A CONVEX DECOMPOSITION THEOREM FOR FOUR-MANIFOLDS

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Abstract. In this article we show that every smooth closed oriented four-manifold admits a decomposition into two submanifolds along common boundary. Each of these submanifolds is a complex manifold with pseudo-convex boundary. This imply, in particular, that every smooth closed simply-connected four-manifold is a Stein domain in the the complement of a certain contractible 2-complex.

1. Introduction

Exact manifold with pseudo-convex boundary (PC manifold, for short) is a compact complex manifold X, which admits strictly pluri-subharmonic Morse function ψ, such that set of maximum points of ψ coincides with the boundary ∂X. We prefer the term PC manifold, since combination of words "*compact Stein* manifold" is likely to precipitate heart palpitations in some mathematicians.

Such manifold admits a symplectic structure $\omega = \frac{i}{2}\partial\overline{\partial}\psi$ and serves as an analogue of closed symplectic manifold. Boundary ∂X of PC manifold X inherits a contact structure ξ, which, in this case, is a distribution of maximal complex subspaces in $\boldsymbol{T}X$ tangent to ∂X. In dimension four analogy between PC manifolds and closed symplectic manifolds is further illustrated by the following two theorems in terms of Seiberg-Witten invariants.

Theorem 1a. (C. Taubes, [T]) *Let (X, ω) be a closed, symplectic four-manifold and K be Chern class of the canonical bundle of an almost complex structure compatible with ω. Then $SW_X(K) = \pm 1$.*

In the relative case Kronheimer and Mrowka proved the following theorem:

Theorem 1b. (P. Kronheimer, T. Mrowka, [KM]) *Let (X, ω) be compact, symplectic manifold and ξ be a positive contact structure on ∂X compatible with ω. Then $SW_X(K) = 1$, where K is the canonical class of ω.*

In particular, it was shown by P. Kronheimer and T. Mrowka that properly embedded surface in PC manifold satisfies Eliashberg-Bennequin inequality analogous to adjunction inequality in the case of closed symplectic manifold, once proper conditions on the boundary of the surface are imposed. Namely, if $F \subset X$ is a properly embedded surface, such that $\alpha = \partial F \subset \partial X$ is a Legendrian knot with respect to induced contact structure on ∂X and f is framing on α induced by a trivialization

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of the normal bundle of F in X, then

$$[tb(\alpha) - f] + |rot(\alpha, F)| \leq -\chi(F),$$

where $tb(\alpha)$ is Thurston-Bennequin framing defined by the vector field along α transversal to α and tangent to contact distribution. To define rotation number observe that $\boldsymbol{T}X|_{\partial X} \cong \xi \oplus \mathbb{C}$ and thus $\xi \cong \Lambda^2 \boldsymbol{T}X|_{\partial X}$. This allows us to view $tb(\alpha)$ as a section of $\Lambda^2 \boldsymbol{T}X|_{\alpha}$. Then $rot(\alpha, F)$ of Legendrian knot α bounding oriented proper surface $F \subset X$ is defined to be an obstruction to extend vector field $tb(\alpha)$ to a non-vanishing section of $\Lambda^2 \boldsymbol{T}X|_F$, i.e.

$$rot(\alpha, F) = c_1(\Lambda^2 \boldsymbol{T}X|_F, tb(\alpha)) \frown [F, \partial F].$$

Details could be found, for example, in [AM].

2. Whitehead multiple of a knot

Suppose (K, f) is a framed knot in an oriented 3-manifold M. *Positive Whitehead multiple* $P_n(K, f)$ of knot K is a band connected sum of n parallel (according to framing f) copies of K equipped with alternating orientations as on Figure 1 (we assume usual orientation of $\mathbb{R}^3$). We shall omit framing f from the notation when

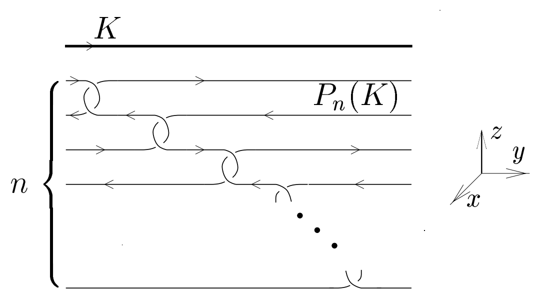

FIGURE 1. Positive Whitehead double of knot K

it is clear from the context or irrelevant to the discussion.

