DEFORMATIONS IN G_2 MANIFOLDS

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ABSTRACT. Here we study the deformations of associative submanifolds inside a G_2 manifold M^7 with a calibration 3-form φ. A choice of 2-plane field Λ on M (which always exists) splits the tangent bundle of M as a direct sum of a 3-dimensional associate bundle and a complex 4-plane bundle $TM = \mathbf{E} \oplus \mathbf{V}$, and this helps us to relate the deformations to Seiberg-Witten type equations. Here all the surveyed results as well as the new ones about G_2 manifolds are proved by using only the cross product operation (equivalently φ). We feel that mixing various different local identifications of the rich G_2 geometry (e.g. cross product, representation theory and the algebra of octonions) makes the study of G_2 manifolds looks harder then it is (e.g. the proof of McLean's theorem [M]). We believe the approach here makes things easier and keeps the presentation elementary. This paper is essentially self-contained.

1. $\mathbf{G}_2$ MANIFOLDS

We first review the basic results about G_2 manifolds, along the way we give a self-contained proof of the McLean's theorem and its generalization [M], [AS1]. A G_2 manifold (M, φ, Λ) with an oriented 2-plane field gives various complex structures on some of the subbundles of TM. This imposes interesting structures on the deformation theory of its associative submanifolds. By using this we relate them to the Seiberg-Witten type equations.

Let us recall some basic definitions (c.f. [B1], [B2],[HL]): Octonions give an 8 dimensional division algebra $\mathbb{O} = \mathbb{H} \oplus l\mathbb{H} = \mathbb{R}^8$ generated by $\langle 1, i, j, k, l, li, lj, lk \rangle$. The imaginary octonions $im\mathbb{O} = \mathbb{R}^7$ is equipped with the cross product operation $\times : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ defined by $u \times v = im(\bar{v}.u)$. The exceptional Lie group G_2 is the linear automorphisms of $im\mathbb{O}$ preserving this cross product. It can also be defined in terms of the orthogonal 3-frames:

$$(1) \qquad G_2 = \{(u_1, u_2, u_3) \in (im\mathbb{O})^3 \mid \langle u_i, u_j \rangle = \delta_{ij},\ \langle u_1 \times u_2, u_3 \rangle = 0\ \}.$$

Date: September 16, 2007.
Key words and phrases. mirror duality, calibration.
First named author is partially supported by NSF grant DMS 0505638.

Alternatively, G_2 is the subgroup of $GL(7,\mathbb{R})$ which fixes a particular 3-form $\varphi_0 \in \Omega^3(\mathbb{R}^7)$, [B1]. Denote $e^{ijk} = dx^i \wedge dx^j \wedge dx^k \in \Omega^3(\mathbb{R}^7)$, then

$$G_2 = \{A \in GL(7,\mathbb{R}) \mid A^*\varphi_0 = \varphi_0 \}.$$

$$\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}. \tag{2}$$

Definition 1. *A smooth 7-manifold M^7 has a G_2 structure if its tangent frame bundle reduces to a G_2 bundle. Equivalently, M^7 has a G_2 structure if there is a 3-form $\varphi \in \Omega^3(M)$ such that at each $x \in M$ the pair $(T_x(M), \varphi(x))$ is isomorphic to $(T_0(\mathbb{R}^7), \varphi_0)$ (pointwise condition). We call (M, φ) a manifold with a G_2 structure.*

A G_2 structure φ on M^7 gives an orientation $\mu_\varphi = \mu \in \Omega^7(M)$ on M, and μ determines a metric $g = g_\varphi = \langle \ , \ \rangle$ on M, and a cross product operation $TM \times TM \mapsto TM$: $(u, v) \mapsto u \times v = u \times_\varphi v$ defined as follows: Let $i_v = v\lrcorner$ be the interior product with a vector v, then

$$\langle u, v\rangle = [(u\lrcorner\, \varphi) \wedge (v\lrcorner\, \varphi) \wedge \varphi]/6\mu.$$

$$\varphi(u, v, w) = (v\lrcorner\, u\lrcorner\, \varphi)(w) = \langle u \times v, w\rangle. \tag{3}$$

Definition 2. *A manifold with G_2 structure (M, φ) is called a G_2 manifold if the holonomy group of the Levi-Civita connection (of the metric g_φ) lies inside of G_2. In this case φ is called integrable. Equivalently (M, φ) is a G_2 manifold if φ is parallel with respect to the metric g_φ, that is $\nabla_{g_\varphi}(\varphi) = 0$; which is in turn equivalent to $d\varphi = 0$, $d(*_{g_\varphi}\varphi) = 0$ (i.e. φ harmonic). Also equivalently, at each point $x_0 \in M$ there is a chart $(U, x_0) \to (\mathbb{R}^7, 0)$ on which φ equals to φ_0 up to second order term, i.e. on the image of U, $\varphi(x) = \varphi_0 + O(|x|^2)$.*

Remark 1. *One important class of G_2 manifolds are the ones obtained from Calabi-Yau manifolds. Let (X, ω, Ω) be a complex 3-dimensional Calabi-Yau manifold with Kähler form ω and a nowhere vanishing holomorphic 3-form Ω, then $X^6 \times S^1$ has holonomy group $SU(3) \subset G_2$, hence is a G_2 manifold. In this case $\varphi = Re\ \Omega + \omega \wedge dt$. Similarly, $X^6 \times \mathbb{R}$ gives a noncompact G_2 manifold.*

