}(F')$$

$$(R'')^* = R(\pi_1(F'')) \cong (S^3)^{2g_2} \quad g_2 = \text{genus}(F'')$$

$$R = R(\pi_1(\partial W_1)) \subset (R')^* \times (R'')^*$$

$$R = \{(\rho', \rho'') \mid \partial \rho' = \partial \rho''\}$$

$$R = \{(\rho', \rho'') \mid \rho'(\lambda) = \rho''(\lambda)\}.$$

In these coordinates, $\tau^\#$ has a simple form:

$$\begin{aligned} \tau^\# : R &\xrightarrow{\quad E \quad} R \\ (\rho', \rho'') &\longmapsto (\rho', \partial \rho' \cdot \rho''). \end{aligned}$$

Note: We choose the base point for ∂W in F' . As above, $u \cdot \rho = u \rho u^{-1}$.

$$R'_- = \{\rho' \in (R')^* \mid \partial \rho' = -1\}$$

$$R''_- = \{\rho'' \in (R'')^* \mid \partial \rho'' = -1\}.$$

Let $R_-(K)$ be $R'_- \ast R''_-$ in R . Since reducible representations are abelian and λ is homologous to zero, we readily deduce that:

$$R'_-(K) \cap S = \emptyset$$

$$\tau^\#|_{R'_-(K)} = \text{identity} \quad \tau^\#|_S = \text{identity}.$$

Let:

$$P_n = (h\tau^{n+1})^\# Q_1.$$

Since K_{n+1} is an homology sphere, we know that:

$$Q_1 \neq P_n \text{ at } 1$$

$$Q_1 \cap P_n \cap S = \{1\}.$$

From the observation concerning $\tau^\#$, we conclude:

$$\tau^\#(Q_1) \cap S = Q_1 \cap S$$

$$\tau^\#(Q_1) \cap R_-(K) = Q_1 \cap R_-(K).$$

In the complement of $R_-(K)$, $\tau^\#$ is given by conjugation of $(R')^*$ by an element of $S^3 \setminus \{-1\}$. Hence, we can employ the natural contraction of $S^3 \setminus \{-1\}$ to $\{1\}$ which arises from the exponential map, (see sections I.1(b) and (c)), to construct an isotopy in the complement of $R_-(K)$ which carries Q_1 onto $\tau^\#(Q_1)$:

$$H : (R \setminus R_-(K)) \times I \longrightarrow R \setminus R_-(K)$$

$$((\rho', \rho''), t) \longmapsto (\rho', (\partial \rho')^t \cdot \rho'').$$

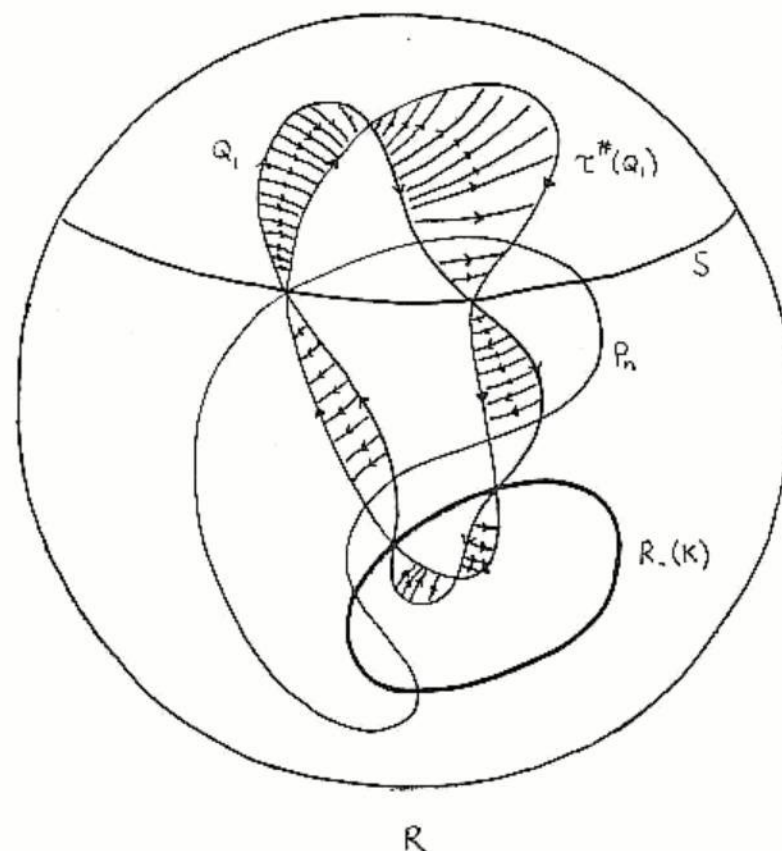


Figure 21

Note: In Figure 21, the shaded region indicates the trace of Q_1 (away from $R_-(K)$) under the isotopy H . It should be observed that H does not extend to $R_-(K)$.

Clearly $H|_{S \times I}$ is identity.

We recall from the proof of Proposition 1.3.3 that the tangent space at 1 to τ^* is spanned by curves of cyclic representations (i.e. the Zariski tangent space at 1 to S is $T_1(R^*)$). Moreover, for any $t \in I$:

$$T_{(1,t)}((R \setminus R_-(K)) \times I) = T_1(R \setminus R_-(K)) \oplus T_t(I).$$

note: In a small neighborhood of 1, the isotopy extends in an obvious way to an open interval containing I . It is with this consideration that we speak of $T_0(I)$ and $T_1(I)$.

From the previous observations we conclude that:

$$D_{(1,t)}(H) = \Pi_1 : T_{(1,t)}((R \setminus R_-(K)) \times I) \longrightarrow T_1(R \setminus R_-(K))$$

where Π_1 is projection onto the first factor. It follows by standard techniques that there exists an open neighborhood of 1 in R , U_n , such that:

$$H((Q_1 \cap U_n) \times I) \cap P_n = \{1\}.$$

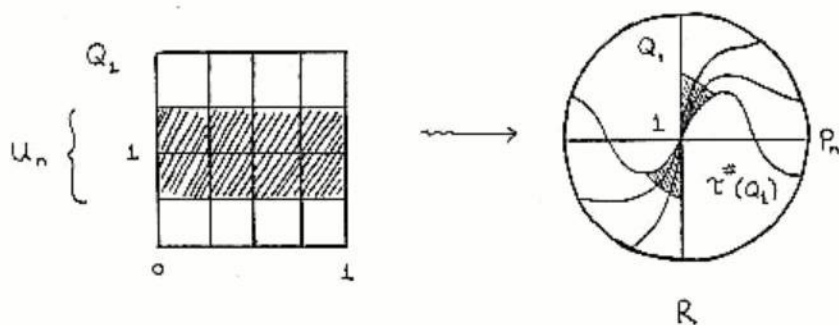


Figure 22

We are now able to prove an analogous result for the trace of Q_1 near S :

Lemma 2.1: There exists a neighborhood of $Q_1 \cap S$ in Q_1 , V_n , such that:

$$H(V_n \times I) \cap P_n = \{1\}.$$

Proof: From the previous observations, it follows that:

$$((Q_1 \cap S) \setminus U) \cap ((P_n \cap S) \setminus U) = \emptyset$$

$$(Q_1 \cap S) \setminus U \text{ and } (P_n \cap S) \setminus U \text{ are compact.}$$

- Moreover, H_t fixes both sets for all t .

- Hence, we may choose a neighborhood of $(Q_1 \cap S) \setminus U_n$ in Q_1 , W_n , such that:

$$H(W_n \times I) \cap P_n = \emptyset.$$

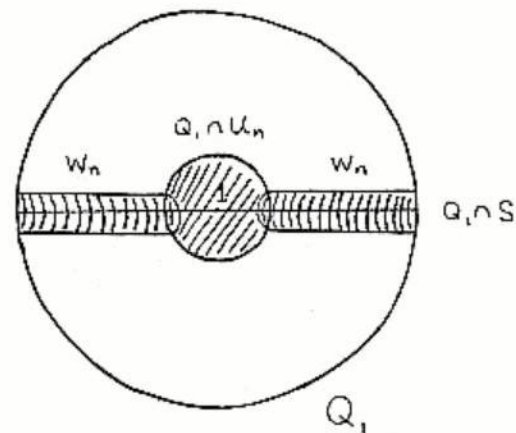


Figure 23

- Finally, let V_n be $(Q_1 \cap U_n) \cup W_n$. $\square$

Note: From the conclusion, we see that we may choose V_n to be SO_3 invariant. Moreover, we can assume that $Q_1 \setminus V_n$ is a manifold with boundary. Clearly, $\hat{H}$ is SO_3 -equivariant. Hence, we have a canonical isotopy $\hat{H}$ defined on $\hat{R}$ in the complement of $\hat{R}_-(K)$ and carrying $\hat{Q}_1$ onto $\hat{\tau}(\hat{Q}_1)$. Moreover:

$$\hat{H}(\hat{Q}_1 \times I) \cap \hat{P}_n \subset \hat{H}((\hat{Q}_1 \setminus \hat{V}_n) \times I) .$$

Note: Recall that $\hat{R} = (R \setminus S)/SO_3$. All quotient spaces, therefore, are contained in this space, (by definition). We will, however, for convenience, speak of $\hat{S}$ and consider that:

$$\begin{aligned} \hat{R} &\subset R/SO_3 & \hat{S} &\subset R/SO_3 \\ \hat{R} \cap \hat{S} &= \emptyset & \hat{R} \cup \hat{S} &= R/SO_3 . \end{aligned}$$

(c) The difference cycle δ

Although $\hat{H}$ is not a compactly supported isotopy, the results of the previous section allow us to employ $\hat{H}$ to define a compact cycle δ which "carries the intersection of $\hat{\tau}(\hat{Q}_1) - \hat{Q}_1$ with $\hat{P}_n$ ". As we shall see, this difference cycle is independent of n (as a genuine cycle in $\hat{R}$) and its intersection with $\hat{P}_n$ is independent of n . This will establish Casson's knot invariant.

Let δ' be the compact cycle:

$$\delta' = \hat{\tau}(\hat{Q}_1 \setminus \hat{V}_n) - \hat{H}(\hat{Q}_1 \setminus \hat{V}_n \times I) - (\hat{Q}_1 \setminus \hat{V}_n) .$$

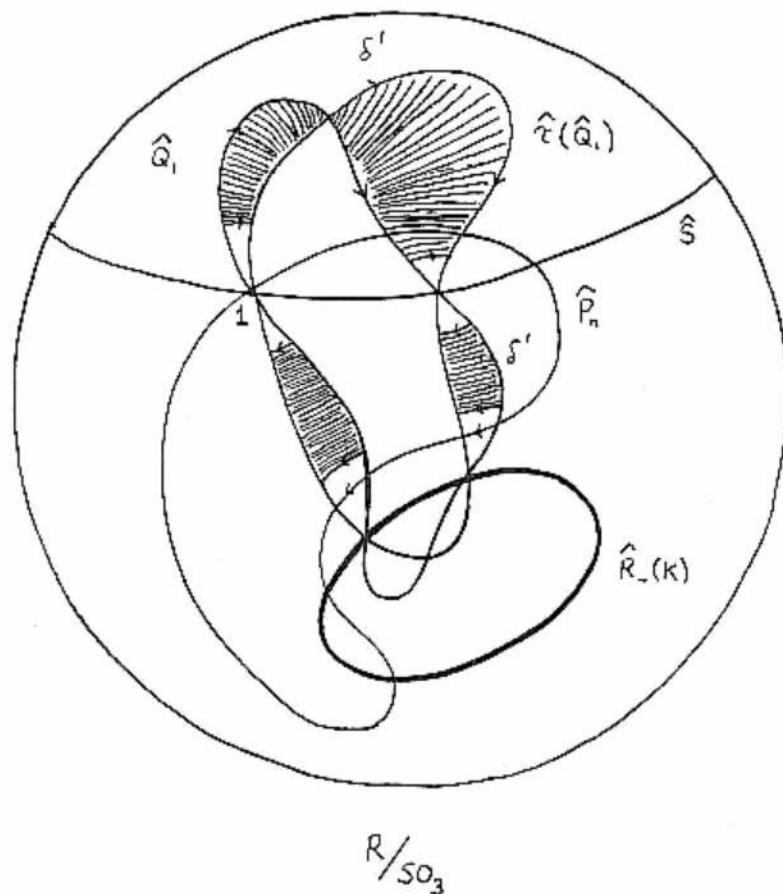


Figure 24

By the discussion in the previous section, we conclude that:

$$\langle \hat{\tau}(\hat{Q}_1), \hat{P}_n \rangle_{\hat{R}} - \langle \hat{Q}_1, \hat{P}_n \rangle_{\hat{R}} = \langle \delta', \hat{P}_n \rangle_{\hat{R}} .$$

Note: δ' is a compact cycle in $\hat{R}$.

We assume that $\hat{V}_n$ is disjoint from $\hat{R}_-(K)$. Let $\hat{N}$ be a compact manifold neighborhood of $\hat{Q}_1 \cap \hat{R}_-(K)$ in $\hat{Q}_1$ with $\hat{N} \cap \hat{V}_n = \emptyset$. Let:

$$\hat{M} = \hat{Q}_1 \setminus (\hat{V}_n \cup \text{interior}(\hat{N})).$$

We choose $\hat{N}$ so that $\hat{M}$ is a manifold. Let

$$\beta = \hat{\tau}(\hat{M}) - \hat{H}(\partial \hat{M} \times I) - \hat{N}.$$

Since β is a compact boundary in $\hat{R}$:

$$\langle \beta, \hat{P}_n \rangle_{\hat{R}} = 0$$

Finally, we define the difference cycle, δ :

$$\delta = \hat{\tau}(\hat{N}) - \hat{H}(\partial \hat{N} \times I) - \hat{N}.$$

Since $\delta = \delta' - \beta$, we conclude that:

$$\langle \hat{\tau}(\hat{Q}_1), \hat{P}_n \rangle_{\hat{R}} - \langle \hat{Q}_1, \hat{P}_n \rangle_{\hat{R}} = \langle \delta, \hat{P}_n \rangle_{\hat{R}}.$$

Clearly, δ is independent of the choice of $\hat{N}$, and independent of n .

1) Independence of framing

From the previous section, we conclude that:

$$\lambda(K_{n+1}) - \lambda(K_n) = -\frac{1}{2} \langle \delta, \hat{P}_n \rangle_{\hat{R}} \cdot S.$$

Employing the fact that $\hat{\tau}(\hat{R})$ is $\hat{R}$ (as an oriented manifold), we have:

$$\lambda(K_{n+1}) - \lambda(K_n) = -\frac{1}{2} \langle \hat{\tau}^{-(n+1)} \delta, \hat{Q}_2 \rangle_{\hat{R}} \cdot S.$$

It suffices to show, therefore, that $\hat{\tau}(\delta)$ is homologous to δ in $\hat{R}$. From the previous section, we see that we can assume that δ is carried by a regular neighborhood of $\hat{R}_-(K)$ and, hence, that δ is in the image of the natural map:

$$i_* : H_{2g}(\hat{R}_-(K)) \longrightarrow H_{2g}(\hat{R}).$$

We have already seen that $\hat{\tau}$ acts trivially on $\hat{R}_-(K)$; (indeed, $\tau^{\#}$ acts trivially on $R_-(K)$). This establishes the result. In summary, we have shown that:

$$\lambda'(K, n) = \lambda(K_{n+1}) - \lambda(K_n) = -\frac{1}{2} \frac{\langle \delta, \hat{Q}_2 \rangle_{\hat{R}}}{\langle \hat{Q}_1, \hat{Q}_2 \rangle_{R^*}} \cdot S$$

for any Heegaard decomposition as in Lemma 1.1. Hence, $\lambda'(K, n)$ is independent of n .

3. Casson's invariant for knots

(a) Casson's invariant for knots

Given a knot K in an homology 3-sphere, we define Casson's invariant by:

$$\lambda'(K) = \lambda(K_{n+1}) - \lambda(K_n) \quad n \in \mathbb{Z}.$$

By the previous section, this is a well defined topological invariant. Moreover:

$$\lambda'(K) = -\frac{1}{2} \frac{\langle \delta, \hat{Q}_2 \rangle_{\hat{R}}}{\langle \hat{Q}_1, \hat{Q}_2 \rangle_{R^*}} \cdot S.$$

Note: In order to compute this invariant, we investigate the difference cycle δ .

The difference cycle, δ

a) Identification of representation spaces

Using an Heegard model as in Lemmes 1.2 and 1.3, we may identify representation spaces as follows:

$$R'^* = R(\Pi_1(F')) \cong (S^1)^{2g_1} \quad g_1 = \text{genus}(F')$$

$$R^* = R(\Pi_1(W_1 \setminus \{0\})) = (R')^* \times (R')^*$$

$$R = R(\Pi_1(\partial W_1)) \subset R^*$$

$$R = \{(\rho', \rho'') \mid \partial \rho' = \partial \rho''\}$$

$$R = \{(\rho', \rho'') \mid \rho'(\lambda) = \rho''(\lambda)\}$$

$$Q_1 = \Delta \subset (R')^* \times (R')^* = R^* \quad (\Delta = \text{diagonal})$$

$$Q_1 = \{(\rho', \rho') \mid \rho' \in (R')^*\}$$

$$Q_2 = h^{\#}(Q_1)$$

$$S' = \text{reducible representations in } (R')^*$$

$$S = \text{reducible representations in } R^*$$

$$S = S' \times S'$$

$$R'_- = \{\rho' \mid \rho' \in (R')^*, \partial \rho' = -I\}$$

$$R_-(K) = R'_- \times R'_-$$

$$Q_1 \cap R_-(K) = \{(\rho', \rho') \mid \partial \rho' = -I\}$$

$$N_\varepsilon = \text{regular neighborhood of } Q_1 \cap R_-(K) \text{ in } Q_1$$

$$N_\varepsilon = \{(\rho', \rho') \mid \Pi - \varepsilon \leq \arg(\partial \rho') \leq \Pi\}$$

$$\tau : R^* \longrightarrow R^*$$

$$(\rho', \rho'') \longmapsto (\rho', \partial \rho' \cdot \rho'')$$

$$H : (R \setminus (R_-(K))) \times I \longrightarrow R \setminus (R_-(K))$$

$$((\rho', \rho''), t) \longmapsto (\rho', (\partial \rho')^t \cdot \rho'')$$

$$\delta_\varepsilon = \tau(\hat{N}_\varepsilon) - \hat{H}(\partial \hat{N}_\varepsilon \times I) - \hat{N}_\varepsilon$$

Note: We shall show that δ_ε converges to a cycle in $\hat{R}_-(K)$.