This construction is a generalization of Whitehead double of a knot, namely $P_2(K) = Wh(K)$.

Here we list some properties of the Whitehead multiple $P_n(K, f)$.

1. The homology class of $P_n(K, f)$ is either that of K if n is odd, or zero if n is even. In fact, for odd values of n, $P_n(K, f)$ is homotopic to K. In both cases, $P_n(K, f)$ can be naturally equipped with a framing, which we also call f. It is framing corresponding to the given framing of K in case of odd n (this is defined correctly because K and $P_n(K, f)$ are in the same homology class). Canonical framing of $P_n(K, f)$ for even n is just a zero framing, which is well-defined for the closed curve homologous to zero.

2. The key property of the above operation is that given a Legendrian knot K in tight contact 3-manifold with zero Thurston-Bennequin invariant we can produce another Legendrian knot in small neighborhood of K homotopic to K and with an arbitrary large Thurston-Bennequin invariant. More precisely, let (K, f) be framed Legendrian knot in tight contact manifold M. Let $K_1, \ldots, K_n$ be parallel copies of K corresponding to framing f. Isotopy class of link $(K_1, \ldots, K_n)$ has a Legendrian representative with $K_1 = K$ and $tb(K_i) - f = -|tb(K) - f|$,

$i = 2, \dots, n$. This is obvious for K being Legendrian unknot with $tb(K) = 1$ and $rot(K) = 0$. The general case follows from the fact that any two Legendrian knots have contactomorphic neighborhoods. Different proof is given in [AM1].

Each band in the construction of $P_n(K, tb(K))$ contributes 1 into $tb(P_n(K, tb(K)))$, thus we have

$$tb(P_n(K, tb(K))) - tb(K) = n - 1.$$

One has to choose n to be odd to make $P_n(K, f)$ homotopic to K.

3. Suppose knot $K \subset M^3$ is a boundary of properly embedded disc D in smooth 4-manifold X with $\partial X = M^3$. Taking a ribbon sum of n copies of disc D, using the same ribbons, which are used to construct $P_n(K)$, we obtain another disc $P_n(D)$. It is bounded by $P_n(K, f)$, where f is the framing induced by trivialization of the normal bundle $\nu_X(D)$ of D in X. Note also, that disc $P_n(D)$ can be constructed in arbitrary small neighborhood of D.

4. If (l_1, l_2) is a Hopf link then link $(P_n(l_1, 0), l_2)$ in S^3 is symmetric (i.e. there is a diffeomorphism of S^3 interchanging components of the link). Moreover, two triads $(B^4, P_n(D_1), D_2)$ and $(B^4, D_1, P_n(D_2))$ are diffeomorphic. Here, D_1 and D_2 is a pair of linear 2-discs in B^4 intersecting at one point.

Paying homage to the popularity of physics terminology first author suggested to call manifold W_n on Figure 2 a positron. Manifold W_n is obtained from B^4 by removing open regular neighborhood of D_1 (this has an effect of turning B^4 to $S^1 \times B^3$) and then attaching a 2-handle to the framed knot $P_n(\partial D_2)$. It is contractible if n is odd and has PC structure if $n \geq 2$.

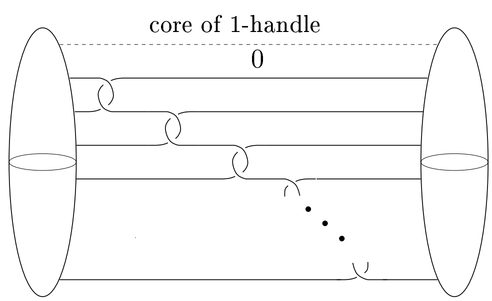

FIGURE 2. Positron

3. Handlebodies of PC manifolds

Handlebodies of four-dimensional PC manifolds are characterized by the following theorem of Ya. Eliashberg.