(b) An equivariant trivialization of N_ε

We may identify Q_1 with $(R')^*$ via the diagonal map:

$$d : (R')^* \longrightarrow R^*$$

$$\rho' \longmapsto (\rho', \rho')$$

$$d : (R')^* \xrightarrow{\cong} Q_1$$

With this identification, we may write:

$$Q_1 \cap R_-(K) \cong R'_-$$

$$N_\varepsilon \cong N_-(\varepsilon) = \{\rho' \mid \Pi - \varepsilon \leq \arg(\partial \rho') \leq \Pi\}$$

Let:

$$N_- = \{\rho' \mid \Pi/2 \leq \arg(\partial \rho') \leq \Pi\}$$

$$N_+ = \{\rho' \mid \text{trace}(\partial \rho') \neq 0\}$$

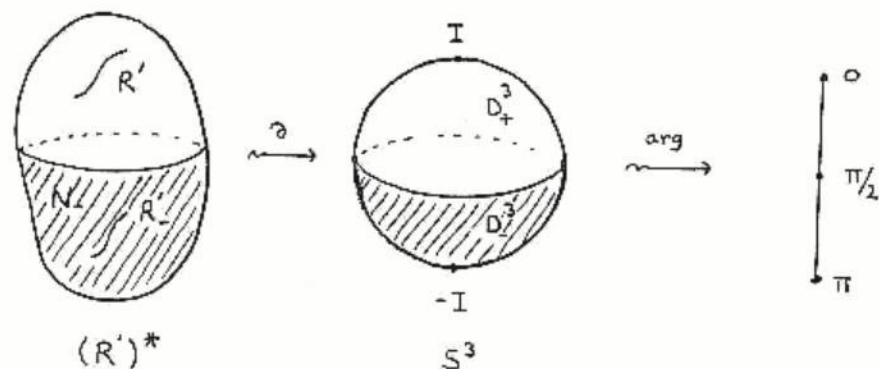


Figure 25

By the results of section III.1(b), we may choose a trivialization:

$$w : R'_+ \times D_-^3 \xrightarrow{\cong} N_-$$

$$w(\rho', A) = A \quad w(\rho', -I) = \rho'.$$

For $\varepsilon \leq \pi/2$, we have $N_-(\varepsilon) \subset N_-$ and the trivialization restricts to:

$$w : R'_+ \times D_-^3(\varepsilon) \xrightarrow{\cong} N_-(\varepsilon).$$

Now, for sufficiently small ε , the exponential map on $(R')^*$ provides a projection which is equivariant with respect to the SO_3 action:

$$\Pi : N_-(\varepsilon) \longrightarrow R'_+.$$

Using a projection of the normal bundle of R'_+ in $(R')^*$, we obtain a diffeomorphism:

$$v : N_-(\varepsilon) \xrightarrow{\cong} R'_+ \times D_-^3(\varepsilon)$$

$$\rho' \longmapsto (\Pi(\rho'), \partial\rho').$$

We may assume that (for sufficiently small ε):

$$vw = \text{identity}$$

w is an equivariant trivialization.

3) Collapsing δ to a cycle in $\hat{R}_-(K)$

As we have already pointed out in section 2(c), δ is carried by $\hat{R}_-(K)$ or more precisely by a regular neighborhood of $\hat{R}_-(K)$. In this section, we collapse δ into $\hat{R}_-(K)$ and obtain an alternative description of δ . Let:

$$D_+^3(\varepsilon) = \{A \in S^3 \mid 0 \leq \arg(A) \leq \varepsilon\}$$

$$S_+^2(\varepsilon) = \{A \in S^3 \mid \arg(A) = \varepsilon\}$$

$$A_\varepsilon = \{A \in S^3 \mid \varepsilon \leq \arg(A) \leq \pi - \varepsilon\}$$

$$S_-^2(\varepsilon) = \{A \in S^3 \mid \arg(A) = \pi - \varepsilon\}$$

$$D_-^3(\varepsilon) = \{A \in S^3 \mid \pi - \varepsilon \leq \arg(A) \leq \pi\}.$$

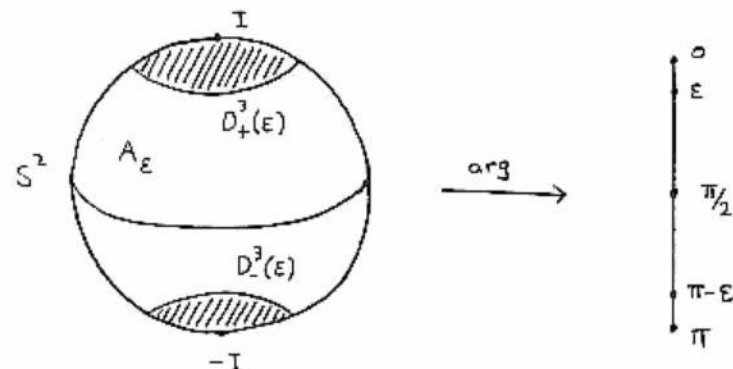


Figure 26

We shall require a trivialization of A_ε obtained via the exponential map in projecting to the "bottom end":

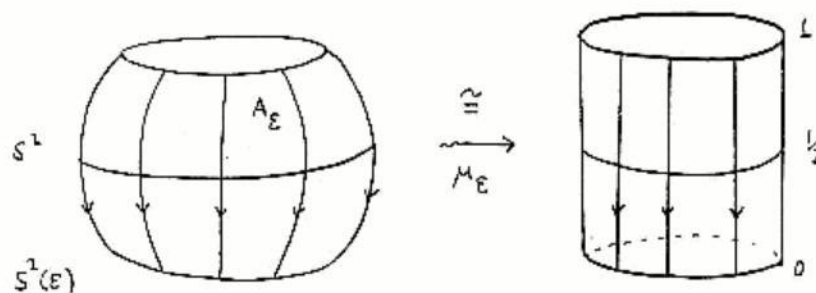


Figure 27

$$\mu_\varepsilon : A_\varepsilon \xrightarrow{\cong} S^2_\varepsilon(\varepsilon) \times [0,1]$$

$$x \longmapsto (x^{S_\varepsilon}, t_\varepsilon)$$

where:

$$\theta = \arg(x)$$

$$s_\varepsilon = (\pi - \varepsilon)/\theta$$

$$t_\varepsilon = (\varepsilon + \theta - \pi)/(2\varepsilon - \pi).$$

We observe that this decomposition of S^3 is preserved under the natural involution:

$$i : (S^3, D^3_+(\varepsilon), S^2_+(\varepsilon), A_\varepsilon, S^2_-(\varepsilon), D^3_-(\varepsilon))$$

$$\longrightarrow (S^3, D^3_-(\varepsilon), S^2_-(\varepsilon), A_\varepsilon, S^2_+(\varepsilon), D^3_+(\varepsilon)).$$

The trivialization of A_ε and the involution i are compatible in that:

$$\mu_\varepsilon(x) = (x_\varepsilon, t_\varepsilon) \text{ iff } \mu_\varepsilon(-x^{-1}) = (x_\varepsilon, 1 - t_\varepsilon).$$

Clearly, we may write:

$$R'_\varepsilon \times S^3 = (R'_\varepsilon \times D^3_-(\varepsilon)) \cup (R'_\varepsilon \times A_\varepsilon) \cup (R'_\varepsilon \times D^3_+(\varepsilon)).$$

The trivialization above allows us to realize δ_ε via a map from $R'_\varepsilon \times S^3$.

Observe that as a cycle:

$$\partial(\hat{N}_\varepsilon \times I) = -\hat{N}_\varepsilon \times \{1\} + (\partial\hat{N}_\varepsilon \times I) + \hat{N}_\varepsilon \times \{0\}.$$

We have also previously remarked that H does not extend across $R-(K)$.

However, in the complement of $R-(K)$, we know that:

$$H_0 = i_0 = \text{inclusion } H_1 = \tau i_1.$$

Hence, we may extend H continuously to:

$$H : (R \times \{0\}) \cup ((R \setminus R_-(K)) \times I) \cup (R \times \{1\}) \longrightarrow R.$$

With this extension in mind, we may write:

$$\delta_\varepsilon = -\hat{H}(\partial(\hat{N}_\varepsilon \times I)).$$

For the purpose of the next lemma, we define the following actions of SO_3 on $R'_\varepsilon \times S^3$ and on $N_\varepsilon \times I$:

$$SO_3 \times (R'_\varepsilon \times S^3) \longrightarrow R'_\varepsilon \times S^3$$

$$(C, (\rho', A)) \longmapsto (C \cdot \rho', CAC^{-1})$$

$$SO_3 \times (N_\varepsilon \times I) \longrightarrow N_\varepsilon \times I$$

$$(C, (\rho, t)) \longmapsto (C \cdot \rho, t).$$

Lemma 4.1: There exists a natural SO_3 -equivariant map:

$$g_\varepsilon : R'_\varepsilon \times S^3 \longrightarrow \partial(N_\varepsilon \times I)$$

such that:

$$\delta_\varepsilon = -\hat{H}(\widehat{g_\varepsilon(R'_\varepsilon \times S^3)}).$$

Note: From the previous observation:

$$\delta_\varepsilon = -\hat{H}(\partial(\hat{N}_\varepsilon \times I)).$$

All we need to do is to construct g_ε as a degree one homeomorphism. Actually, we shall ensure that g_ε converges as ε tends to zero. In this way, we shall realize δ_ε as a class in $\hat{R}(K)$.

Proof: We construct g_ε "in pieces".

$$(1) \quad g_\varepsilon : R'_- \times D_-^3(\varepsilon) \longrightarrow N_\varepsilon \times \{0\}$$

$$(\rho', x) \longmapsto (w(\rho', x), w(\rho', x), 0) .$$

Observe that μ_ε is the natural inclusion on the intersection of $D_-^3(\varepsilon)$ and A_ε :

$$\mu_\varepsilon : S_-^2(\varepsilon) \longrightarrow S_-^2(\varepsilon) \times \{0, 1\}$$

$$x \longmapsto (x, 0) .$$

Hence, we may extend g_ε as follows:

$$(2) \quad g_\varepsilon : R'_- \times A_\varepsilon \xrightarrow{\cong} \mathbb{R}N_\varepsilon \times I$$

$$(\rho', x) \longmapsto (w(\rho', x_\varepsilon), w(\rho', x_\varepsilon), t_\varepsilon)$$

where:

$$\mu_\varepsilon(x) = (x_\varepsilon, t_\varepsilon) .$$

From the compatibility of μ_ε and i , we observe that on $R'_- \times A_\varepsilon$:

$$g_\varepsilon(\rho', x) = (w(\rho', x_0), w(\rho', x_0), t)$$

If

$$g_\varepsilon(\rho', -x^{-1}) = (w(\rho', x_0), w(\rho', x_0), 1 - t) .$$

Hence, we may extend g_ε as follows:

$$(3) \quad g_\varepsilon : R'_- \times D_+^3(\varepsilon) \xrightarrow{\cong} N_\varepsilon \times \{1\}$$

$$(\rho', x) \longmapsto (w(\rho', -x^{-1}), w(\rho', -x^{-1}), 1) .$$

Clearly, g_ε is an SO_3 -equivariant, degree one homeomorphism. $\square$

Let:

$$f_\varepsilon = Hg_\varepsilon : R'_- \times S^3 \longrightarrow R .$$

Then:

$$f_\varepsilon = -\widehat{f}_\varepsilon(R'_- \times S^3) .$$

Our next goal is to study the limiting behavior of f_ε as ε tends to zero. With this in mind, we define:

$$f_- : R'_- \times S^3 \longrightarrow R_-(K)$$

$$(\rho', u) \longmapsto (\rho', u^{-1} \cdot \rho') .$$

Clearly, f_- is SO_3 -equivariant.

Lemma 4.2: $\lim_{\varepsilon \rightarrow 0} f_\varepsilon = f_-$

Proof: We check the limit "in pieces". Let $(\rho', x) \in R'_- \times S^3$.

Case (1): $x = -I$

$$f_\varepsilon(\rho', x) = f_\varepsilon(\rho', -I) = i_0(g_\varepsilon(\rho', -I))$$

$$= i_0(w(\rho', -I), w(\rho', -I), 0)$$

$$= (\rho', \rho')$$

$$f_-(\rho', x) = (\rho', (-I)^{-1} \cdot \rho') = (\rho', \rho') .$$

Hence:

$$\lim_{\varepsilon \rightarrow 0} f_\varepsilon(\rho', x) = f_-(\rho', x) .$$

Case (2): $x = 1$

$$\begin{aligned} f_{\varepsilon}(\rho', x) &= \tau i_{\varepsilon} g_{\varepsilon}(\rho', 1) \\ &= \tau(w(\rho', -I^{-1}), w(\rho', -I^{-1})) \\ &= \tau(\rho', \rho') \\ &= (\rho', (\partial \rho') \cdot \rho') \\ &= (\rho', \rho') \\ f_{\varepsilon}(\rho', x) &= (\rho', I^{-1} \cdot \rho') = (\rho', \rho') . \end{aligned}$$

Hence:

$$\lim_{\varepsilon \rightarrow 0} f_{\varepsilon}(\rho', x) = f_{\varepsilon}(\rho', x) .$$

Case (3): $x \neq 1$

Choose δ sufficiently small such that $x \in A_{\varepsilon}$ for all $\varepsilon < \delta$. Then, if $\varepsilon < \delta$:

$$\begin{aligned} f_{\varepsilon}(\rho', x) &= H g_{\varepsilon}(\rho', x) \\ &= H(w(\rho', x_{\varepsilon}), w(\rho', x_{\varepsilon}), t_{\varepsilon}) . \end{aligned}$$

Let:

$$\rho_{\varepsilon} = w(\rho', x_{\varepsilon}) \quad \text{so} \quad \partial \rho_{\varepsilon} = x_{\varepsilon} .$$

Then:

$$\begin{aligned} f_{\varepsilon}(\rho', x) &= (\rho_{\varepsilon}, (\partial \rho_{\varepsilon})^{t_{\varepsilon}} \cdot \rho_{\varepsilon}) \\ f_{\varepsilon}(\rho', x) &= (\rho_{\varepsilon}, (x_{\varepsilon})^{t_{\varepsilon}} \cdot \rho_{\varepsilon}) . \end{aligned}$$

Clearly:

$$\lim_{\varepsilon \rightarrow 0} x_{\varepsilon} = -I .$$

Hence:

$$\lim_{\varepsilon \rightarrow 0} \rho_{\varepsilon} = w(\rho', -I) = \rho' .$$

Claim: $\lim_{\varepsilon \rightarrow 0} (x_{\varepsilon})^{t_{\varepsilon}} = -x^{-1} .$

Note: We recall that the natural contraction on $S^3 \setminus \{-I\}$ does not extend to S^3 .

Proof of Claim:

Recall that $x_{\varepsilon} = x^{s_{\varepsilon}}$ and $\arg(x^{s_{\varepsilon}}) = \pi - \varepsilon$. Hence:

$$s_{\varepsilon} \arg(x) = \pi - \varepsilon .$$

Let:

$$\theta = \arg(x), \quad \text{so that} \quad s_{\varepsilon} = \frac{\pi - \varepsilon}{\theta} .$$

Now, by definition, $t_{\varepsilon} = (\varepsilon + \theta - \pi)/(2\varepsilon - \pi)$. Now:

$$(x_{\varepsilon})^{t_{\varepsilon}} = x^{s_{\varepsilon} \cdot t_{\varepsilon}} .$$

Moreover:

$$s_{\varepsilon} t_{\varepsilon} = \frac{(\pi - \varepsilon) \cdot (\varepsilon + \theta - \pi)}{\theta \cdot (2\varepsilon - \pi)}$$

so that:

$$\lim_{\varepsilon \rightarrow 0} s_{\varepsilon} t_{\varepsilon} = \frac{\pi - \theta}{\theta} .$$

Hence:

$$\lim_{\varepsilon \rightarrow 0} (x_{\varepsilon})^{t_{\varepsilon}} = x^{(\pi - \theta)/\theta} .$$

If we write $x = C \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} C^{-1}$, we compute that:

$$x^{(\pi-\theta)/\theta} = C \begin{pmatrix} e^{i(\pi-\theta)} & 0 \\ 0 & e^{-i(\pi-\theta)} \end{pmatrix} C^{-1}$$

that is:

$$x^{(\pi-\theta)/\theta} = -C \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} C^{-1}$$

$$x^{(\pi-\theta)/\theta} = -x^{-1} \quad \square$$

Having established the claim, we conclude that:

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} f_{\varepsilon}(\rho', x) &= (\rho', -x^{-1} \cdot \rho') \\ &= (\rho', x^{-1} \cdot \rho') \\ &= f_{-}(\rho', x) \end{aligned}$$

from steps (1), (2) and (3), we conclude that:

$$\lim_{\varepsilon \rightarrow 0} f_{\varepsilon} = f_{-} \quad \square$$

Let:

$$\begin{aligned} f : R' \times S^3 &\longrightarrow R_{-}(K) \\ (\rho', u) &\longmapsto (\rho', u \cdot \rho') \end{aligned}$$

that:

$$\delta_{-} = \widehat{f(R' \times S^3)} \in H_{3g}(\hat{R}_{-}(K))$$

Consider the natural map:

$$i_{*} : H_{3g}(\hat{R}_{-}(K)) \longrightarrow H_{3g}(\hat{R})$$

Corollary 4.3: The difference cycle δ is given as:

$$\delta = i_{*}(\delta_{-}) = i_{*} \widehat{f(R' \times S^3)}$$

Proof: From Lemma 4.2, there is a continuous map:

$$\hat{F} : (\widehat{R' \times S^3}) \times [0, \alpha] \longrightarrow \hat{R}$$

with:

$$\hat{F}_{\varepsilon} = \hat{F}_{\varepsilon} \quad 0 < \varepsilon < \alpha \quad \hat{F}_0 = \hat{F}_{-}$$

- Since:

$$\delta_{\varepsilon} = -\widehat{f_{\varepsilon}(R' \times S^3)} \quad 0 < \varepsilon < \alpha$$

$$\text{we conclude that } \delta = -i_{*}(\widehat{f_{-}(R' \times S^3)})$$

- But the inversion map on S^3 is orientation reversing:

$$\begin{aligned} \text{inv} : S^3 &\longrightarrow S^3 \\ u &\longmapsto u^{-1} \end{aligned}$$

- This implies that the following map is orientation reversing:

$$\begin{aligned} h : R' \times S^3 &\longrightarrow R' \times S^3 \\ (\rho', u) &\longmapsto (\rho', u^{-1}) \end{aligned}$$

Moreover, the natural map is an SO_3 bundle map:

$$\Pi : \widehat{R_- \times S^3} \longrightarrow \widehat{R_- \times S^3}.$$

Hence, since h is SO_3 equivariant, $\tilde{h}$ is also orientation reversing.

Clearly, $\hat{f} = \tilde{f} \cdot \tilde{h}$.

Hence, $\hat{f}(\widehat{R_- \times S^3}) = -\tilde{f}(\widehat{R_- \times S^3})$ and $\delta = i_*(\delta_-)$. $\square$

(d) The cycle δ_- in $\tilde{R}(K)$ as a pullback

In this section, we give an alternative description of δ_- as a pullback of a fibre bundle with total space $\tilde{R}(K)$. We begin by considering the map $\tilde{f}$ of the previous section:

$$\begin{aligned} \tilde{f} : \widehat{R_- \times S^3} &\longrightarrow \tilde{R}(K) = \widehat{R_- \times R_-} \\ [(\rho', u)] &\longmapsto [(\rho', u \cdot \rho')] \end{aligned}$$

Note: From hereon $[]$ will denote equivalence class or orbit under the appropriate SO_3 action.