Theorem 2. (Ya. Eliashberg, [E]; see also [G]) *Let $X = B^4 \cup (1\text{-handles}) \cup (2\text{-handles})$ be four-dimensional handlebody with one 0-handle and no 3- or 4-handles. Then:*

- *The standard PC structure on B^4 can be extended over 1-handles, so that manifold $X_1 = B^4 \cup (1\text{-handles})$ has pseudo-convex boundary.*

- *If each 2-handle is attached to ∂X_1 along a Legendrian knot with framing one less then Thurston-Bennequin framing of this knot, then the complex structure on X_1 can be extended over 2-handles to a complex structure on X, which makes X a PC manifold.*

In this section we shall study "partial handlebodies" obtained by attaching 2-handles on top of a PC manifold. Let Z be a PC manifold and h be a two handle attached to ∂Z along a Legendrian knot $K \subset \partial Z$ with framing f. If $tb(K) \geq f+1$ then by C^0-small smooth isotopy of K we can decrease Thurston-Bennequin invariant of K and make it equal to $f+1$. Therefore, by a theorem of Eliashberg, manifold $Z \cup h$ possesses PC structure.

However, in general, it is not possible to increase Thurston-Bennequin invariant of K by isotopy, and equip $Z \cup h$ with the structure of PC manifold. Thus, we make the following definition: the *defect* $\mathcal{D}(h^2)$ of a 2-handle h^2 attached to a Legendrian knot K on the boundary of PC manifold with framing f is a number $\max\{f+1-tb(K), 0\}$. If we have several 2-handles $h^2_1, \ldots, h^2_n$ attached to a Legendrian link on the boundary of PC manifold Z, then the defect $\mathcal{D}(Z \cup \cup_i h^2_i)$ of this partial handlebody built on top of Z is a sum of the defects of individual handles. So if the defect is zero, the PC structure extends over 2-handles. Given such partial handlebody $Z \cup \cup_i h_i$ with a base Z being a PC manifold , we can build another $Z' \cup \cup_i h'_i$ with lesser defect, so that Z and $Z \cup \cup_i h_i$ are homotopy equivalent to Z' and $Z' \cup \cup_i h'_i$, respectively. To see this, let h_i be a handle

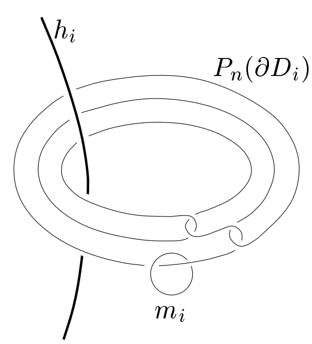

FIGURE 3. Manifold $Z \cup h_i$

with a non-zero defect. Let D_i be a cocore of handle h_i and m_i be a meridian of $P_n(\partial D_i)$ in ∂Z, see Figure 3. We may view D_i as a disk in Z. Note that $(P_n(\partial D_i), m_i)$ is a small (in a chart) Hopf link in ∂Z. Manifold Z' is obtained by removing $Nd_Z(P_n(D_i))$ — a tubular neighborhood of $P_n(D_i)$ in Z, from Z and attaching a 2-handle to $P_k(m_i)$. We assume that k is odd and $k \geq 3$. Manifold Z' is boundary connected sum of Z and a positron W_k, hence it is PC manifold homotopy equivalent to Z. New handle h'_i is attached to the connected sum of attaching circle of h_i and P_n(core of 1-handle in positron), thus it's defect is n less than defect of h_i. Manifold $Z' \cup h'_i$ is shown on Figure 4.

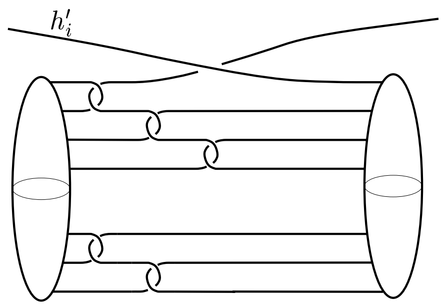

FIGURE 4. Manifold $Z' \cup h_i'$ for $n = 4$, $k = 3$.

4. Convex Decomposition Theorem

We will use above construction to prove the following convex decomposition theorem.