Proposition 4.4: $\tilde{f}$ is a 2-fold cover of its image.

Proof: It suffices to show that $\tilde{f}$ is 2-to-1.

- But $\tilde{f}[(\rho'_1, u_1)] = \tilde{f}[(\rho'_2, u_2)]$ iff $\rho'_2 = C\rho'_1 C^{-1}$ and $u_2 \rho'_2 u_2^{-1} = C u_1 \rho'_1 u_1^{-1} C^{-1}$ for some fixed $C \in S^3$.

- This latter condition, however, implies that:

$$\rho'_1 = (C^{-1} u_2^{-1} C u_1) \rho'_1 (u_1^{-1} C^{-1} u_2 C).$$

- Since R_- consists of irreducible representations and $\rho'_1 \in R_-$, we conclude from Proposition I.2.3 that:

$$C^{-1} u_2^{-1} C u_1 = \pm I.$$

- Hence, $\rho'_2 = C\rho'_1 C^{-1}$ and $u_2 = \pm C u_1 C^{-1}$.

- Thus, $[(\rho'_2, u_2)] = [(\rho'_1, \pm u_1)]$.

- Now suppose that $[(\rho'_1, -u_1)] = [(\rho'_1, u_1)]$. Then as above:

$$\rho'_1 = C\rho'_1 C^{-1} \text{ and } -u_1 = C u_1 C^{-1}, \quad C \in S^3.$$

- Once again, by Proposition I.2.3, we conclude that:

$$C = \pm I \text{ and } -u_1 = u_1.$$

- This is clearly impossible, and hence $\tilde{f}$ is 2-to-1. $\square$

The previous proof indicates a natural factorization of $\tilde{f}$ by a 2-fold cover and an inclusion. We define the following action of SO_3 :

$$\begin{aligned} SO_3 \times (R_- \times SO_3) &\longrightarrow R_- \times SO_3 \\ (C, (\rho', A)) &\longmapsto (C\rho', CAC^{-1}). \end{aligned}$$

We have the natural map:

$$\begin{aligned} \Pi : \widehat{R_- \times S^3} &\longrightarrow \widehat{R_- \times SO_3} \\ (\rho', A) &\longmapsto (\rho', [A]) \end{aligned}$$

which induces a map of orbit spaces:

$$\hat{\Pi} : \widehat{R_- \times S^3} \longrightarrow \widehat{R_- \times SO_3}.$$

Finally, we can define a map:

$$g : R'_- \times SO_3 \longrightarrow R_-(K)$$

$$(\rho', u) \longmapsto (\rho', u \cdot \rho')$$

which induces a map of orbit spaces:

$$\hat{g} : \widehat{R'_- \times SO_3} \longrightarrow \hat{R}_-(K)$$

Corollary 4.5: (1) $\hat{f} = \hat{g} \cdot \hat{\Pi}$

- (2) $\hat{g}$ is an embedding
- (3) $\hat{\Pi}$ is a 2-fold cover
- (4) $\delta_- = 2 \cdot \widehat{g(R'_- \times SO_3)}$

Proof: This is an immediate consequence of Proposition 4.4. $\square$

Now we have an obvious projection map:

$$p : \hat{R}_-(K) \longrightarrow \hat{R}'_- \times \hat{R}'_-$$

$$[(\rho', \rho'')] \longmapsto ([\rho'], [\rho''])$$

Proposition 28: p is a fibre bundle with fibre SO_3 .

Proof: It suffices to show that the fibre of p is SO_3 .

- Suppose $p([\rho'_1, \rho''_1]) = ([\rho'], [\rho''])$.
- Then $\rho'_1 = C\rho' C^{-1}$ and $\rho''_1 = D\rho'' D^{-1}$ $C, D \in S^3$.
- Hence, $[(\rho'_1, \rho''_1)] = [(\rho', E\rho'' E^{-1})]$, $E = C^{-1}D \in S^3$.
- In other words, $p^{-1}([\rho'], [\rho'']) = \{[(\rho', E\rho'' E^{-1})] \mid E \in S^3\}$.

- But $[\rho', E_1 \rho'' E_1^{-1}] = [\rho', E_2 \rho'' E_2^{-1}]$ iff $E_2 = * E_1$.

- Hence, $p^{-1}([\rho'], [\rho'']) \cong SO_3$. $\square$

Let $\hat{\Delta}$ be the diagonal in $\hat{R}'_- \times \hat{R}'_-$.

Proposition 4.7: $p^{-1}(\hat{\Delta}) = \widehat{g(R'_- \times SO_3)}$.

Proof: $p\hat{g}([\rho', u]) = p([\rho', u \cdot \rho']) = ([\rho'], [\rho'])$.

- Hence, $\widehat{g(R'_- \times SO_3)} \subset p^{-1}(\hat{\Delta})$.

- On the other hand, suppose $p([\rho', \rho'']) \in \hat{\Delta}$.

- Then $[\rho'] = [\rho'']$ or $\rho'' = u \cdot \rho$, $u \in SO_3$. $\square$

Hence, we have a pullback diagram:

$$\begin{array}{ccc} \widehat{R'_- \times SO_3} & \xrightarrow{\hat{g}} & \hat{R}_-(K) \\ p \cdot \hat{g} \downarrow & & \downarrow p \\ \hat{\Delta} & \xrightarrow{\text{inc}} & \hat{R}'_- \times \hat{R}'_- \end{array}$$

Corollary 4.8: $\delta_- = 2 \cdot p^{-1}(\hat{\Delta})$

Proof: This is an immediate consequence of the previous proposition. $\square$

Note: $p^{-1}(\hat{\Delta})$ has a natural orientation inherited from $\hat{\Delta}$ via p . We may think of this in two ways (at least).

(1) duality

Suppose $p : M \longrightarrow N$, where M and N are manifolds. Suppose $\hat{\Delta}$ is an oriented submanifold of N . Suppose p is transverse to $\hat{\Delta}$. Then $p^{-1}(\hat{\Delta})$ is a submanifold of M with a natural orientation given by:

$$p^{-1}(\hat{\Delta}) = Dp^*p^{-1}(\hat{\Delta})$$

where D is Poincare duality.

(2) transversality

Under the same assumptions as in (1), we may orient the tangent spaces of $p^{-1}(\hat{\Delta})$ compatibly by the rules:

$$T_x(p^{-1}(\hat{\Delta})) = T_x(F_x) \oplus T_x(\hat{\Delta})$$

where $x = p(\tilde{x})$ and $F_x = p^{-1}(x)$. (Compare this with the discussion in section IV.1.)

These two descriptions are, of course, equivalent.

5. Casson's invariant for fibered knots

(a) Fibered knots

We recall that a knot K in a 3-manifold M is a fibered knot with monodromy f if the knot space $M^3(K)$ fibres over S^1 with fibre F' (where F' is a Seifert surface for K) and monodromy f :

$$M^3(K) = M^3 \setminus \text{interior } (N(K))$$

$$N(K) = \text{closed tubular nbd. of } K \text{ in } M^3$$

$$M^3(K) = (F' \times [-1,1]) / \sim \quad \partial F' = K$$

$$\partial F' \text{ is homotopic to } K \text{ in } N(K)$$

$$M^3(K) = \frac{F' \times J}{\langle (x,1) \sim (f(x),-1) \rangle} \quad J = [-1,1] \quad f: F' \xrightarrow{f} F'$$

From this fibration of $M^3(K)$, we obtain a strongly preferred Heegard decomposition for K_D , (as in Lemmas 1.2 and 1.3), as follows. Let:

$$W_1 = F' \times [0,1] \quad W_2 = F' \times [-1,0]$$

Let:

$$A_1 = \partial F \times [0,1] \quad \text{and} \quad A_2 = \partial F' \times [-1,0]$$

Then:

$$\partial W_1 = (F' \times \{0\}) \cup A_1 \cup (F' \times \{1\})$$

$$\partial W_2 = (F' \times \{0\}) \cup A_2 \cup (F' \times \{-1\})$$

We consider the following homeomorphism:

$$\begin{aligned} q: \partial W_1 &\xrightarrow{q} \partial W_2 \\ (x,0) &\longmapsto (x,0) \quad x \in F' \\ (x,t) &\longmapsto (x,-t) \quad x \in \partial F' \quad 0 \leq t \leq 1 \\ (x,1) &\longmapsto (f(x),-1) \end{aligned}$$

Then we see that:

$$M^3 = \frac{W_1 \cup W_2}{\langle q(z) \sim z \mid z \in \partial W_1 \rangle}$$

If we identify W_2 with W_1 via the map:

$$\begin{aligned} r: W_2 &\longrightarrow W_1 \\ (x,t) &\longmapsto (x,-t) \end{aligned}$$

then we obtain an Heegard model for M^3 :

$$M^3 = (W,h) \quad W = F' \times I$$

where:

$$\partial W = F' \cup A \cup F''$$

$$F' = F' \times \{0\} \quad A = \partial F' \times 1 \quad F'' = F' \times \{1\}$$

$$h : \partial W \xrightarrow{\cong} \partial W$$

$$x \longmapsto x \quad x \in F' \cup A$$

$$x \longmapsto f(x) \quad x \in F''$$

$$K = \partial F' \times \frac{1}{2}.$$

Note: This decomposition satisfies Lemmas 1.2 and 1.3.

(b) Casson's invariant for fibered knots in homology 3-spheres
and Lofschetz numbers

Given a fibered knot K in an homology 3-sphere M , let (W, h) be an Heegaard decomposition for K_M as in the previous section. As in section 4(a), we may identify representation spaces as follows:

$$(R')^* = R(\pi_1(F')) \cong (S^3)^{2g_1} \quad g_1 = \text{genus}(F')$$

$$R^* = R(\pi_1(\partial W_1 \setminus \{0\})) = (R')^* \times (R')^*$$

$$R = R(\pi_1(\partial W_1)) \subset R^*$$

$$R = \{(\rho', \rho'') \mid \partial \rho' = \partial \rho''\}$$

$$R = \{(\rho', \rho'') \mid \rho'(\lambda) = \rho''(\lambda)\}$$

$$Q_1 = \{(\rho', \rho') \mid \rho' \in (R')^*\}$$

$$Q_2 = \{(\rho', f^{\#}(\rho')) \mid \rho' \in (R')^*\}$$

S' = reducible representations in $(R')^*$

S = reducibles in $R^* = S' \times S'$

$$R'_- = \{\rho' \mid \rho' \in (R')^*, \partial \rho' = -I\}$$

$$R_-(K) = R'_- \times R'_-.$$

In addition, we have the following objects:

$$Q_{2-} = Q_2 \cap R_-(K) = \{(\rho', f^{\#}(\rho')) \mid \rho' \in R'_-\}$$

$$Q_{2-} = \Gamma_{f^{\#}} = \text{graph of } f^{\#} : R'_- \longrightarrow R'_-$$

$$\Gamma_{\hat{f}} = \text{graph of } \hat{f} : \hat{R}'_- \longrightarrow \hat{R}'_-.$$

Proposition 5.1: Q_2 is transverse to $R_-(K)$ in R .

Proof: First, we make a dimension count:

$$\dim(R^*) = 6g_1 + 6g_1 = 12g_1$$

$$\dim(R) = 12g_1 - 3$$

$$\dim(Q_2) = 6g_1$$

$$\dim(R_-(K)) = (6g_1 - 3) + (6g_1 - 3) = 12g_1 - 6$$

$$\dim(Q_{2-}) = 6g_1 - 3.$$

- Hence, we see that:

$$\dim(R) = \dim(Q_2) + \dim(R_-(K)) - \dim(Q_{2-}).$$

- Next, we compare tangent spaces at a point of intersection of Q_2 and $R_-(K)$, x :

$$T_x(Q_2) = \{(X, D_{X_1} f^\#(X)) \mid X \in T_{X_1}((R')^*)\}$$

$$T_x(R_-(K)) = \{(u, v) \mid u, v \in T_{X_1}(R')\}$$

$$T_x(Q_{2-}) = \{(y, D_y f^\#(y)) \mid y \in T_{X_1}(R_-)\}.$$

- Clearly, $T_x(Q_{2-}) = T_x(Q_2) \cap T_x(R_-(K))$.

- The dimension count and intersection calculation establish transversality in R . $\square$

Note: We did not use the assumption that M is an homology 3-sphere in establishing this last proposition. This assumption is equivalent to the following condition on the monodromy:

$$f_\# - I : H_1(F', \mathbb{Z}) \longrightarrow H_1(F', \mathbb{Z}) \text{ is nonsingular.}$$

As in the proof of Proposition III.1.1, we may interpret this in terms of $D_X f^\#$.

But we made no assumptions about this derivative.

Proposition 5.2: $\hat{Q}_{2-}$ has an orientable normal bundle in $\hat{Q}_2$.

Proof: By Proposition 5.1, $\hat{Q}_{2-}$ is a submanifold of $\hat{Q}_2$, (as well as of $\hat{R}_-(K)$). All manifolds in question are orientable. $\square$

We recall from sections 3 and 4 that:

$$\lambda'(K) = \pm \frac{1}{2} \langle \delta, \hat{Q}_2 \rangle_{\hat{R}}$$

$$\delta = i_*(\delta_-) \quad \delta_- = 2 \cdot \widehat{g(R' \times SO_3)} = 2p^{-1}(\hat{\Delta})$$

$$\begin{aligned} \hat{g} : \widehat{R' \times SO_3} &\longrightarrow \widehat{R' \times H_-} \\ [(\rho', u)] &\longmapsto [(\rho', u \cdot \rho')] \end{aligned}$$

$$\begin{aligned} p : \widehat{R' \times R'} &\longrightarrow \hat{R}' \times \hat{R}' \\ [(\rho', \rho'')] &\longmapsto ([\rho'], [\rho'']) \end{aligned}$$

$$\hat{\Delta} = \text{diagonal of } \hat{R}' \times \hat{R}'.$$

Construct an isotopy of $\hat{R}_-(K)$ which carries δ_- to a submanifold $\tilde{\delta}_-$ with:

$$\tilde{\delta}_- \text{ transverse to } \hat{Q}_{2-}.$$

Extend this isotopy of $\hat{R}$ and let:

$$\tilde{\delta} = i_*(\tilde{\delta}_-).$$

Then:

$$\tilde{\delta} \text{ is isotopic to } \delta.$$

Furthermore, by Proposition 5.1:

$$\tilde{\delta} \text{ is transverse to } \hat{Q}_2.$$

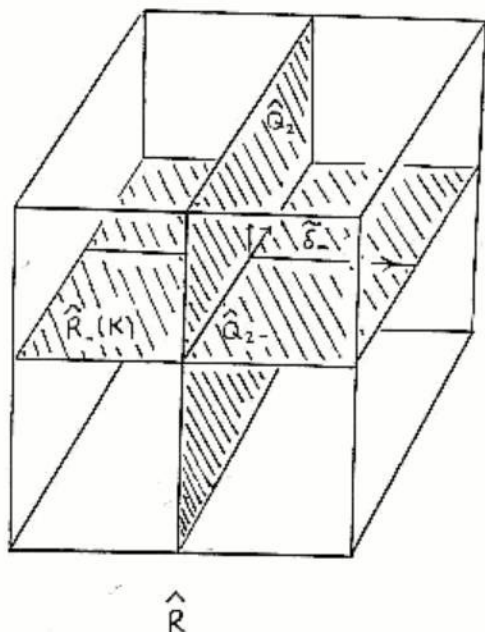


Figure 28

Hence, by Proposition 5.2, we conclude that:

$$\langle \delta, \hat{Q}_2 \rangle_{\hat{R}} = \pm \langle \delta_-, \hat{Q}_2 \rangle_{\hat{R}_-(K)} .$$

Applying the above observations:

$$\langle \delta, \hat{Q}_2 \rangle_{\hat{R}} = \pm 2 \langle \rho^{-1}(\hat{\Delta}), \hat{\Gamma}_{f^\#} \rangle_{\hat{R}_-(K)} .$$

Now we wish to apply the projection p . First, we need to observe that:

Proposition 5.3: $\hat{\Gamma}_{f^\#}$ is a lift of $\Gamma_{\hat{P}}$.

Proof: It suffices to construct a section σ of the restriction of p to $\hat{\Gamma}_{f^\#}$:

$$p : \hat{\Gamma}_{f^\#} \longrightarrow \Gamma_{\hat{P}} \\ [(\rho', f^\#(\rho'))] \longmapsto ([\rho'], [f^\#(\rho')]) .$$

We need only to show that the obvious map is well defined:

$$\sigma : \Gamma_{\hat{P}} \longrightarrow \hat{\Gamma}_{f^\#} \\ ([\rho'], [f^\#(\rho')]) \longmapsto [(\rho', f^\#(\rho'))] .$$

But this is immediate from the observation:

$$\rho' = u \cdot \rho'' \text{ iff } f^\#(\rho') = u \cdot f^\#(\rho'') . \quad \square$$

We need the following simple lemma, which will allow us to compare:

$$\langle p^{-1}(\hat{\Delta}), \hat{\Gamma}_{f^\#} \rangle_{\hat{R}_-(K)} \quad \text{and} \quad \langle \hat{\Delta}, \Gamma_{\hat{P}} \rangle_{\hat{R}' \times \hat{R}'} .$$

Lemma 5.4: Let $p : E \longrightarrow B$ be a map of connected, oriented manifolds. Suppose that Δ and Γ are oriented submanifolds of B , that p is transverse to Δ and that Γ lifts to an oriented submanifold of E , $\hat{\Gamma}$. Then:

$$\langle p^{-1}(\Delta), \hat{\Gamma} \rangle_E = \langle \Delta, \Gamma \rangle_B .$$

Note: We have already used this in "projecting" to $\hat{R}_-(K)$ from $\hat{R}$. The lemma follows readily from the same local transversality argument. However, we shall give an homological argument, via duality, for the sake of variety.

Proof: Note that, under the standard identifications of $H_0(E)$ and $H_0(B)$ with $\mathbb{Z}$, p_* is the identity:

$$\begin{aligned} p_* &= \text{id} : H_0(E, \mathbb{Z}) \longrightarrow H_0(B, \mathbb{Z}) \\ \langle p^{-1}(\Delta), \tilde{\Gamma} \rangle_E &= p^*(D^{-1}(\Delta)) \cap \tilde{\Gamma} \in H_0(E) = \mathbb{Z} \\ p^*(D^{-1}(\Delta)) \cap \tilde{\Gamma} &= p_*(p^*(D^{-1}(\Delta)) \cap \tilde{\Gamma}) \\ &= D^{-1}(\Delta) \cap p_*(\tilde{\Gamma}) . \end{aligned}$$

- But $\tilde{\Gamma}$ is a lift of Γ . That is, there is a section:

$$\sigma : \Gamma \longrightarrow \tilde{\Gamma} \quad p \circ \sigma = \text{id} .$$

- Hence, $p_*(\tilde{\Gamma}) = p_* \sigma_*(\Gamma) = \Gamma$.

- Finally, $p^*(D^{-1}(\Delta)) \cap \tilde{\Gamma} = D^{-1}(\Delta) \cap \Gamma$. $\square$

Applying this last lemma, we conclude that:

$$\langle p^{-1}(\hat{\Delta}), \hat{\Gamma}_{\hat{f}} \rangle_{\hat{R}_-(K)} = \langle \hat{\Delta}, \hat{\Gamma}_{\hat{f}} \rangle_{\hat{R}'_+ \times \hat{R}'_-} .$$

By definition, the right hand side is the Lefschetz number of $\hat{f}$:

$$\langle \hat{\Delta}, \hat{\Gamma}_{\hat{f}} \rangle_{\hat{R}'_+ \times \hat{R}'_-} = L(\hat{f} : \hat{R}'_- \longrightarrow \hat{R}'_+) .$$

Corollary 5.5: If K is a fibered knot with monodromy f in an homology 3-sphere M , then:

$$\lambda'(K) = * L(\hat{f} : \hat{R}'_- \longrightarrow \hat{R}'_+) .$$

(c) Fibered knots of genus one

We recall that a knot K in a 3-manifold M is a fibered knot of genus g_1 if K has a fibration such that the genus of the fiber is g_1 .

Note: We do not assume that g_1 is minimal.

Suppose that g_1 is 1. Then:

$$\Pi_1(F') = \langle a, b \rangle = \text{free group of rank } 2 .$$

Consider the following representation:

$$\begin{aligned} \rho_0 : \Pi_1(F') &\longrightarrow \text{SU}(2, \mathbb{C}) \\ a &\longmapsto \begin{bmatrix} 0 & +1 \\ -1 & 0 \end{bmatrix} \\ b &\longmapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} . \end{aligned}$$

Then:

$$\partial \rho_0 = \rho_0(a b a^{-1} b^{-1}) = -I$$

$$\rho_0 \in R'_-$$

Lemma 5.6: If $g_1 = 1$, then $\hat{R}' = \{[\rho_0]\}$.

Note: By a dimension count, we know that $\hat{R}'_+$ is a manifold of dimension 0. Moreover, we know it is connected, since it is a quotient of R'_+ and R'_+ is a retraction of the connected space $(R')^* \setminus S'$:

$$\begin{aligned} (R')^* &= (S^3)^2 & \dim((R')^*) &= 6 \\ S' &= R' & \dim(S') &= 4 . \end{aligned}$$

e shall, however, give a direct proof.

roof: Suppose that $\rho \in R'$.

Let $\rho(a) = A$ and $\rho(b) = B$. Then:

$$ABA^{-1}B^{-1} = -I \text{ or } B = -ABA^{-1}.$$

Hence, $\text{tr}(B) = -\text{tr}(B)$ or $\text{tr}(B) = 0$.

This implies that $\arg(B) = \pi/2$.

Therefore, at the risk of conjugating ρ , we may assume that:

$$B = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Now solving the consequence, $AB = -BA$, of the relation above, we conclude that:

$$A = \begin{bmatrix} 0 & e^{i\theta} \\ -e^{i\theta} & 0 \end{bmatrix} \quad \theta - \text{arbitrary}.$$

Let $C = \begin{bmatrix} -i\theta/2 & 0 \\ 0 & e^{i\theta/2} \end{bmatrix}$. Then we compute:

$$CAC^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$CBC^{-1} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Hence, ρ is conjugate to ρ_0 . $\square$

Corollary 5.7: If K is a fibered knot of genus one in an homology 3-sphere, then $|\lambda'(K)| = 1$.

Proof: This is an immediate consequence of Corollary 5.5 and Lemma 5.6. $\square$

Corollary 5.8: Let K be the trefoil or figure eight knot in S^3 . Then $\lambda'(K) = \pm 1$.

Note: Since λ and λ' have only been defined up to a choice of sign S , we now choose S such that:

$$\lambda'(\text{trefoil knot}) = 1.$$

We have not attempted to calculate S .

6. Dehn surgery on unlinks

(a) Dehn surgery on unlinks

Let M^3 be an homology sphere. Let (K, L) be a link of two components in M^3 . Then:

$$M^3(K, L) = M^3 \setminus \text{interior}(N(K \cup L)).$$

If (μ, λ) and (μ', λ') are standard meridian-longitude pairs for K and L respectively, then:

$$H_1(M^3(K, L)) = \mathbb{Z}\mu \oplus \mathbb{Z}\mu'.$$

Let $M^3(K, L; p/q, r/s)$ denote the manifold obtained from M^3 by performing p/q surgery to K and r/s surgery to L (as in section 1(a)). This is an homology 3-sphere precisely when $p = r = 1$ (assuming $p, r \neq 0$). As before, we denote these homology spheres as follows:

$$(K_m, L_n) = M^3(K, L; \frac{1}{m}, \frac{1}{n}) .$$

Now suppose that K and L have zero linking number. That is to say:

$$\lambda = \lambda' = 0 \quad \text{in} \quad H_1(M^3(K, L)) .$$

Now K is naturally contained in L_n . This assumption implies that (μ, λ) is a standard meridian longitude pair for K in L_n . Hence, it is clear that:

$$(K_m, L_n) = L_n(K, \frac{1}{m}) = K_m(L, \frac{1}{n}) .$$

Note: In general, (K_m, L_n) is $L_n(K, p/q)$ for some surgery coefficient p/q which depends on m, n and the linking number of K and L .

(b) Preferred Heegard decompositions for surgeries on boundary links

Suppose that (K, L) is a boundary link. That is, K and L bound disjoint Seifert surfaces. (In particular, K and L have zero linking number.) In a manner similar to the construction of section 1(b), we may construct a particularly convenient Heegard decomposition compatible with the various surgeries on (K, L) .

Proposition 6.1: There exists an Heegard decomposition for $M^3(W_1, W_2)$ such that:

$$W'_1 = F' \times I \quad L = \partial F' \times \{\frac{1}{2}\}$$

and K is a separating curve in F' .

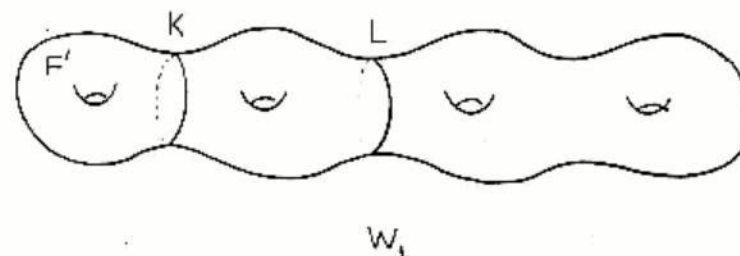


Figure 29

Proof: Choose a pair of disjoint Seifert surfaces for K and L , F_0 and F_1 . Thicken F_0 to $F_0 \times I$, with $F_0 \times I$ disjoint from F_1 . Let F_2 be the boundary of $F_0 \times I$ and form the connected sum of F_2 and F_1 by tubing F_2 and F_1 together. By choosing the tubing away from K and L , we obtain a surface, F' , with the following properties:

$$L = \partial F' \quad K \text{ is separating in } F' .$$

The rest of the proof continues along the lines of the proofs of Lemmas 1.1 and 1.2. In order to preserve the condition that K is separating, we only need to insure that all 1-handles are attached in the complement of the component of $F' \setminus K$ bounded by K . This is easily accomplished by sliding the attaching discs off this surface. $\square$

As in section 1(b), we identify this decomposition with an Heegard model (W, h) . Let τ_K and τ_L denote the Dehn twists about K and L respectively. We obtain the analog of Proposition 1.3.

Proposition 5.2: $(W, h \cdot \tau_K^m \tau_L^n)$ is an Heegard model for (K_m, L_n) .

7 Casson's invariant for unlinks

(a) Casson's invariant for unlinks $\lambda^*(K, L)$

Let (K, L) be an unlink of two components in an homology 3-sphere M^3 . That is, assume that K and L have zero linking number. As previously observed, K is naturally contained in L_n , L is naturally contained in K_m . We denote these various knots as follows:

$$(K, L_n) = K \text{ in } L_n \quad (K_m, L) = L \text{ in } K_m.$$

Similarly, we use $\lambda^*(K, L_n)$ and $\lambda^*(K_m, L)$ to denote Casson's invariant for the knots (K, L_n) and (K_m, L) respectively.

Casson's invariant for the unlink (K, L) in M^3 is defined by the rule:

$$\begin{aligned} \lambda^*(K, L) &= \lambda(K_{m+1}, L_{n+1}) - \lambda(K_m, L_{n+1}) \\ &\quad - \lambda(K_{m+1}, L_n) + \lambda(K_m, L_n). \end{aligned}$$

by grouping the terms in two distinct ways, we see:

$$\begin{aligned} \lambda^*(K, L) &= \lambda^*(K, L_{n+1}) - \lambda^*(K, L_n) \\ &= \lambda^*(K_{m+1}, L) - \lambda^*(K_m, L). \end{aligned}$$

This shows that $\lambda^*(K, L)$ is independent of m and n respectively. Hence, it is independent of m and n ; $\lambda^*(K, L)$ is well defined.

b) Triviality for boundary links

Let (K, L) be a boundary link in an homology sphere M^3 . Choose an

Heegard decomposition for M^3 as in Proposition 6.1 with corresponding Heegard models as in Proposition 6.2.

Following the argument of section 2(b), we conclude that:

$$\lambda(K_m, L_n) = \frac{1}{2} \langle \hat{Q}_1, \tau_K^m \tau_L^n(\hat{Q}_2) \rangle_{\hat{R}} \cdot S.$$

Let $h_K = \tau_K^{-1}$ and $h_L = \tau_L^{-1}$. Then:

$$\begin{aligned} \lambda^*(K, L) &= \lambda(K_1, L_1) - \lambda(K_0, L_1) - \lambda(K_1, L_0) + \lambda(K_0, L_0) \\ &= \frac{1}{2} \langle \hat{h}_K(\hat{h}_L(\hat{Q}_1) - \hat{Q}_1) - \hat{h}_L(\hat{Q}_1) + \hat{Q}_1, \hat{Q}_2 \rangle \cdot S \\ &= \left\{ \frac{1}{2} \langle \hat{h}_K(\hat{h}_L(\hat{Q}_1) - \hat{Q}_1), \hat{Q}_2 \rangle - \frac{1}{2} \langle \hat{h}_L(\hat{Q}_1) - \hat{Q}_1, \hat{Q}_2 \rangle \right\} \cdot S. \end{aligned}$$

Recall that we have a difference cycle:

$$\delta_L = \hat{\tau}_L(\hat{Q}_1) - \hat{Q}_1, \quad \hat{\tau}_L(\delta_L) = \delta_L.$$

Then:

$$\delta_L = \hat{h}_L(\delta_L) = \hat{Q}_1 - \hat{h}_L(\hat{Q}_1).$$

Hence, we conclude that:

$$\lambda^*(K, L) = \frac{1}{2} (\langle \delta_L, \hat{Q}_2 \rangle - \langle \hat{h}_K(\delta_L), \hat{Q}_2 \rangle) \cdot S.$$

Proposition 7.1: $\hat{h}_K(\delta_L) = \delta_L$ (rationally).

Proof: Recall that δ_L is carried by $\hat{R}_-(L)$, (section 2(c)). Here, we have:

$$R_-(L) = R'_- \times R''_-.$$

Let F' be the component of $F \setminus L$ which contains K and denote the restriction of h_K to F' by h_K . Then the action of h_K on $R_-(L)$ is given by the rule:

$$h_K^\# : R_-(L) \longrightarrow R_-(L)$$

$$(\rho', \rho'') \longmapsto (h_K^\#(\rho'), \rho'') .$$

In addition we have the projection:

$$p : \hat{R}_-(L) \longrightarrow \hat{R}' \times \hat{R}' .$$

From Corollary 4.8, we know that:

$$\delta_L = 2p^{-1}(\hat{\Delta})$$

where $\hat{\Delta}$ is the diagonal of $\hat{R}' \times \hat{R}'$. In terms of Poincaré duality, this equality may be expressed as follows:

$$D^{-1}(\delta_L) = 2 \cdot p^*(D^{-1}(\hat{\Delta})) .$$

From the results of the next chapter (Theorem VI.2.4) and the fact that h_K acts trivially on the homology of F' , we conclude that:

$$(\hat{h}_K)_* = \text{id} : H_k(\hat{R}'_+; \mathbb{Q}) \longrightarrow H_k(\hat{R}'_-; \mathbb{Q}) .$$

Hence, by the Künneth formula and the fact that h_K acts trivially on the second factor:

$$(\hat{h}_K)_* = \text{id} : H_k(\hat{R}'_+ \times \hat{R}'_-) \longrightarrow H_k(\hat{R}'_- \times \hat{R}'_-)$$

where $\hat{h}_K$ is the obvious map induced from h_K . Since the rational homology and cohomology are dual:

$$(\hat{h}_K)^* = \text{id} : H^k(\hat{R}'_+ \times \hat{R}'_-) \longrightarrow H^k(\hat{R}'_- \times \hat{R}'_-) .$$

Since the following diagram commutes:

$$\begin{array}{ccc} H^k(\hat{R}'_+ \times \hat{R}'_-) & \xrightarrow{(\hat{h}_K)^*} & H^k(\hat{R}'_- \times \hat{R}'_-) \\ p^* \downarrow & & \downarrow p^* \\ H^k(\hat{R}'_+(L)) & \xrightarrow{(\hat{h}_K)^*} & H^k(\hat{R}'_-(L)) \end{array}$$

we conclude from the previous remarks that:

$$\begin{aligned} (\hat{h}_K)^*(D^{-1}(\delta_L)) &= D^{-1}(\delta_L) \\ D^{-1}((\hat{h}_K)_*(\delta_L)) &= D^{-1}(\delta_L) \\ (\hat{h}_K)_*(\delta_L) &= \delta_L . \quad \square \end{aligned}$$

Corollary 7.2: If (K, L) is a boundary link in an homology 3 sphere M^3 , then $\lambda^*(K, L) = 0$.

Proof: This follows immediately from the previous identity:

$$\lambda^*(K, L) = \frac{1}{2} \langle \delta_L, \hat{Q}_2 \rangle - \langle \hat{h}_K(\delta_L), \hat{Q}_2 \rangle = 0$$

and Proposition 7.1. $\square$

(c) Casson's invariant for unlinks, $\lambda^*(K, C, C')$

Let (K, C, C') be an unlink of three components in an homology 3-sphere M^3 . That is, assume that every pair of components has zero linking number. As in previous sections, we adopt notation:

$$\langle K_n, C_k, C'_m \rangle = M^3(K, C, C'; \frac{1}{n}, \frac{1}{k}, \frac{1}{m})$$

$$\langle K_n, C_k, C' \rangle = C' \text{ in } \langle K_n, C_k \rangle$$

$$\langle K_n, C, C' \rangle = \langle C, C' \rangle \text{ in } K_n$$

and the various modified forms.

Casson's invariant is then defined by the rule:

$$\begin{aligned} \lambda^*(K, C, C') &= \lambda(K_{n+1}, C_{k+1}, C'_{m+1}) - \lambda(K_{n+1}, C_k, C'_{m+1}) \\ &\quad - \lambda(K_{n+1}, C_{k+1}, C'_m) + \lambda(K_{n+1}, C_k, C'_m) \\ &\quad - \lambda(K_n, C_{k+1}, C'_{m+1}) + \lambda(K_n, C_k, C'_{m+1}) \\ &\quad + \lambda(K_n, C_{k+1}, C'_m) - \lambda(K_n, C_k, C'_m) \end{aligned}$$

By grouping the terms in various ways, we see:

$$\begin{aligned} \lambda^*(K, C, C') &= \lambda^*(K_{n+1}, C, C') - \lambda^*(K_n, C, C') \\ &= \lambda^*(K, C_{k+1}, C') - \lambda^*(K, C_k, C') \\ &= \lambda^*(K, C, C'_{m+1}) - \lambda^*(K, C, C'_m) \end{aligned}$$

Pr: The fact that all these terms are defined follows from the assumption at each pair of components is unlinked. This is an easy exercise which we leave to the reader.

These various formulations show that $\lambda^*(K, C, C')$ is well defined independently of n, k and m .

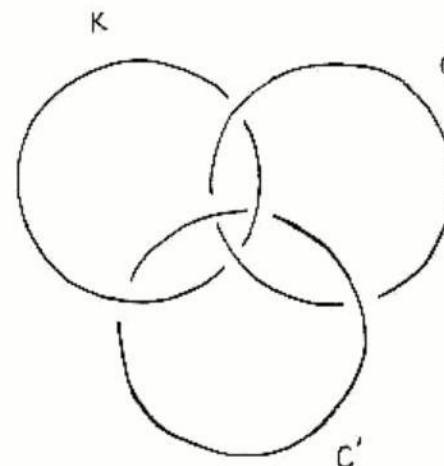


Figure 30

B. Casson's invariant and the Alexander polynomial

(a) The Alexander polynomial $\Delta_K(t)$

Let K be a knot (or link) in an homology 3 sphere M^3 . Let F be a Seifert surface in M^3 for K . Let V be a matrix for the linking form:

$$\begin{aligned} \langle \cdot, \cdot \rangle : H_1(F, \mathbb{Z}) \times H_1(F, \mathbb{Z}) &\longrightarrow \mathbb{Z} \\ (x, y) &\longmapsto \text{link}(x, y^+) \end{aligned}$$

where y^+ is the "plus push off" of y ([Ro]). Then we can define the Alexander polynomial of K by the rule:

$$\Delta_K(t) = \det(V - tV^T) .$$

If we change the Seifert surface, the polynomial changes by at most a unit multiple:

$$\Delta_{F'}(t) = \pm t^n \Delta_F(t) \quad n \in \mathbb{Z}.$$

We shall denote any such polynomial as $\tilde{\Delta}_K$. We recall that $\tilde{\Delta}_K$ is symmetric up to a unit:

$$\tilde{\Delta}_K(t^{-1}) = t^{-r} \tilde{\Delta}_K(t)$$

where r is the rank of $H_1(F, \mathbb{Z})$, $\beta_1(F)$. In particular, if K is a knot, $r = 2g$, where g is the genus of F . Hence, we obtain a symmetrized Alexander polynomial by symmetrizing:

$$\Delta_K(t) = t^{-g} \tilde{\Delta}_K(t)$$

$$\Delta_K(t^{-1}) = \Delta_K(t)$$

$$\Delta_K(t) = a_0 + a_1(t + t^{-1}) + \dots + a_m(t^m + t^{-m}).$$

We recall that $V - V^T$ is the intersection form:

$$V - V^T = (,)_F : H_1(F, \mathbb{Z}) \times H_1(F, \mathbb{Z}) \longrightarrow \mathbb{Z}.$$

Hence, we have the following proposition:

Proposition 8.1: Let K be a knot or link. Then:

$$\tilde{\Delta}_K(1) = 1 \quad \text{if } K \text{ is a knot}$$

$$\tilde{\Delta}_K(1) = 0 \quad \text{if } K \text{ is not a knot}.$$

Proof: For a connected surface of one boundary component, the intersection form is the standard symplectic form which has determinant one. On the other hand, a connected surface of more than one boundary component has a degenerate intersection form, since any boundary component represents a nontrivial element of the kernel. $\square$

Note: This and subsequent results are well-known for knots and links in S^3 . In particular, they follow from results in Kauffman [K] for the Conway polynomial.