Theorem 3. *Let $X = X_1 \cup_\partial X_2$ be a decomposition of a closed smooth oriented 4-manifold into a union of two compact, smooth, codimension zero submanifolds X_1 and X_2 along common boundary. Suppose each X_i, $i = 1, 2$, has a handlebody without 3- and 4-handles. Then there exist another decomposition $X = \tilde{X}_1 \cup_\partial \tilde{X}_2$, such that manifolds $\tilde{X}_1$ and $-\tilde{X}_2$ admit structures of PC manifolds and each $\tilde{X}_i$ is homotopy equivalent to X_i, $i = 1, 2$.*

Proof: Consider handlebodies of X_i, $i = 1, 2$, with the properties stated in the assumption of the theorem. Let Y_i be a union of 0- and 1-handles in X_i. According to theorem of Eliashberg (Theorem 2, above) Y_i is a PC manifold. Complex structures on Y_i are chosen so that complex orientation coincides with the induced orientation on $Y_1 \subset X$ and is opposite on $Y_2 \subset X$. Since every curve in contact manifold is isotopic to a Legendrian curve via smooth C^0-small isotopy, we can assume that 2-handles in handlebodies of X_i are attached to Legendrian knots in ∂Y_i. Let h be a 2-handle in X_1 with a non-zero defect, D be the cocore of h, $d = \partial D$, m be the meridian of $P_n(d)$ and F be a trivial embedded disc in X_2 bounded by m.

As in the construction in previous section, we remove $P_n(D)$ from X_1, and attach handle to $P_k(m)$ with framing 0, reducing defect of h and, therefore, total defect of X_1 by n. Manifold X_1' can be built inside of X, namely

$$X_1' = [\, X_1 \setminus Nd_{X_1}(P_n D)\,] \cup \overline{Nd_{X_2}(P_k F)}.$$

Here $Nd_X(Y)$ stands for a tubular neighborhood of Y in X. Its complement X_2' is obtained from X_2 by attaching a new 2-handle $g = \overline{Nd_{X_1}(P_n D)}$ and removing neighborhood of $P_k(F)$. Since X_1 and $-X_2$ induce the same orientation on their common boundary, positive Whitehead multiple is the same whether it is considered in ∂X_1 or $\partial(-X_2)$. If we choose n to be defect of h and k to be odd and greater then or equal to defect of g (after Legendrianization of attaching circle), than total defect of X_1 is reduced by n and defect of X_2 is not increased. By applying this procedure to every 2-handle of X_1 with non-zero defect, we obtain manifold $\bar{X}_1$ with pseudo-convex boundary, and it's complement $\bar{X}_2$ has a defect less then or equal to the defect of X_2. To finish the proof one has to apply the same procedure to decomposition $-X = (-\bar{X}_2) \cup (-\bar{X}_1)$ to obtain decomposition $X = \tilde{X}_1 \cup_\partial \tilde{X}_2$, with $\tilde{X}_1$ and $-\tilde{X}_2$ being PC manifolds homotopy equivalent to X_1 and $-X_2$, respectively. □

Corollary. *Every closed simply-connected four-manifold X possesses a structure of complex manifold with pseudo-convex boundary in the complement of some compact contractible submanifold (which is also PC manifold).*

Proof: Consider arbitrary handle decomposition of X. Let Y_1 be a union of all 0- and 1-handles in X. The fundamental group of Y_1 is free and the natural map $\pi_1(\partial Y_1) \to \pi_1(Y_1)$ is an isomorphism.

We shall rearrange handlebody of X by introducing pairs of dual 2- and 3-handles and handle addition, so that among 2-handles of new handlebody of X it is possible to choose a set $\{h_i\}$ such that $X_1 = Y_1 \cup \cup_i h_i$ is a contractible manifold.

More detailed description of this construction is as follows: Let $\{x_1, \ldots, x_l\}$ be a free basis of $\pi_1(\partial Y_1)$. If we fix paths from a base-point to attaching spheres of 2-handles then they represent elements of $\pi_1(\partial Y_1)$, say $y_1, \ldots, y_L$. Since X is simply-connected, $\{y_1, \ldots, y_L\}$ normally generate $\pi_1(\partial Y_1) \cong \pi_1(Y_1)$. Thus, each $\tilde{x}_i$, $i = 1, \ldots, l$ is a product of elements adjoint to $y_1, \ldots, y_L$. Introduce a pair of canceling 2- and 3-handles. We can slide this new 2-handle over handles corresponding to the elements $y_1, \ldots, y_L$ in the decomposition of $\tilde{x}_i$, so that its attaching circle become homotopic to $\tilde{x}_i$. Union X_1 of manifold Y_1 and 2-handles obtained by construction above for every element in $\{\tilde{x}_1, \ldots, \tilde{x}_l\}$ is a contractible submanifold of X.