Corollary 8.2: Let K be a knot with Seifert surfaces F and F' . Then:

$$\Delta_{F'}(t) = t^n \Delta_F(t) \quad n \in \mathbb{Z}.$$

Note: According to the classical definition of the Alexander polynomial as an elementary divisor, it is only well defined up to a unit. However, for any link in S^3 , any two Seifert matrices are related by the moves described in [BZ], pg. 236. Hence, the above corollary also holds for links in S^3 .

(b) The second derivative of $\Delta_K(t)$

We have the following identities for links of two components in S^3 :

Proposition 8.3: If K is a link of two components in S^3 and $\ell(K)$ is the linking number of these components, then:

$$\tilde{\Delta}_K'(1) = \ell(K).$$

Note: The conclusion is independent of the choice of Seifert surface. As we have previously observed, since K is a link in S^3 , if F' is another Seifert surface for K , then:

$$\Delta_{F'}(t) = t^n \tilde{\Delta}_K(t) \quad n \in \mathbb{Z}.$$

Hence, we compute that:

$$\Delta_{F'}(1) = n \cdot \tilde{\Delta}_K(1) + \tilde{\Delta}_K'(1) = \tilde{\Delta}_K'(1).$$

Proof of Proposition 8.3

The proof is by induction on the geometric linking number of the components of K (i.e. the minimal number of elementary crossings of the components of K required to homotope K to a splittable link).

Hence, we first consider the situation where K is a splittable link (K_1, K_2) . Let S^2 be a separating 2-sphere and F_1 and F_2 be Seifert surfaces for K_1 and K_2 respectively with F_1 and F_2 contained in components of $S^2 \setminus S^2$.

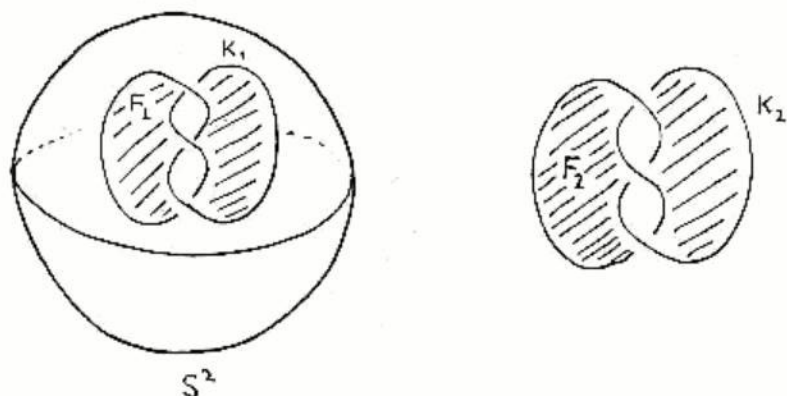


Figure 31

Tubing F_1 and F_2 together along an arc in $S^2 \setminus (F_1 \cup F_2)$, we obtain Seifert surface for K , F .

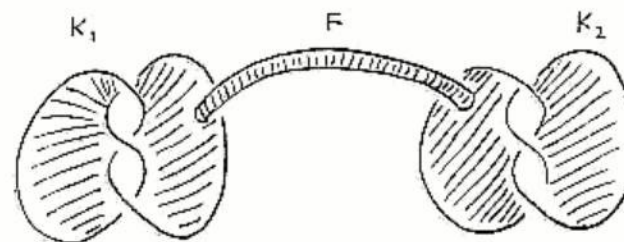


Figure 32

The corresponding Seifert matrix has the form:

$$V = \begin{matrix} & \begin{matrix} F_1^+ & F_2^+ & K_2^+ \end{matrix} \\ \begin{matrix} F_1 \\ F_2 \\ K_2 \end{matrix} & \begin{bmatrix} v_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{matrix}.$$

Hence, $\tilde{\Delta}_K(t) = 0$. On the other hand, $\sharp(K) = 0$. This establishes the result for split links.

Hence, we may assume, by standard arguments, that there is a projection of K with at least one elementary linking of the components of K , such that an elementary crossing at this elementary linking reduces the geometric linking of K .

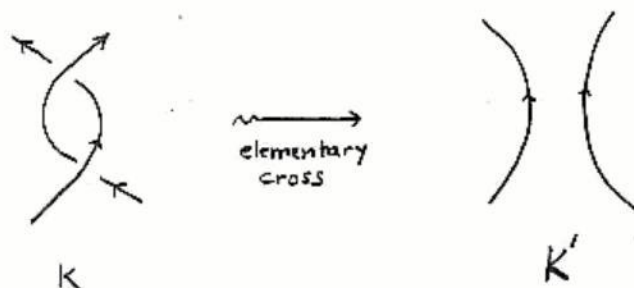


Figure 33

Using the standard Seifert surface for this projection, F , we can choose a basis for $H_1(F, \mathbb{Z})$, $(a, b, a_1, \dots, a_n)$, containing curves a and b as depicted in Figure 34.

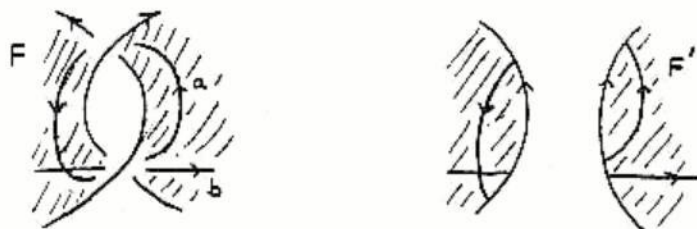


Figure 34

Note: If the associated surface for K', F' is not connected, then we can tube the two components together. By adding this tube as a handle to F , we insure that the curve b exists.

Then the corresponding Seifert matrix for K has the form:

$$V = \begin{matrix} & a^+ & b^+ & (F')^+ \\ \begin{matrix} a \\ b \\ F' \end{matrix} & \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & * \\ 0 & * & V' \end{bmatrix} \end{matrix}$$

where V' is the Seifert matrix for F' . Then we compute that:

$$V - tV^T = \begin{bmatrix} (t-1) & -1 & 0 \\ t & (t-1) & * \\ 0 & * & V' - t(V')^T \end{bmatrix}.$$

We observe that the matrix obtained by deleting the first row and column is an Alexander matrix for the knot L obtained by slicing F transverse to a (i.e. by deleting the generator a):



Figure 35

Hence, we have the following identity:

$$\tilde{\Delta}_K(t) = (t-1)\tilde{\Delta}_L(t) + t\tilde{\Delta}_{K'}(t) .$$

Computing derivatives, we conclude that:

$$\tilde{\Delta}'_K(1) = \tilde{\Delta}'_L(1) + \tilde{\Delta}'_{K'}(1) + \tilde{\Delta}'_{K'}(1) .$$

From the previous section, we know that:

$$\tilde{\Delta}'_L(1) = 1 \quad \tilde{\Delta}'_{K'}(1) = 0 .$$

By induction, we conclude that:

$$\tilde{\Delta}'_{K'}(1) = \sharp(K') .$$

On the other hand, we readily see that:

$$\sharp(K) = \sharp(K') + 1 .$$

Since, we have the desired result:

$$\tilde{\Delta}'_K(1) = \sharp(K) \quad \square$$

Now suppose that K_- , K_+ and K_0 are links in S^3 which have projections which differ at a single crossing of K_- as depicted below.

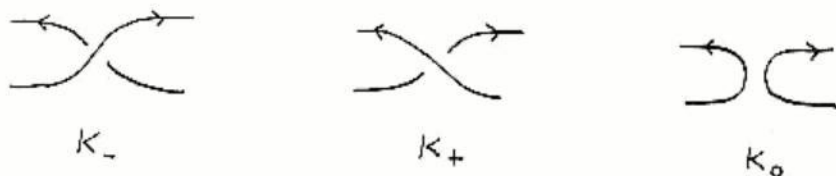


Figure 36

Lemma 8.4: Let K_- be a knot in S^3 . Let K_+ and K_0 be as above. Then the link K_0 is a link of two components, and:

$$\frac{1}{2} \Delta_{K_+}^*(1) - \frac{1}{2} \Delta_{K_-}^*(1) = \sharp(K_0) .$$

Proof: K_0 has two components for homological reasons. Now choose the Seifert surfaces as follows:

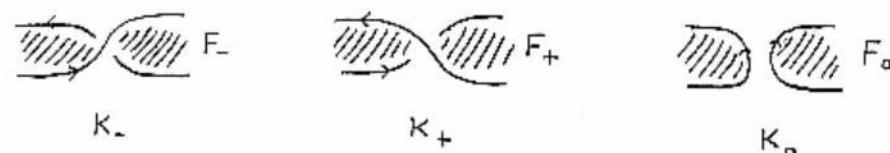


Figure 37

So we can arrange the corresponding Seifert matrices V_+ , V_- , V_0 as follows:

$$V_+ = \begin{bmatrix} a_{11} & * \\ * & V_0 \end{bmatrix} \quad V_- = \begin{bmatrix} a_{11}+1 & * \\ * & V_0 \end{bmatrix}$$

$$\tilde{\Delta}_+(t) = \det(V_+ - tV_+^T) , \quad \tilde{\Delta}_-(t) = \det(V_- - tV_-^T)$$

$$\tilde{\Delta}_0(t) = \det(V_0 - tV_0^T) , \quad \text{hence:}$$

$$\tilde{\Delta}_+(t) - \tilde{\Delta}_-(t) = (t-1)\tilde{\Delta}_0(t) .$$

Observe that F_- is homeomorphic to F_+ . Hence, $\text{genus}(F_-) = \text{genus}(F_+) = g$.

To obtain the symmetrized Alexander polynomials, we have to multiply by $t-\bar{t}$.

Hence:

$$\Delta_{K_+}(t) - \Delta_{K_-}(t) = (t-1)t^{-g_{\Delta_0}}(t) .$$

Let $\varphi(t) = (t-1)t^{-g_{\Delta_0}}(t)$. Computing derivatives:

$$\varphi'(1)/2 = -g_{\Delta_0}'(1) + \tilde{\Delta}_0'(1) .$$

Hence, by Propositions 8.1 and 8.3:

$$\frac{1}{2} \Delta_{K_+}'(1) - \frac{1}{2} \Delta_{K_-}'(1) = \sharp(K_0) . \quad \square$$

(c) Casson's invariant and the Alexander polynomial

We are now ready to prove the main result relating Casson's invariant to the classical Alexander polynomial. We begin with a simple lemma.

Lemma 8.5: Let K be a knot in an homology 3-sphere M^3 . Then there exists a knot L in S^3 such that $\lambda^*(K) = \lambda^*(L)$ and $\Delta_K(t) = \Delta_L(t)$.

Proof: It is well known that S^3 may be obtained by ± 1 surgeries on the components of a link in M^3 :

$$S^3 = (C_{\pm 1}^1, \dots, C_{\pm 1}^n)$$

$$C = (C^1, \dots, C^n) \text{ is a link in } M^3 .$$

The proof is by induction on n . It is obvious for $n = 0$.

Now choose Seifert surfaces for K and C^n , F and F' . By general position, we may assume that there exist one dimensional spines, Σ and Σ' , for F and F' which are disjoint. We may isotope F into a small regular neighborhood of Σ . By a separate isotopy, we may isotope F' into a small regular neighborhood of Σ' . Hence, we may assume that F and F' are

disjoint. (Of course, this may require crossings of K and C , but not self crossings of C .) In other words, we may assume that (K, C^n) is a boundary link in M^3 . Hence, $\lambda^*(K, C^n) = 0$ by Corollary 7.2. That is:

$$\lambda^*(K) = \lambda^*(K; C_{\pm 1}^n) .$$

On the other hand, since K and C^n bound disjoint Seifert surfaces in $M^3 \setminus (K, C^n)$, we may also conclude that:

$$\Delta_K(t) = \Delta_{(K; C_{\pm 1}^n)}(t)$$

where $(K; C_{\pm 1}^n)$ is the knot K considered as a knot in the surgered manifold $M^3(C^n; \pm 1)$. If we denote this manifold by N^3 :

$$S^3 = (D_{\pm 1}^1, \dots, D_{\pm 1}^{n-1})$$

$$D = (D^1, \dots, D^{n-1}) \text{ is the link } (C^1, \dots, C^{n-1}) \text{ considered as a link in } N^3 .$$

Hence, by induction, there is a knot L in S^3 such that:

$$\lambda^*(K; C_{\pm 1}^n) = \lambda^*(L)$$

$$\Delta_{(K; C_{\pm 1}^n)}(t) = \Delta_L(t) .$$

This establishes the desired result. $\square$

Next we prove an analog of Lemma 8.4.

Lemma 8.6: Let K_- be a knot in S^3 . Let K_+ and K_0 be as above. Then K_0 is a link of two components and:

$$\lambda'(K_+) - \lambda'(K_-) = \sharp(K_0) .$$

Proof: Suppose that K is a knot in S^3 . Now suppose that (C, C') is a link in $S^3 \setminus K$ such that C and C' bound disjoint discs, D and D' , each meeting K geometrically twice and algebraically zero times. In addition, assume that $K \cap D$ and $K \cap D'$ do not separate each other in K :

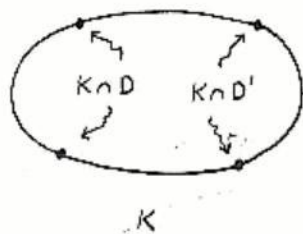


Figure 38

In particular, (C, C') is a boundary link in K_n , for C' and C bound obvious disjoint punctured tori in $S^3 \setminus K$:

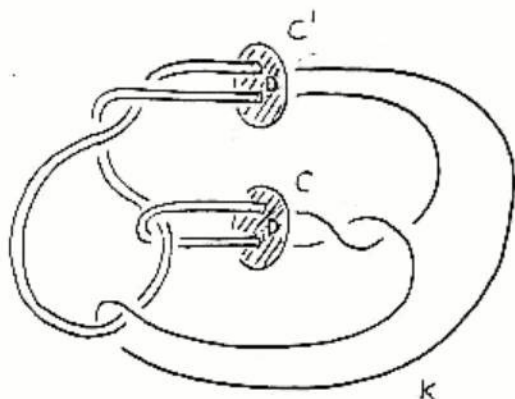


Figure 39

Hence, $\lambda^*(K_n, C, C') = 0$ for all n . From this, we conclude that:

$$\lambda^*(K, C, C') = 0 .$$

This in turn implies that:

$$\lambda^*(K, C_1, C') = \lambda^*(K, C') .$$

This means that changing K with a twist across D does not change the value of $\lambda^*(K, C')$. But, by changing K across such discs D , we can change K to the following knot K^m :

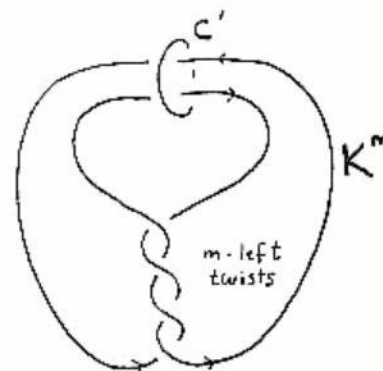


Figure 40

Let C^* denote the curve depicted below:

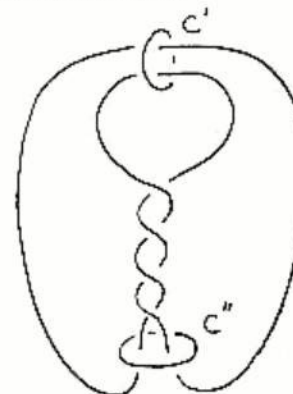


Figure 41

We have the following computation:

$$\begin{aligned}
 \lambda^*(K, C') &= \lambda^*(K^m, C') \\
 &= \lambda^*(K^m; C'_1) - \lambda^*(K^m; C'_0) \\
 &= \lambda^*(K^m; C'_1) \\
 &= \sum_{j=1}^m (\lambda^*(K^j; C'_1) - \lambda^*(K^{j-1}; C'_1)) \\
 &= \sum_{j=1}^m (\lambda^*(K^0, C'_1, C''_j) - \lambda^*(K^0, C'_1, C''_{j-1})) \\
 &= \sum_{j=1}^m (\lambda^*(K^0, C'_1, C'')) \\
 &= m(\lambda^*(K^0, C'_1, C'') - \lambda^*(K^0, C'_1, C''_0)) \\
 &= m(\lambda^*(K'; C'_1)) \\
 &= m(\lambda^*(\text{trefoil knot})) = 4m.
 \end{aligned}$$

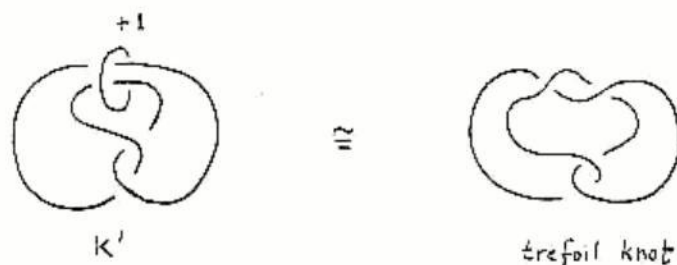


Figure 42

Hence, from the previous comments:

$$\lambda^*(K, C'_1) - \lambda^*(K, C'_0) = 4m$$

$$\lambda^*(K; C'_1) - \lambda^*(K) = 4m.$$

We may assume that K is K_+ and K_- is the knot obtained from K by applying a left twist across D' . That is K_- is $(K; C'_1)$. Let K_0 denote the two component link obtained from K by cutting across D' and connecting "the other way":

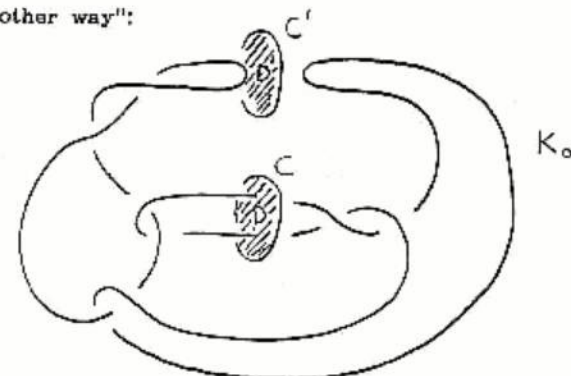


Figure 43

The move of twisting across the disc D only affects one component of K_0 . Hence, it clearly leaves $\ell(K_0)$ invariant. For the standard form K^m , however, it is evident that $\ell(K^m) = -m$:

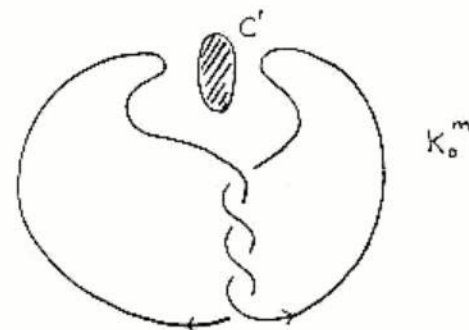


Figure 44

It follows that:

$$\lambda'(K_+) - \lambda'(K_-) = \varepsilon \cdot \ell(K_0)$$

$$\varepsilon = \lambda'(\text{trefoil knot}) .$$

As previously observed, we may "normalize" λ' so that:

$$\lambda'(\text{trefoil knot}) = +1 .$$

For this normalization, we have the identity:

$$\lambda'(K_+) - \lambda'(K_-) = \ell(K_0) . \quad \square$$

We now turn to the main theorem.

Theorem 8.7: If K is a knot in an homology 3 sphere M^3 , then:

$$\lambda'(K) = \frac{1}{2} \Delta_K''(1) = \sum_{n=1}^{\infty} n^2 a_n$$

where $\Delta_K(t) = a_0 + a_1(t + t^{-1}) + \dots + a_m(t^m + t^{-m})$.

Proof: By Lemma 8.5, we assume that K is a knot in S^3 . The proof is by induction on the number of crossings in a projection of K which must be changed in order to convert K to an unknot. Since the theorem is true for the unknot, we may consider a projection of K and a single crossing which we need to change. Let K_- denote K , K_+ the knot obtained by changing this crossing and K_0 the usual associated link of two components. Then:

$$\lambda'(K_+) - \lambda'(K_-) = \ell(K_0)$$

$$\frac{1}{2} \Delta_{K_+}''(1) - \frac{1}{2} \Delta_{K_-}''(1) = \ell(K_0) .$$

On the other hand, by induction:

$$\lambda'(K_+) = \frac{1}{2} \Delta_{K_+}''(1) .$$

The result follows immediately. $\square$

With this result, we are able to establish property (5) for $\lambda(M)$. Let $\varphi(K)$ be the arf invariant of K .

Property (5): $\lambda(M^3) \equiv \mu(M^3) \pmod{2}$

Proof: From Theorem 7 of [R], (after symmetrizing), and Theorem 8.7 above, we conclude that:

$$\varphi(K) \equiv \lambda'(K) \pmod{2}$$

for any knot in S^3 . The argument of Lemma 8.5 permits us to deduce the same fact for any knot in any homology sphere. (The arf invariant also comes from a Seifert matrix.)

On the other hand, Theorem 2 of [G] implies that:

$$\varphi(K) \equiv \mu(K_+) - \mu(K_0) \pmod{2} .$$

This yields the identity:

$$\lambda(K_+) - \lambda(K_0) \equiv \mu(K_+) - \mu(K_0) \pmod{2} .$$

Since we can write M^3 as a sequence of ± 1 surgeries on a link in S^3 :

$$M^3 = S^3(C_{\pm 1}, \dots, C_{\pm n})$$

and we know that:

$$\lambda(S^3) \equiv \mu(S^3) \equiv 0 \pmod{2}$$

the property follows by induction on n using the identity above. $\square$

CHAPTER VI: THE TOPOLOGY OF THE SPACE OF REPRESENTATIONS

1. The topology of R^g

(a) Identification of spaces

As in previous sections, we identify R^* with $(S^3)^{2g}$. Since we shall be inducting on g at various points, we denote the boundary map as follows:

$$\partial_g : R^* \longrightarrow S^3$$

$$(A_1, \dots, A_g, B_1, \dots, B_g) \longmapsto \prod_{i=1}^g [A_i, B_i] .$$

Likewise, we use the following notation:

$$R_+^g = R^g = \partial_g^{-1}(I) \quad N_+^g = \partial_g^{-1}(D_+^3)$$

$$R_-^g = \partial_g^{-1}(-I) \quad N_-^g = \partial_g^{-1}(D_-^3)$$

Clearly:

$$R^* = N_+^g \cup N_-^g .$$

The singular set of $\partial_g S$ is contained in R_+ . Hence, ∂_g is regular on $R^* \setminus S$. In particular, we have a fibre bundle with fibre R_- :

$$\partial_g| : N_-^g \longrightarrow D_-^3 .$$

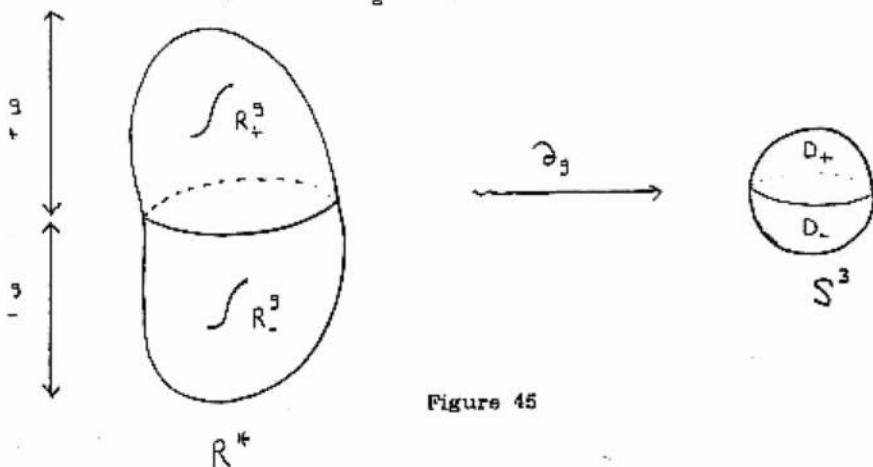


Figure 45

Hence, there is a trivialization:

$$w_g : (R_-^g \times D_-^3, R_-^g \times S^2) \xrightarrow{\cong} (N_-^g, \partial N_-^g)$$

with the following properties:

- (1) $\partial_g \cdot w_g = p_2$ (projection onto second factor)
- (2) $w_g|_{R_-^g \times (-I)}$ is the usual inclusion $R_-^g \subset N_-^g$.

(b) The topology of R^g

Lemma 1.1: R_-^g is connected for $g = 1$

simply connected for $g \geq 2$.

Proof: The restriction of the boundary map:

$$\partial_g : (S^3)^{2g} \setminus R_+^g \longrightarrow S^3 \setminus \{I\}$$

is a locally trivial fibration with fiber R_-^g . It suffices to show that $(S^3)^{2g} \setminus R_+^g$ is connected for $g = 1$ and simply connected for $g \geq 2$.

Consider the decomposition of R_+^g :

$$R_+^g = R_0 \cup R_1, \quad R_0 = R \setminus S, \quad R_1 = S .$$

Let Δ be S^1 in S^3 (i.e. Δ is the set of diagonal matrices). Consider the map:

$$\alpha : \Delta^{2g} \times S^3 \longrightarrow (S^3)^{2g}$$

$$((A_1, B_1, \dots, A_g, B_g), T) \longmapsto (TA_1 T^{-1}, TB_1 T^{-1}, \dots, TA_g T^{-1}, TB_g T^{-1}) .$$

Then:

$$\text{im}(\alpha) = R_1 .$$

Since diagonal matrices commute, α factors to:

$$\bar{\alpha} : \Delta^{2g} \times (S^3/\Lambda) \longrightarrow (S^3)^{2g} \\ \downarrow \Pi \\ (S^1)^{2g} \times S^2.$$

Hence, $\dim(R_1) = 2g + 2$, which implies $\text{codim}(R_1) = 4g - 2$. Since $\text{codim}(R_0) = 3$, the result follows by general position. $\square$

We now want to relate the topology of R_-^{g+1} to the topology of R_-^g . Write:

$$(S^3)^{2(g+1)} = (S^3)^2 \times (S^3)^{2g}.$$

Let p_1, p_2 be the projections onto $(S^3)^2$ and $(S^3)^{2g}$. So we can write:

$$\partial_{g+1}(x) = \partial_1(p_1(x)) \cdot \partial_g(p_2(x))$$

$$R_-^{g+1} = \{(y_1, y_2) \in (S^3)^2 \times (S^3)^{2g} \mid \partial_g(y_2) = -\partial_1(y_1)^{-1}\}.$$

Recall the involution of S^3 sending A to $-A^{-1}$:

$$i : S^3 \longrightarrow S^3 \\ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \longmapsto \begin{pmatrix} -\bar{a} & b \\ -\bar{b} & -a \end{pmatrix}.$$

Hence, if $A \in D_+^3$, then $-A^{-1} \in D_-^3$.

Now construct imbeddings φ, ψ :

$$\begin{array}{ccc} R_-^1 \times N_+^g & & N_+^1 \times R_-^g \\ \varphi \searrow & & \swarrow \psi \\ R^* = (S^3)^2 \times (S^3)^{2g} \end{array}$$

given by:

$$\varphi(x_1, x_2) = (w_1(x_1, -f_g(x_1)^{-1}), x_2)$$

$$\psi(z_1, z_2) = (z_1, w_g(z_2, -f_g(z_1)^{-1}))$$

Then we conclude that:

$$\text{im}(\varphi) = \{(y_1, y_2) \in (S^3)^2 \times (S^3)^{2g} \mid f_g(y_2) \in D_+^3, f_g(y_2) = -f_1(y_1)^{-1}\}$$

$$\text{im}(\psi) = \{(y_1, y_2) \in (S^3)^2 \times (S^3)^{2g} \mid f_g(y_2) \in D_-^3, f_g(y_2) = -f_1(y_1)^{-1}\}$$

$$R_-^{g+1} = \text{im}(\varphi) \cup \text{im}(\psi).$$

Consider the diffeomorphism:

$$\lambda : R_-^1 \times \partial N_+^g \longrightarrow \partial N_+^1 \times R_-^g$$

$$\lambda(x_1, x_2) = (w_1(x_1, -f_g(x_2)^{-1}), p_1 w_g^{-1}(x_2)).$$

Then we get the following commutative diagram:

$$\begin{array}{ccc} R_-^1 \times \partial N_+^g & \xrightarrow[\cong]{\lambda} & \partial N_+^1 \times R_-^g \\ \varphi \searrow & & \swarrow \psi \\ R_-^1 \times S^2 \times R_-^g & & \text{im } \varphi \cap \text{im } \psi \end{array}$$

Hence:

$$R_-^{g+1} = R_-^1 \times N_+^g \cup_{\lambda} N_+^1 \times R_-^g.$$

Furthermore, since $A = -A^{-1}$ for $A \in S^2$, we can write:

$$\lambda(x_1, x_2) = (w_1(x_1, f_g(x_2)), p_1 w_g^{-1}(x_2)).$$

λ is the composition of:

$$R_-^1 \times \partial N_+^g \xrightarrow{1 \times \bar{w}_1^{-1}} R_-^1 \times S^2 \times R_-^g \xrightarrow{w_1 \times 1} \partial N_+^1 \times R_-^g.$$

We can describe the gluing by the diagram:

$$\begin{array}{ccccc} & & R_-^1 \times N_+^g & & \\ & \nearrow 1 \times \bar{w}_1 & & \nwarrow & \\ R_-^1 \times S^2 \times R_-^g & & & & R_-^{g+1} \\ & \searrow \bar{w}_1 \times 1 & & \nearrow & \\ & & N_+^1 \times R_-^g & & \end{array}$$

where $\bar{w}_g$ and $\bar{w}_1$ are w_g and w_1 followed by inclusions.

We have a principal fibre bundle with fibre SO_3 :

$$R_-^g \longrightarrow \hat{R}_-^g.$$

We have previously shown that $\hat{R}_-^1$ is a point. Hence, R_-^1 is homeomorphic to SO_3 .

The Mayer-Vietoris sequence yields:

$$\begin{array}{ccc} \cdots \longrightarrow H_r(R_-^1 \times S^2 \times R_-^g) & \xrightarrow{\lambda_r} & H_r(R_-^1 \times N_+^g) \oplus H_r(N_+^1 \times R_-^g) \\ & & \downarrow \\ \cdots \xleftarrow{\lambda_{r-1}} H_{r-1}(R_-^1 \times S^2 \times R_-^g) & \xleftarrow{\beta} & H_{r-1}(R_-^{g+1}) \end{array}$$

We conclude that:

$$\begin{aligned} H_r(R_-^1 \times S^2 \times R_-^g) &\cong H_r(R_-^g) \oplus [H_2(S^2) \otimes H_{r-2}(R_-^g)] \oplus H_{r-3}(S^2 \times R_-^g) \\ H_r(R_-^1 \times N_+^g) &\cong H_r(N_+^g) \oplus H_{r-3}(N_+^g) \\ H_r(N_+^1 \times R_-^g) &\cong H_r(R_-^g) \oplus (H_2(N_+^1) \otimes H_{r-2}(R_-^g)) \oplus (H_3(N_+^1) \otimes H_{r-3}(R_-^g)). \end{aligned}$$

We can easily check that:

$$\lambda_r(\alpha, \beta, \gamma) = ((\bar{w}_g)_r \alpha + (\bar{w}_g)_r \beta, (\bar{w}_g)_{r-1} \gamma) \oplus (\alpha, \beta, D)$$

where $(\bar{w}_g)_r$ denotes the map induced in r -th homology. We deduce:

$$\ker \lambda_r \cong \ker (\bar{w}_g)_{r-3}.$$

Now, consider the exact sequence:

$$0 \longrightarrow H_4(S^3 \times S^3, N_+^1) \xrightarrow{\beta} H_3(N_+^1) \xrightarrow{i_*} H_3(S^3 \times S^3) \longrightarrow \cdots$$

Since the usual generators of $H_3(S^3 \times S^3)$ lie in N_+^1 , i_* is onto. In addition, we compute that:

$$\begin{aligned} H_4(S^3 \times S^3, N_+^1) &\cong H_4(N_-^1, \partial N_-^1) \quad (\text{excision}) \\ &\cong H_4(\mathbb{RP}^3 \times D^3, \mathbb{RP}^3 \times S^2) \\ &\cong H_1(\mathbb{RP}^3) \quad (\text{Thom isomorphism}) \\ &\cong \mathbb{Z}_2. \end{aligned}$$

Hence:

$$H_3(N_+^1) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_2.$$

Also, since i_* is onto, we have an exact sequence:

$$0 \longrightarrow H_3(S^3 \times S^3, N_+^1) \xrightarrow{\beta} H_2(N_+^1) \longrightarrow 0.$$

This implies that:

$$H_2(N_+^1) \cong H_3(S^3 \times S^3, N_+^1) \cong H_3(\mathbb{RP}^3 \times D^3, \mathbb{RP}^3 \times S^2) \cong H_0(\mathbb{RP}^3).$$

Hence:

$$H_2(N_+^1) = \mathbb{Z}.$$

Similarly:

$$H_1(N_+^1) = 0.$$

By using $\mathbb{Q}$ coefficients in homology, we get:

$$\text{coker } \lambda_r \cong H_r(N_+^g) \oplus \text{coker}(\bar{w}_g)_{r-3} \oplus 2 H_{r-3}(R_-^g).$$

Hence: $H_r(R_-^{g+1}) \cong \text{coker } \lambda_r \oplus \ker \lambda_{r-1}$,

$$H_r(R_-^{g+1}) \cong H_r(N_+^g) \oplus \text{coker}(\bar{w}_g)_{r-3} \oplus 2 H_{r-3}(R_-^g) \oplus \ker(\bar{w}_g)_{r-4}.$$

$$H_r(R_-^{g+1}) \cong H_r(N_+^g) \oplus 2 H_{r-3}(R_-^g) \oplus H_{r-3}(N_+^g, \mathbb{J}N_+^g).$$

For a fixed $z \in R_-^1$, we can define the map:

$$\alpha : N_+^g \xrightarrow{\cong} \{z\} \times N_+^g \subset R_-^1 \times N_+^g \xrightarrow{\varphi} R_-^{g+1}.$$

It follows that:

$$\alpha_* : H_2(N_+^g) \longrightarrow H_2(R_-^{g+1})$$

is an isomorphism. So:

$$\alpha_* : H^2(R_-^{g+1}) \longrightarrow H^2(N_+^g)$$

is an isomorphism. Let $\mu_1, \dots, \mu_{2g}$ be the generators of $H^3(N_+^g)$ induced from the obvious generators of $(S^3)^{2g}$ by the inclusion:

$$N_+^g \longrightarrow (S^3)^{2g}.$$

Then:

$$\alpha^*(\mu_i) = \begin{cases} 0 & \text{for } i = 1, 2 \\ \mu_{i-2} & \text{for } 3 \leq i \leq 2g+2 \end{cases}.$$

This fact will be used later.

Theorem 1.2: The following inclusions are surjective:

$$(a) \quad H_r(N_+^g) \longrightarrow H_r(R^*) \quad \text{for } r \leq 3g+2$$

$$(b) \quad H_r(N_-^g) \longrightarrow H_r(R^*) \quad \text{for } r \leq 3g-1.$$

Proof: (a) We first prove the case: $g = 1$. Since R^* is $S^3 \times S^3$, the only relevant case is $r = 3$:

$$H_3(N_+^1) \longrightarrow H_3(R^*) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

The obvious generators, $S^3 \times I$ and $I \times S^3$, are contained in R_+^1 . So they lie in N_+^1 .

We then show the case: $g = 2$. In this case, R^* is $(S^3)^4$. So the relevant cases are $r = 3, 6$:

$$H_3(R^*) \cong \mathbb{Z}^4, \quad H_6(R^*) \cong \mathbb{Z}^6.$$

Clearly, all the generators of $H_3(R^*)$ and 4 of the 6-generators of $H_6(R^*)$ lie in R_+^2 , and, therefore, in N_+^2 . It suffices to prove that given:

$$(S^3)^2 = S^3 \times S^3 \times I \times I \longrightarrow (S^3)^4$$

and the induced homology homomorphism:

$$H_2((S^3)^2) \longrightarrow H_6((S^3)^4)$$

the image of the generator lies in the image of $H_6(N_+^2)$. Consider the commutative diagram:

$$\begin{array}{ccccc}
H_6(N_+^1) & \longrightarrow & H_6((S^3)^2) & \longrightarrow & H_6((S^3)^2, N_+^1) \\
\downarrow & & \downarrow & & \downarrow i_* \\
H_6(N_+^2) & \longrightarrow & H_6((S^3)^4) & \longrightarrow & H_6((S^3)^4, N_+^2) .
\end{array}$$

suffices to prove that $i_* = 0$. By excision, it suffices to prove that:

$$H_6(N_-^1, \partial N_-^1) \xrightarrow{i_*} H_6(N_-^2, \partial N_-^2)$$

is the zero homomorphism.

Let k be the map given by the commutative diagram:

$$\begin{array}{ccc}
(R_-^g \times D_-^3, R_-^g \times S^2) & \xrightarrow{k} & (R_-^{g+1} \times D_-^3, R_-^{g+1} \times S^2) \\
\approx \downarrow w_g & & \approx \downarrow w_{g+1} \\
(N_-^g, \partial N_-^g) & \xrightarrow{i} & (N_-^{g+1}, \partial N_-^{g+1}) .
\end{array}$$

By the properties of the trivializations of section (a):

$$p_2 k = p_2 w_{g+1}^{-1} i w_g = \partial_{g+1} i w_g = \partial_g w_g = p_2 .$$

Consider:

$$p_1 k : R_-^g \times D_-^3 \longrightarrow R_-^{g+1} .$$

Then:

$$p_1 k \approx \left\{ p_1 k \mid_{R_-^g \times (-I)} \right\} \pi = i\pi \quad (\text{inclusion})$$

here π is the obvious projection:

$$\pi : R_-^g \times D_-^3 \longrightarrow R_-^g \times \{-I\} .$$

Hence, $k = i \circ \pi$.