Manifolds X_1 and $X_2 = \overline{X \setminus X_1}$ have only handles of indexes less then 3 in their handle decompositions. Thus, Theorem 3 can be applied to decomposition $X = X_1 \cup_\partial X_2$, which finishes proof of the corollary. □

Above corollary implies that every closed simply-connected manifold posseses structure of Stein domain in the complement of certain contractible two-dimensional complex, hence every closed embedded surface F in the complement of this complex satisfies adjunction inequality:

$$-\chi(F) \geq F \cdot F + K \frown F.$$

5. Corks with pseudo-convex boundary

Theorem 4. (See [CH, M]) *Let M_1, M_2 be two closed, smooth, simply-connected, h-cobordant 4-manifolds. Then:*

$$M_1 = N \cup_{\varphi_1} A_1, \qquad M_2 = N \cup_{\varphi_2} A_2,$$

where N is simply-connected, A_1 and A_2 are contractible and diffeomorphic to each other and $\varphi_i : \partial A_i \to \partial N$, $i = 1, 2$, are some diffeomorphisms.

Manifolds A_1 and A_2 have come to be known as *corks*, [K].

The proof of Theorem 3 can be adapted to show that the corks and manifold N in the theorem above can be made pseudo-convex.

Theorem 5. *Decompositions in Theorem 4 can be made pseudo-convex.*

Proof: Start with decompositions provided by Theorem 4. Note that by the construction, see [M], manifolds N, A_1 and A_2 have handlebodies without handles of indexes 3 and 4. We divide the rest of the proof into three steps:

1. Make N pseudo-convex.

Apply procedure from the proof of Theorem 3 to both decompositions $M_1 = N \cup_{\varphi_1} A_1$, and $M_2 = N \cup_{\varphi_2} A_2$, simultaneously. We obtain new decompositions $M_1 = N' \cup_{\varphi_1'} A_1'$, and $M_2 = N' \cup_{\varphi_2'} A_2'$, where N' is pseudo-convex and homotopy equivalent to N; A_1', A_2' are contractible, but not necessarily diffeomorphic to each other.

2. Make A_1' and A_2' pseudo-convex.

Consider decompositions resulting from step 1 "up side down":

$$\begin{aligned} -M_1 &= (-A_1') \cup_{\varphi_1'^{-1}} (-N') \text{ and} \\ -M_2 &= (-A_2') \cup_{\varphi_2'^{-1}} (-N'). \end{aligned}$$

Suppose h is a 2-handle in handlebody of $-A_1'$ with non-zero defect. Let D be a cocore of h, $d = \partial D$, m be a meridian of $P_n(d)$ and F be a trivial embedded disk in N bounded by $\varphi_1'(m)$ (note that m is unknot in $\partial A_1 \cong \partial N$). We set

$$\begin{aligned} A_1'' &= [A_1' \setminus Nd(P_n D)] \cup \overline{Nd(P_k F)}, \\ N'' &= [N' \setminus Nd(P_k F)] \cup \overline{Nd(P_n D)}. \end{aligned}$$

Hence $M_1 = N'' \cup A_1''$. Now we have to find manifold A_2'', so that $N'' \cup A_2'' = M_2$ and defect of A_2'' is not greater than defect of A_2'. Note that N'' is obtained from N' by attaching a 2-handle along $\varphi_1'(P_n(d))$ and then removing Whitehead multiple of its cocore. Thus, $\partial N''$ is the result of the surgery of $\partial N'$ along the link (l_1, l_2), where $l_1 = \varphi_1'(P_n(d))$ and $l_2 = P_k(\text{meridian of } l_1)$. Take

$$A_2'' = [A_2' \cup h'] \setminus Nd(\, P_k(\text{cocore of } h')\,),$$

where h' is a 2-handle attached along $\varphi_2'^{-1} \circ \varphi_1'(P_n(d))$. Boundary of A_2'' is obtained by the surgery of $\partial A_2'$ along link $(\varphi_2'^{-1}(l_1), \varphi_2'^{-1}(l_1))$, therefore φ_2' extends to $\varphi_2'' : \partial A_2'' \to \partial N''$. Manifold $N'' \cup_{\varphi_2''} A_2''$ is diffeomorphic to connected sum of $N' \cup_{\varphi_2'} A_2'$ and the double of positron W_k. It is easy to see that the double of a positron is diffeomorphic to S^4, which implies that $N'' \cup_{\varphi_2''} A_2'' \cong N_2$. Now, if we choose k greater then $\max\{\mathcal{D}(\overline{Nd(P_k F)}), \mathcal{D}(h')\}$ (here $\overline{Nd(P_k F)}$ is considered as a 2-handles attached to N') and take n to be the defect of h, then defect of A_1' is reduced by n and defects of A_2' and N_2' are not increased. Apply the above construction to every 2-handle with non-zero defect in A_1' and A_2'. The resulting decompositions