So it suffices to show that the following is zero:

$$(i \times 1)_* : H_6(R_-^1 \times D_-^3, R_-^1 \times S^2) \longrightarrow H_6(R_-^2 \times D_-^3, R_-^2 \times S^2) .$$

Since R_-^1 is $\mathbb{RP}^3$ and:

$$H_*(R_-^2 \times S^2) \longrightarrow H_*(R_-^2 \times D_-^3)$$

is onto, the long exact homology sequences give:

$$\begin{array}{ccccc}
0 & \longrightarrow & H_6(R_-^1 \times D_-^3, R_-^1 \times S^2) & \xrightarrow{j} & H_5(R_-^1 \times S^2) \\
& & \downarrow (i \times 1)_* & & \downarrow (i \times 1)_* \\
0 & \longrightarrow & H_6(R_-^2 \times D_-^3, R_-^2 \times S^2) & \xrightarrow{j} & H_5(R_-^2 \times S^2) .
\end{array}$$

So it suffices to prove the right hand vertical map is zero. In turn, it suffices to prove that:

$$i_* : H_5(R_-^1) \longrightarrow H_5(R_-^2)$$

is zero. This map factors through the inclusion:

$$i_* : H_5(R_-^1) \longrightarrow H_5(S^3 \times S^3) .$$

Recall that R_-^1 is $\mathbb{RP}^3$ and:

$$\begin{aligned}
R_-^1 &= \{(A, B) \in S^3 \times S^3 \mid ABA^{-1}B^{-1} = -I\} \\
&= \{(A, B) \in S^3 \times S^3 \mid ABA^{-1}B^{-1} = -I, \operatorname{tr}(A) = 0, \operatorname{tr}(B) = 0\} \\
&\subset S^3 \times S^2 \subset S^3 \times S^3 .
\end{aligned}$$

Therefore, $i_* = 0$.

We now assume that $g \geq 3$. By induction, we assume that (a) is proved - 1. Let $s \leq g$. Consider a standard generator of $H_{3s}((S^3)^{2g})$:

$$\Sigma = \Sigma_1 * \cdots * \Sigma_{2g}$$

where each Σ_i is either S^3 or $\{I\}$ and exactly s such factors of Σ are S^3 .

Lemma 1: $\Sigma_{2k+1} * \Sigma_{2k+2} = S^3 * \{I\}$ or $\{I\} * S^3$ for some integer k with $0 \leq k \leq g-1$.

For simplicity, we assume that $k = 0$. By assumption:

$$\Sigma \text{ lies in } H_3(R_+^1) \otimes H_{3s-3}((S^3)^{2g-1}).$$

By induction, therefore:

$$\Sigma \text{ lies in the image of } H_3(R_+^1) \otimes H_{3s-3}(N_+^{g-1}).$$

By the image of the obvious imbedding:

$$R_+^1 * N_+^{g-1} \longrightarrow (S^3)^2 * (S^3)^{2g-2} = (S^3)^{2g}$$

clearly lies in N_+^g . Hence, Σ lies in the image of $H_{3s}(N_+^g)$.

Since at least one factor must be $\{I\}$, the only remaining case to treat is the following.

Lemma 2: $\Sigma_{2k+1} * \Sigma_{2k+2} * \Sigma_{2k+3} * \Sigma_{2k+4} = S^3 * S^3 * \{I\} * \{I\}$ or $\{I\} * \{I\} * S^3 * S^3$

for some integer k with $0 \leq k \leq g-2$.

For simplicity, we assume that $k = 0$. By assumption:

$$\Sigma \text{ lies in } H_4((S^3)^4) \otimes H_{3s-6}((S^3)^{2g-4}).$$

By induction, therefore:

$$\Sigma \text{ lies in the image of } H_4(N_+^2) \otimes H_{3s-6}(N_+^{g-2}).$$

Therefore, consider the imbedding.

$$g : N_+^2 * N_+^{g-2} \longrightarrow (S^3)^4 * (S^3)^{2g-4} = (S^3)^{2g}.$$

Since:

$$N_+^k = F_k^{-1}(D_+)$$

let:

$$N_+^k(z) = F_k^{-1}(D_+(z))$$

where:

$$D_+(z) = \{A \in S^3 \mid \text{tr}(A) \geq 2 - z\}$$

where $0 < z < 2$. N_+ can be deformation retracted onto $N_+(z)$ and for some small positive z , $N_+(z)$ is contained in a regular neighborhood of R_+^k . Since:

$$g(R_+^2 * R_+^{g-2}) \subset R_+^g,$$

the image of g deformation retracts into N_+^g . Hence, Σ comes from $H_{3s}(N_+^g)$.

We now prove (b). Suppose $s \leq g-1$. Let Σ be a generator of $H_{3s}((S^3 * S^3)^g)$. Then, for some k , $\Sigma_{2k+1} = \Sigma_{2k+2} = I$. Suppose $k = 0$. Then $\Sigma = (I * I) * \Sigma_3 * \cdots * \Sigma_{2g}$ is homologous to $\Sigma' = x_0 * \Sigma_3 * \cdots * \Sigma_{2g}$ for $x_0 \in R_-^1$. Consider the imbedding:

$$\varphi : (S^3)^{2g-2} \longrightarrow (S^3)^2 \times (S^3)^{2g-2}$$

with $\varphi(y) = (x_0, y)$. Then:

$$\varphi(N_+^{g-1}) \subset N_-^g.$$

Hence, Σ lies in the image of φ and the following commutes:

$$\begin{array}{ccc} H_{3S}((S^3)^{2g-2}) & \xrightarrow{\varphi_*} & H_{3S}((S^3)^{2g}) \\ \uparrow i_* & & \uparrow i_* \\ H_{3S}(N_+^{g-1}) & \xrightarrow{\varphi_*} & H_{3S}(N_-^g) \end{array}$$

Consequently, (a) implies (b). $\square$

We introduce the following notation:

$$\begin{aligned} h_r^g &= \dim H_r(R_-^g) \\ m_r^g &= \dim H_r((S^3)^{2g}) = \begin{cases} \begin{bmatrix} 2g \\ r/3 \end{bmatrix} & \text{if } r \equiv 0(3) \\ 0 & \text{if } r \not\equiv 0(3) \end{cases} \end{aligned}$$

Theorem 1.3: If $r \leq 3g+1$, we have:

$$h_r^{g+1} = h_{r-2}^g + 2h_{r-3}^g + h_{r-4}^g + m_r^g - m_{r-4}^g.$$

Proof: Since: $H_r(N_+^g) \longrightarrow H_r((S^3)^{2g})$ is onto for $r \leq 3g+2$:

$$\begin{aligned} H_r(N_+^g) &\cong H_r((S^3)^{2g}) \oplus H_{r+1}((S^3)^{2g}, N_+^g) \\ &\stackrel{\text{exc}}{=} H_r((S^3)^{2g}) \oplus H_{r+1}(N_-^g, \partial N_-^g) \\ &\stackrel{\text{Thom}}{=} H_r((S^3)^{2g}) \oplus H_{r-2}(R_-^g), \quad (\text{since } N_-^g \cong R_-^g \times D^3) \end{aligned}$$

when $r \leq 3g+1$. Since:

$$H_r(N_-^g) \longrightarrow H_r((S^3)^{2g})$$

is onto when $r \leq 3g-1$, we conclude that:

$$H_r(N_-^g) \cong H_r((S^3)^{2g}) \oplus H_{r+1}(N_+^g, \partial N_+^g)$$

when $r \leq 3g-2$.

Hence:

$$\begin{aligned} \dim H_r(N_+^g) &= h_{r-2}^g + m_r^g \\ \dim H_{r+1}(N_+^g, \partial N_+^g) &= h_r^g - m_r^g. \end{aligned}$$

From the discussion preceding Theorem 1.2:

$$h_r^{g+1} = (h_{r-2}^g + m_r^g) + 2h_{r-3}^g + (h_{r-4}^g - m_{r-4}^g) \quad \square$$

Theorem 1.4:

$$\begin{aligned} \text{(a)} \quad h_r^g &= \begin{cases} \begin{bmatrix} s/3 \\ j=s-g+1 \end{bmatrix} \begin{bmatrix} 2g \\ 2j \end{bmatrix} & \text{if } r = 2s, \quad s \leq \left\lfloor \frac{3g}{2} \right\rfloor - 1 \\ \begin{bmatrix} (s-1)/3 \\ j=s-g+1 \end{bmatrix} \begin{bmatrix} 2g \\ 2j+1 \end{bmatrix} & \text{if } r = 2s+1, \quad s \leq \left\lfloor \frac{3(g-1)}{2} \right\rfloor \end{cases} \\ \text{(b)} \quad h_r^g &= h_{s-g-1}^g \quad \text{if } r \geq 3g-1 \end{aligned}$$

Proof: The proof is by induction on g . For $g=1$, R_-^1 is $\mathbb{RP}^3$ and (a) and (b) hold. By Theorem 1.3, for $r \leq 3g+1$:

$$h_{r-2}^{g+1} = h_{r-2}^g + 2h_{r-3}^g + h_{r-4}^g + m_{r-4}^g - m_{r-5}^g .$$

, after the appropriate restriction on m , we have:

$$h_{em}^{g+1} = h_{em-2}^g + 2h_{em-3}^g + h_{em-4}^g + \left[\begin{smallmatrix} 2g \\ 2m \end{smallmatrix} \right]$$

$$h_{em+1}^{g+1} = h_{em-1}^g + 2h_{em-2}^g + h_{em-3}^g - \left[\begin{smallmatrix} 2g \\ 2m-1 \end{smallmatrix} \right]$$

$$h_{em+2}^{g+1} = h_{em}^g + 2h_{em-1}^g + h_{em-2}^g$$

$$h_{em+3}^{g+1} = h_{em+1}^g + 2h_{em}^g + h_{em-1}^g + \left[\begin{smallmatrix} 2g \\ 2m+1 \end{smallmatrix} \right]$$

$$h_{em+4}^{g+1} = h_{em+2}^g + 2h_{em+1}^g + h_{em}^g - \left[\begin{smallmatrix} 2g \\ 2m \end{smallmatrix} \right]$$

$$h_{em+5}^{g+1} = h_{em+3}^g + 2h_{em+2}^g + h_{em+1}^g .$$

will prove formula (a) by induction on g . We give the proof when $r = em$.

1 the other cases are proved similarly. From above:

$$h_{em}^{g+1} = h_{2(em-1)}^g + 2h_{2(em-2)+1}^g + h_{2(em-2)}^g + \left[\begin{smallmatrix} 2g \\ 2m \end{smallmatrix} \right] .$$

sume:

$$3m \leq \left[\frac{3(g+1)}{2} \right] - 1 .$$

is implies:

$$3m-2 \leq 3m-1 \leq \left[\frac{3g}{2} \right] - 1 ,$$

$$3m-2 \leq \left[\frac{3(g-1)}{2} \right] .$$

nce, by induction, we have:

$$\begin{aligned} h_{em}^{g+1} &= \sum_{j=3m-g}^{m-1} \left[\begin{smallmatrix} 2g \\ 2j \end{smallmatrix} \right] + 2 \sum_{j=3m-g-1}^{m-1} \left[\begin{smallmatrix} 2g \\ 2j+1 \end{smallmatrix} \right] + \sum_{j=3m-g-1}^{m-1} \left[\begin{smallmatrix} 2g \\ 2j \end{smallmatrix} \right] + \left[\begin{smallmatrix} 2g \\ 2m \end{smallmatrix} \right] \\ &= \sum_{i=3m-g}^m \left[\begin{smallmatrix} 2g \\ 2i-2 \end{smallmatrix} \right] + \sum_{i=3m-g}^m 2 \left[\begin{smallmatrix} 2g \\ 2i-1 \end{smallmatrix} \right] + \sum_{i=3m-g}^{m-1} \left[\begin{smallmatrix} 2g \\ 2i \end{smallmatrix} \right] + \left[\begin{smallmatrix} 2g \\ 2m \end{smallmatrix} \right] \\ &= \sum_{i=3m-g}^m \left[\left[\begin{smallmatrix} 2g \\ 2i-2 \end{smallmatrix} \right] + \left[\begin{smallmatrix} 2g \\ 2i-1 \end{smallmatrix} \right] + \left[\begin{smallmatrix} 2g \\ 2i-1 \end{smallmatrix} \right] + \left[\begin{smallmatrix} 2g \\ 2i \end{smallmatrix} \right] \right] + \left[\begin{smallmatrix} 2g \\ 2m \end{smallmatrix} \right] - \left[\begin{smallmatrix} 2g \\ 2m \end{smallmatrix} \right] \\ &= \sum_{i=3m-g}^m \left[\left[\begin{smallmatrix} 2g+1 \\ 2i-1 \end{smallmatrix} \right] + \left[\begin{smallmatrix} 2g+1 \\ 2i \end{smallmatrix} \right] \right] = \sum_{i=3m-g}^m \left[\begin{smallmatrix} 2g+2 \\ 2i \end{smallmatrix} \right] \\ &= \sum_{i=3m-(g+1)+1}^m \left[\begin{smallmatrix} 2(g+1) \\ 2i \end{smallmatrix} \right] , \text{ which is (a) .} \end{aligned}$$

To prove (b), notice that:

$$\dim(R_{-}^g) = 6g - 3 .$$

Hence, Poincare duality gives (b). $\square$

Corollary 1.5: $h_{3g-1}^g = h_{3g-2}^g = 0$ and $h_1^g = 0$.

Proof: The first equality comes from (b) of Theorem 1.4. If g is odd:

$$3g-2 = 2 \left[\frac{3(g-1)}{2} \right] + 1 .$$

If g is even:

$$3g-2 = 2 \left[\frac{3g}{2} - 1 \right] .$$

Hence, Theorem 1.4(a) gives $h_{3g-2}^g = 0$. Likewise, $h_1^g = 0$. $\square$

Consider the fibration:

$$\begin{array}{ccc} SO_3 & \longrightarrow & R_{-}^g \\ & & \downarrow p \\ & & \tilde{R}_{-}^g . \end{array}$$

Lemma 1.6: If $r \leq 3g - 2$, then:

$$p_* : H_r(\hat{R}_-^g; \mathbb{Q}) \longrightarrow H_r(\hat{R}_-^g; \mathbb{Q})$$

an injection.

Proof: Since $SO_3 = \mathbb{RP}^3$ is a $\mathbb{Q}$ -homology 3-sphere, there is a Thom-Gysin exact sequence:

$$\longrightarrow H^r(\hat{R}_-^g) \xrightarrow{p^*} H^r(\hat{R}_-^g) \xrightarrow{\rho} H^{r-3}(\hat{R}_-^g) \xrightarrow{u_X} H^{r+1}(\hat{R}_-^g) \longrightarrow \dots$$

where all cohomologies are taken with $\mathbb{Q}$ -coefficients and:

$$X \in H^3(\hat{R}_-^g) \dots$$

we also make the convention that the rest of the homologies in this proof are taken with $\mathbb{Q}$ coefficients.

Since:

$$H^*(X; \mathbb{Q}) \cong \text{Hom}(H_*(X); \mathbb{Q})$$

it suffices to prove that:

$$p^* : H^r(\hat{R}_-^g) \longrightarrow H^r(\hat{R}_-^g)$$

is onto for $r \leq 3g - 2$. In turn, it is enough to prove that the map:

$$u_X : H^{r-3}(\hat{R}_-^g) \longrightarrow H^{r+1}(\hat{R}_-^g)$$

is injective for $r \leq 3g - 2$. We prove this by descending induction on r , starting with the cases $r = 3g - 2$ and $r = 3g - 3$. Consider:

$$H^{3g-2}(\hat{R}_-^g) \longrightarrow H^{3g-3}(\hat{R}_-^g) \xrightarrow{u_X} H^{3g-1}(\hat{R}_-^g) \xrightarrow{p^*} H^{3g-1}(\hat{R}_-^g)$$

By Corollary 1.5, the two outside groups are zero, so u_X is an isomorphism. Similarly:

$$u_X : H^{3g-4}(\hat{R}_-^g) \longrightarrow H^{3g-2}(\hat{R}_-^g)$$

is onto. Since $\hat{R}_-^g$ is a Kähler manifold [N1], there is $w \in H^2(\hat{R}_-^g)$ such that:

$$u_W : H^r(\hat{R}_-^g) \longrightarrow H^{r+2}(\hat{R}_-^g)$$

is injective for $r \leq 3g - 4$. Hence:

$$H^{3g-6}(\hat{R}_-^g) \xrightarrow{u_W} H^{3g-4}(\hat{R}_-^g) \xrightarrow{u_W} H^{3g-2}(\hat{R}_-^g)$$

is injective and:

$$\dim H^{3g-6}(\hat{R}_-^g) \leq \dim H^{3g-2}(\hat{R}_-^g).$$

Hence:

$$u_X : H^{3g-6}(\hat{R}_-^g) \longrightarrow H^{3g-2}(\hat{R}_-^g)$$

is an isomorphism.

Now assume $r \leq 3g - 4$ and:

$$u_X : H^{3g-3}(\hat{R}_-^g) \longrightarrow H^{3g-1}(\hat{R}_-^g)$$

is injective for all $r \leq 3g - 2$. Consider:

$$u_W u_X : H^{r-3}(\hat{R}_-^g) \longrightarrow H^{r+1}(\hat{R}_-^g).$$

It can be written as compositions in two different ways

$$\begin{aligned} H^{r-3}(\hat{R}_-^g) &\xrightarrow{u_X} H^{r+1}(\hat{R}_-^g) \xrightarrow{u_W} H^{r+3}(\hat{R}_-^g) \\ H^{r-3}(\hat{R}_-^g) &\xrightarrow{u_W} H^{r-1}(\hat{R}_-^g) \xrightarrow{u_X} H^{r+1}(\hat{R}_-^g). \end{aligned}$$

the second factorization, both homomorphisms are injective by the properties of w and by induction. Hence:

$$H^{r-1}(\hat{R}_-^g) \xrightarrow{u_X} H^{r+1}(\hat{R}_-^g) \text{ is injective.} \quad \square$$

ollary 1.7: $p^* : H^r(\hat{R}_-^g) \longrightarrow H^r(R_-^g)$ is an isomorphism for $r \neq 3$ and 2.

pf: Consider again the Thom-Gysin sequence:

$$\longrightarrow H^{r-4}(\hat{R}_-^g) \xrightarrow{u_X} H^r(\hat{R}_-^g) \xrightarrow{p^*} H^r(\hat{R}_-^g) \longrightarrow H^r(R_-^g) \longrightarrow \dots$$

course, the first term is zero. Hence, p^* is injective for $r \neq 3$. On the other hand, by Theorem 1.6, p^* is surjective for $r \neq 3$ (provided $3g-2$). $\square$

ollary 1.8: For $r \leq 3g-2$:

$$\dim H_r(\hat{R}_-^g) = \sum_{j=0}^{\lfloor r/4 \rfloor} h_{r-4j}^g.$$

pf: By Theorem 1.6, for $r \leq 3g-2$, we have a short exact sequence:

$$0 \longrightarrow H^{r-4}(\hat{R}_-^g) \xrightarrow{u_X} H^r(\hat{R}_-^g) \xrightarrow{p^*} H^r(R_-^g) \longrightarrow 0.$$

$$k_r^g = \dim H^r(\hat{R}_-^g).$$

$$k_r^g = k_{r-4}^g + h_r^g.$$

s gives the required formula. $\square$

Consider the maps:

$$i_+ : N_+^g \longrightarrow N_+^{g+1} \quad i_+(x) = (x, I, I)$$

$$i_- : R_-^g \longrightarrow R_-^{g+1} \quad i_-(x) = (x, I, I)$$

$$j : S^2 \longrightarrow R_-^g * S^2 \quad j(x) = (\langle \langle 2, I, \dots, I \rangle, x \rangle).$$

Lemma 1.9: The maps $\bar{w}_g \circ j$ and i_+ induce isomorphisms on H^2 . If $g \geq 2$, i_- also induces an isomorphism.

Proof: Since, by Theorem 1.2:

$$i_* : H_3(N_+^g) \longrightarrow H_3((S^3)^{2g})$$

is surjective:

$$i^* : H^3((S^3)^{2g}) \longrightarrow H^3(N_+^g)$$

is injective. Hence, Mayer-Vietoris gives:

$$0 \longrightarrow H^2(N_+^g) \oplus H^2(N_-^g) \xrightarrow{\cong} H^2(R_-^g * S^2) \longrightarrow 0.$$

The map is induced by:

$$x \longmapsto (\bar{w}_g(x), w_b(x)).$$

Since the image of:

$$w_g^* : H^2(N_-^g) \longrightarrow H^2(R_-^g * S^2)$$

is the kernel of j^* , $(\bar{w}_g \circ j)^*$ is an isomorphism.

Consider the diagram:

$$\begin{array}{ccccccc} S^2 & \xrightarrow{j} & R_-^g * S^2 & \xrightarrow{\bar{w}_g} & N_+^g & \xrightarrow{\alpha} & R_-^{g+1} \\ \downarrow 1 & & \downarrow i_- \times 1 & & \downarrow i_+ & & \downarrow i_- \\ S^2 & \xrightarrow{j} & R_-^{g+1} * S^2 & \xrightarrow{\bar{w}_{g+1}} & N_+^{g+1} & \xrightarrow{\alpha} & R_-^{g+2} \end{array}$$

commutes except on the middle square which is homotopy commutative, i.e.:

$$(\bar{w}_{g+1})^{-1} \cdot i_+ \cdot \bar{w}_g : R_-^g \times S^2 \longrightarrow R_-^{g+1} \times S^2$$

homotopic to $i_- \times 1$. (This follows from the proof of Theorem 1.2(a) by observing that the homotopy of the map:

$$R_-^g \times D_-^3 \xrightarrow{k} R_-^{g+1} \times D_-^3$$

the inclusion:

$$R_-^g \times D_-^3 \xrightarrow{i_- \times 1} R_-^{g+1} \times D_-^3$$

be taken to fix the second factor. Hence, it restricts to the desired homotopy on $R_-^g \times S^2$.)

Hence, we have the commutative diagram:

$$\begin{array}{ccccc} H^2(S^2) & \xleftarrow[\cong]{(\bar{w}_g \cdot j)^*} & H^2(N_+^g) & \xleftarrow[\cong]{\alpha^*} & H^2(R_-^{g+1}) \\ \uparrow = & & \uparrow i_+^* & & \uparrow i_-^* \\ H^2(S^2) & \xleftarrow[\cong]{(\bar{w}_{g+1} \cdot j)^*} & H^2(N_+^{g+1}) & \xleftarrow[\cong]{\alpha^*} & H^2(R_-^{g+1}) \end{array}$$

i_+ and i_- induce isomorphisms on H^2 . $\square$

Let λ be the generator of $H^2(N_+^1)$. Also, let λ denote the generators $H^2(N_+^g)$ and $H^2(R_-^g)$ for $g \geq 2$ induced by the isomorphism of the above maps. Let $\mu_1, \dots, \mu_{2g}$ be the elements of $H^2(N_+^g)$ and $H^2(R_-^g)$ induced by inclusions:

$$N_+^g \longrightarrow (S^2)^{2g} \longleftarrow R_-^g$$

of $(S^2)^{2g}$. Likewise, using Corollary 1.7, let these symbols denote classes in $H^*(R_-^g)$, $g \geq 2$.

We have natural inclusions:

$$N_+^{2g} \times N_+^1 \longrightarrow (S^2)^{2g} \times (S^2)^2 = (S^2)^{2(g+1)}.$$

As in the proof of Theorem 1.2(a), Case 2, we can deform this map into a map:

$$\ell : N_+^g \times N_+^1 \longrightarrow N_+^{g+1}.$$

Lemma 1.10: (1) $\ell^*(\lambda^P) = \lambda^P \otimes 1 + p\lambda^{p-1} \otimes \lambda$

$$(2) \quad \ell^*(\mu_i) = \begin{cases} \mu_i \otimes 1 & \text{for } i \leq 2g \\ 1 \otimes \mu_{i-2g} & \text{for } i = 2g+1, 2g+2 \end{cases}$$

Proof: Part (2) is trivial. To prove part (1), we begin by observing that the discussion preceding Theorem 5.1 implies:

$$H_1(N_+^g) \cong H_1(R_-^{g+1}) = 0 \quad \text{for } g \geq 1.$$

Hence, we compute that:

$$H^2(N_+^g \times N_+^1) \cong H^2(N_+^g) \oplus H^2(N_+^1).$$

We may write the pullback of λ as follows:

$$\ell^*(\lambda) = a(\lambda \otimes 1) \oplus b(1 \otimes \lambda).$$

Now we have the following diagrams:

$$\begin{array}{ccc} N_+^g & \xrightarrow{i'} & N_+^g \times N_+^1 \\ & \searrow i & \downarrow \ell \\ & & N_+^{g+1} \end{array} \quad \begin{array}{l} i(x) = (x, 1) \\ i'(x) = (x, 1) \end{array}$$

$$\begin{array}{ccc} N_-^1 & \xrightarrow{j'} & N_-^g \times N_-^1 \\ & \searrow j & \downarrow \ell \\ & & N_-^{g+1} \end{array} \quad \begin{array}{l} j(x) = (1, x) \\ j'(x) = (1, x) \end{array}$$

choosing the deformation to ℓ carefully, we may insure that these maps commute up to homotopy. Hence, we have commutative diagrams:

$$\begin{array}{ccc} H^2(N_+^{g+1}) & \xrightarrow{\ell^*} & H^2(N_+^g \times N_+^1) \\ & \searrow i^* & \downarrow i'^* \\ & & H^2(N_+^g) \end{array}$$

$$\begin{array}{ccc} H^2(N_-^{g+1}) & \xrightarrow{\ell^*} & H^2(N_-^g \times N_-^1) \\ & \searrow j^* & \downarrow j'^* \\ & & H^2(N_-^g) \end{array}$$

definition, $i^*(\lambda) = \lambda$ and $j^*(\lambda) = \lambda$. Hence, we conclude that:

$$\lambda = i^*(\lambda) = (i')^* \ell^*(\lambda) = a\lambda$$

$$\lambda = j^*(\lambda) = (j')^* \ell^*(\lambda) = b\lambda$$

This implies that $a = 1$ and $b = 1$:

$$\ell^*(\lambda) = \lambda \otimes 1 \otimes 1 \otimes \lambda.$$

Since $H^*(N_+^1) = 0$, by the proof of Theorem 1.3, we conclude, by induction on p , that:

$$\ell^*(\lambda^p) = \lambda^p \otimes 1 \otimes p\lambda^{p-1} \otimes \lambda. \quad \square$$

Proposition 1.11: The elements $\lambda^p \mu_{q_1} \cdots \mu_{q_r}$ in $H^*(N_+^g)$, where $p \geq 0$, $1 \leq q_1 < \cdots < q_r \leq 2g$, and $p + r \leq g$, are linearly independent.

Proof: The proof is by induction on g . Assume that the result holds for $g-1 \geq 1$. Recall:

$$\ell : N_+^{g-1} \times N_+^1 \longrightarrow N_+^g.$$

Let Σ be a linear combination of elements of the form $\lambda^p \mu_{q_1} \cdots \mu_{q_r}$. Assume $\Sigma = 0$. Look at the component of $\ell^*(\Sigma)$ in:

$$H^*(N_+^{g-1}) \otimes H^2(N_+^1).$$

By induction on g and Lemma 1.10, we conclude that the only terms in Σ which can have nonzero coefficients are of one of the following forms:

(a) the ones with $p = 0$

(b) the ones containing μ_{2g-1} or μ_{2g} .

We can find an automorphism of N_+^{g-1} which takes μ_{2g-1} and μ_{2g} to any μ_{i-1} , μ_{2i} for all i , $1 \leq i \leq g$. Hence, we can replace (b) by:

(b') the ones containing at least one of μ_{2i-1} , μ_{2i} for all i , $1 \leq i \leq g$.

Hence, all terms of Σ satisfy (a) and are in the form:

$$\mu_{q_1} \cdots \mu_{q_r} \quad r \leq g.$$

at the inclusion induces an injection:

$$H^r((S^3)^{2g}) \longrightarrow H^r(N_+^g)$$

or $r \leq 3g + 2$ and such elements are linearly independent in $H^*((S^3)^{2g})$. Hence, they are linearly independent in $H^*(N_+^g)$. The case of genus one is true by inspection. $\square$

Corollary 1.12: The element λ generates $H^2(N_+^g)$. The element λ^2 generates $H_+^4(N^g)$.

Proof: From Theorem 1.2 and the long exact sequence for a pair, we deduce the short exact sequence:

$$0 \longrightarrow H_{r+1}((S^3)^{2g}, N_+^g) \longrightarrow H_r(N_+^g) \longrightarrow H_r((S^3)^{2g}) \longrightarrow 0$$

for $r \leq 3g + 1$. Hence, we deduce that:

$$\begin{aligned} H_2(N_+^g) &\cong H_2((S^3)^{2g}, N_+^g) \\ &\cong H_2(N_-^g, N_+^g) \quad (\text{excision}) \\ &\cong H_0(R_-^g) \quad (\text{Thom}). \end{aligned}$$

Consequently, $H^2(N_+^g)$ has rank one and the result follows from Proposition 1.11.

In a similar manner, we deduce that:

$$H_4(N_+^g) \cong H_4(R_-^g).$$

By Theorem 1.4, we conclude that the rank of $H^4(N_+^g)$ is zero in genus one and one in genus ≥ 2 . Again, the result follows from Proposition 1.11. $\square$

Proposition 1.13: The elements $\lambda^p \mu_{q_1} \cdots \mu_{q_r}$, $p \geq 0$, $1 \leq q_1 < \cdots < q_r \leq 2g$, $p + r \leq g - 1$ are linearly independent in $H^*(R_-^g)$.

Proof: This is obvious for $g = 1$. Let Σ be a linear combination of elements as above such that $\Sigma = 0$. As above, the only terms with non-zero coefficients are in the form (b)'. But $p + r \leq g - 1$ implies that there are no such terms. $\square$

Corollary 1.14: The elements $\lambda^p \mu_{q_1} \cdots \mu_{q_r}$, $p \geq 0$,

$$1 \leq q_1 < q_2 < \cdots < q_r \leq 2g,$$

$p + r \leq g - 1$ generate $H^*(R_-^g)$.

Proof: By Theorem 1.4, we know the rank h_r^g of $H^r(R_-^g)$ satisfies:

$$h_r^g \leq \#\{\lambda^p \mu_{q_1} \cdots \mu_{q_r} \mid p \geq 0, 1 \leq q_1 < \cdots < q_r \leq 2g, p + r \leq g - 1\}.$$

Hence, the result follows. $\square$

Corollary 1.15: The element λ generates $H^2(R_-^g)$. The element λ^2 generates $H^4(R_-^g)$.

of: Both claims are obvious for $g = 1$. The first claim follows from
 ollary 1.14 for $g \geq 2$. The second claim follows for $g \geq 3$. On the
 or hand, by Theorem 1.4:

$$H^*(R_-^2) = 0.$$

ce, the second claim is also true when $g = 2$. $\square$

ollary 1.16: For $r \leq 3g - 2$, $H^r(\hat{R}_-)$ is generated by $\lambda, \mu_1, \dots, \mu_{2g}$ and
 (as a cohomology ring).

of: For $r \leq 3g - 2$, by Theorem 1.6, we have the exact sequence:

$$0 \longrightarrow H^{r-4}(\hat{R}_-^g) \xrightarrow{u_X} H^r(\hat{R}_-^g) \xrightarrow{p^*} H^r(R_-^g) \longrightarrow 0,$$

re $\chi \in H^4(\hat{R}_-^g)$. Hence, the proof follows by induction on r . $\square$

Since $\hat{R}_-$ is a nonsingular projective variety of complex dimension
 -3 [N1], it follows from the Hard Lefschetz Theorem ([GH], page 122)
 t we have a stronger result:

ollary 1.17: $H^*(\hat{R}_-)$ is generated by $\lambda, \mu_1, \dots, \mu_{2g}$ and χ (as a ring).

The action of the Torelli group

The action of the Torelli group

Recall that every base point preserving, homeomorphism, $f : (F^*, 0) \longrightarrow$
 $, 0)$ induces an homomorphism of R^* :

$$\begin{aligned} f^* : R^* &\longrightarrow R^* \\ p &\longmapsto p \circ f_* \end{aligned}$$

re f_* is the induced homomorphism on $H_1(F^*, 0)$. If in addition, f is

orientation preserving, then f preserves the boundary of F^* and we have a
 commutative diagram:

$$\begin{array}{ccc} R^* & \xrightarrow{d} & S^1 \\ f^* \downarrow & & \downarrow \text{id} \quad (f\text{-orient. pres.}) \\ R^* & \xrightarrow{d} & S^1 \end{array}$$

Note: If f is orientation reversing, then we must replace the right hand
 vertical arrow by the inversion map, $A \longmapsto A^{-1}$.

By Proposition 7, it follows that f^* preserves the orientation of R^* .

Note: If f is orientation reversing, this assertion is dependent on the
 parity of g .

From the previous diagram and remark, we obtain from every base-point
 preserving, orientation preserving homeomorphism, f , the following restric-
 tions:

$$\begin{aligned} f^* : R_+^g &\longrightarrow R_+^g \\ f^* : N_+^g &\longrightarrow N_+^g \end{aligned}$$

From the compatibility with ∂ and the previous remark, it follows that all
 these induced maps preserve orientations.

Since these maps commute with the SO_3 action, we obtain orientation
 preserving maps of orbit spaces:

$$\hat{f} : \hat{R}_+^g \longrightarrow \hat{R}_+^g$$

$$\hat{f} : \hat{N}_+^g \longrightarrow \hat{N}_+^g$$

The mapping class group $\Gamma(F^*, 0)$ is defined by:

$$\Gamma(F^*, 0) = \pi_0(\text{Homeo}^+(F^*, 0))$$

where $\text{Homeo}^+(F^*, 0)$ is the group of orientation preserving homeomorphisms of $(F^*, 0)$ with the compact open topology. We denote the mapping class of f by f . It is evident that the maps induced from f depend only on the mapping class of f . Hence, we have well defined orientation preserving actions of $\Gamma(F^*, 0)$ on these various spaces.

The Torelli group $T(F^*, 0)$ is defined by:

$$T(F^*, 0) = \{f \in \Gamma(F^*, 0) \mid f_* = \text{id} : H_1(F^*, \mathbb{Z}) \longrightarrow H_1(F^*, \mathbb{Z})\}$$

The above actions, of course, restrict to $T(F^*, 0)$.

(b) Triviality on the homology of $\hat{R}_+^g$

We wish to show that the Torelli group acts trivially on the rational homology of $\hat{R}_+^g$. Since rational homology and cohomology are dual, we can consider rational cohomology. On the other hand, by Corollary 1.17, it suffices to consider dimensions 2, 3 and 4. We begin with dimension 3. (All homology and cohomology will be assumed to be rational.)

Lemma 2.1: $T(F^*, 0)$ acts trivially on $H^3(\hat{R}_+^g)$.

Proof: By Proposition 1.3.1, $T(F^*, 0)$ acts trivially on $H^3(R^*)$. Since R_-^g is homotopy equivalent to N_-^g , Theorem 1.2 implies that we have a surjection:

$$H_2(R_-^g) \xrightarrow{i_*} H_2(R^*)$$

On the other hand, by Theorem 1.4, the rank of $H_2(R_-^g)$, h_2^g , is $2g$. Since this is clearly the rank of $H_2(R^*)$, the inclusion induces an isomorphism. Hence, we have an isomorphism:

$$H^3(R^*) \xrightarrow[\cong]{i^*} H^3(R_-^g)$$

By Corollary 1.7, the natural projection induces an isomorphism (provided $g \geq 2$):

$$H^3(\hat{R}_+^g) \xrightarrow[\cong]{p^*} H^3(R_-^g)$$

Since these identifications are compatible with the actions, we deduce that $T(F^*, 0)$ acts trivially on $H^3(\hat{R}_+^g)$ provided $g \geq 2$. The case of genus one is obvious. $\square$

We now consider dimension 2.

Lemma 2.2: $T(F^*, 0)$ acts trivially on $H^2(\hat{R}_+^g)$.

Proof: As usual, we may assume that $g \geq 2$. By Corollary 1.7, it suffices to consider $H_2(R_-^g)$. Tracing through the identification:

$$H_4(R_+^g) \xrightarrow[\cong]{} H_2(R_-^g)$$

given in the proof of Corollary 1.12, we find that this isomorphism is also compatible with the actions. (The reader should convince himself of this

section for the Thom isomorphism. Here we use the fact that the actions are orientation preserving.) Hence, we may reduce the problem to $H_*(N_+^R)$. By Corollary 1.12, on the other hand, it suffices to consider $H_2(N_+^R)$.

Again, using the identifications outlined in Proposition 1.11, we have:

$$H_2(N_+^R) \xrightarrow{\cong} H_0(R_-^R)$$

which reduces the proof to $H_0(R_-^R)$. But certainly $T(F^*, 0)$ acts trivially on $H_0(R_-^R)$. $\square$

Finally, we consider dimension 4.

Lemma 2.3: $T(F^*, 0)$ acts trivially on $H^*(\hat{R}_-^R)$.

Proof: From Corollary 1.17, Lemma 2.1 and Lemma 2.2, the proof is reduced to considering the action of $T(F^*, 0)$ on the element x . But x is the Euler-class of the bundle:

$$\begin{array}{ccc} SO_3 & \longrightarrow & R_-^R \\ & & \downarrow \\ & & \hat{R}_-^R \end{array}$$

orientable sphere-bundle. Since $T(F^*, 0)$ acts orientably on this bundle, we conclude that $T(F^*, 0)$ preserves x . $\square$

As observed at the beginning of this part, we can deduce the following result from Corollary 1.17, Lemma 2.1, 2.2, and 2.3.

Theorem 2.4: $T(F^*, 0)$ acts trivially on $H^*(\hat{R}_-^R)$.

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