$$\begin{aligned} M_1 &= N'' \cup_{\varphi_1''} A_1'', \\ M_2 &= N'' \cup_{\varphi_2''} A_2'' \end{aligned}$$

are pseudo-convex, but corks are not diffeomorphic to each other. To fix that is the subject of the next step.

3. Make A_1'' diffeomorphic to A_2''.

It is shown in [M] that manifolds A_1, A_2 have the property that their doubles $A_i \cup_\partial (-A_i)$ as well as their union $A_1 \cup_{\varphi_1^{-1} \circ \varphi_2} (-A_2)$ are diffeomorphic to S^4. Not difficult but rather technical calculation shows that this properties are preserved under modifications from steps 1 and 2 above. Thus we have

$$A_i'' \cup_\partial (-A_i'') \cong S^4,\ i = 1, 2;$$
$$A_1'' \cup_{\varphi_1^{-1}\circ\varphi_2} (-A_2'') \cong S^4.$$

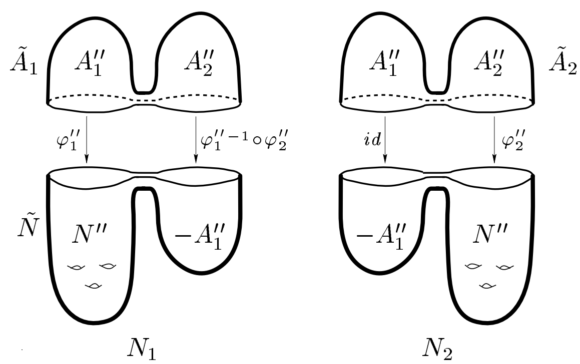

FIGURE 5. Convex decompositions with diffeomorphic corks.

We define

$$\begin{aligned} \tilde{N} &= N'' \natural (-A_1''), \\ \tilde{A}_1 &= A_1'' \natural A_2'', \\ \tilde{A}_2 &= A_2'' \natural A_1'', \end{aligned}$$

where $X \natural Y$ stands for boundary connected sum of X and Y. Since boundary connected sum of PC manifolds is a PC manifold, $\tilde{N}$, $\tilde{A}_1$ and $\tilde{A}_2$ have pseudo-convex boundary and, obviously, $\tilde{A}_1 \cong \tilde{A}_2$. Now we calculate

$$\begin{aligned} \tilde{N} \cup \tilde{A}_1 &= [N'' \natural (-A_1'')] \cup_{\varphi_1 \natural [\varphi_1^{-1}\circ\varphi_2]} [A_1'' \natural A_2''] \\ &\cong [N'' \cup_{\varphi_1} A_1] \# [(-A_1'') \cup_{\varphi_1^{-1}\circ\varphi_2} A_2''] \\ &\cong N_1 \# S^4 \cong N_1. \end{aligned}$$

Analogously,

$$\begin{aligned} \tilde{N} \cup \tilde{A}_2 &= [N'' \natural (-A_1'')] \cup_{\varphi_2 \natural id} [A_2'' \natural A_1''] \\ &\cong [N'' \cup_{\varphi_2} A_2] \# [(-A_1'') \cup_{id} A_1''] \\ &\cong N_2 \# S^4 \cong N_2. \end{aligned}$$

This gives us convex decomposition of N_1 and N_2 with diffeomorphic corks $\tilde{A}_1$ and $\tilde{A}_2$ and finishes the proof of Theorem 5. □

6. Questions and remarks

We would like to conclude with some questions and remarks.

Question 1. Is it always possible to find convex decomposition as in Theorem 3, so that contact structures on $\partial\tilde{X}_1$ and $\partial(-\tilde{X}_2)$ coincide. (Authors can show that contact structures on $\partial\tilde{X}_1$ and $\partial(-\tilde{X}_2)$ can be made homotopic.)

Question 2. Does Theorem 3 (and possibly positive answer to Question 1) pose any restriction to the genera of embedded surfaces in four-manifold.

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