Definition 3. *Let (M, φ) be a G_2 manifold. A 4-dimensional submanifold $X \subset M$ is called* coassociative *if $\varphi|_X = 0$. A 3-dimensional submanifold $Y \subset M$ is called* associative *if $\varphi|_Y \equiv vol(Y)$; this condition is equivalent to $\chi|_Y \equiv 0$, where $\chi \in \Omega^3(M, TM)$ is the tangent bundle valued 3-form given by:*

$$\langle \chi(u, v, w), z\rangle = *\varphi(u, v, w, z). \tag{4}$$

Equivalence of these conditions follows from the 'associator equality' of [HL]

$$\varphi(u,v,w)^2 + |\chi(u,v,w)|^2/4 = |u\wedge v\wedge w|^2.$$

Sometimes χ is also called the triple cross product operation and denoted by $\chi(u,v,w) = u\times v\times w$. By imitating the definition of χ, we can view the usual cross product as a tangent bundle valued 2-form $\psi\in\Omega^2(M,TM)$ defined by

$$\langle\psi(u,v),w\rangle = \varphi(u,v,w). \tag{5}$$

As in the case of φ, χ can be expressed in terms of cross product and metric

$$\chi(u,v,w) = -u\times(v\times w) - \langle u,v\rangle w + \langle u,w\rangle v \tag{6}$$

(c.f. [H], [HL], [K]). From (6) and the identity $u\times v = u.v + \langle u,v\rangle$, the reader can easily check that $2\chi(u,v,w) = (u.v).w - u.(v.w)$, which shows that the associative submanifolds of (M,φ) are the manifolds where the octonion multiplication of the tangent vectors is "associative".

Associative (and coassociative) submanifolds of a G_2 manifold (M,φ) are calibrated by the closed forms φ (and $*\varphi$). When (M,φ) is a manifold with G_2 structure (i.e. when φ and $*\varphi$ are not necessarily closed) we will call them *almost associative* (or *almost coassociative*) submanifolds. We call a 3-plane $E\subset TM$ *associative plane* if $\varphi|_E \equiv vol(E)$, so associative submanifolds Y^3 are submanifolds whose tangent planes are associative. Associative planes are closed under cross product operation, more specifically, from (2) and (3) we see that an associative 3-plane $E\subset TM$ is a plane generated by three orthonormal vectors in the form $\langle u,v,u\times v\rangle$; and also if $V=E^\perp$ is its orthogonal complement (coassociative), the cross product induces maps:

$$E\times V\to V, \text{ and } V\times V\to E, \text{ and } E\times E\to E. \tag{7}$$

Note that (4) implies that the 3-form χ assigns a normal vector to every oriented 3-plane in TM, [AS1], which is zero on the associative planes. Therefore, we can view χ as a section of the 4-plane bundle $\mathbb{V}=\mathbb{E}^\perp\to G_3(M)$ over the Grassmannian bundle of orientable 3-planes in TM, where $\mathbb{V}$ is the orthogonal bundle to the canonical bundle $\mathbb{E}\to G_3(M)$. In particular, χ gives a normal vector field on all oriented 3-dimensional submanifolds $f: Y^3\hookrightarrow (M,\varphi)$, which is zero if the submanifold is associative. This gives an interesting first order flow $\partial f/\partial t = \chi(f_* vol(Y))$ (which is called χ-flow in [AS2]). We conjecture that this flow pushes $f(Y)$ towards associative submanifolds.

Finally, a useful fact which will be used later is the following: The $SO(3)$-bundle $\mathbb{E}$ is the reduction of the $SO(4)$-bundle $\mathbb{V}$ by the projection to the first factor $SO(4) = (SU(2)\times SU(2))/\mathbf{Z}_2 \to SU(2)/\mathbf{Z}_2 = SO(3)$, i.e. $\mathbb{E}=\Lambda^2_+\mathbb{V}$.

2. 2-frame fields of $\mathbf{G}_2$ manifolds

By a theorem of Emery Thomas, all orientable 7-manifolds admit non-vanishing 2-frame fields [T], in particular they admit non-vanishing oriented 2-plane fields. Using this, we get a useful additional structure on the tangent bundle of G_2 manifolds.

Lemma 1. *A non-vanishing oriented 2-plane field Λ on a manifold with G_2-structure (M,φ) induces a splitting of $TM = \mathbf{E}\oplus\mathbf{V}$, where $\mathbf{E}$ is a bundle of associative 3-planes, and $\mathbf{V}=\mathbf{E}^{\perp}$ is a bundle of coassociative 4-planes. The unit sections ξ of the bundle $\mathbf{E}\to M$ give complex structures J_ξ on $\mathbf{V}$.*

Proof. Let $\Lambda = \langle u, v\rangle$ be the 2-plane spanned by the basis vectors of an orthonormal 2-frame $\{u,v\}$ in M. Then we define $\mathbf{E}=\langle u,v,u\times v\rangle$, and $\mathbf{V}=\mathbf{E}^{\perp}$. We can define the complex structure on $\mathbf{V}$ by $J_\xi(x) = x\times\xi$. □

Similar complex structures were studied in [HL]. The complex structure $J_\Lambda(z)=\chi(u,v,z)$ of [AS1] turns out to coincide with $J_{v\times u}$ because by (6):

$$\chi(u,v,z)=\chi(z,u,v)=-z\times(u\times v)-\langle z,u\rangle v+\langle z,v\rangle u = J_{v\times u}(z). \tag{8}$$

J_ξ also defines a complex structure on the bigger bundle $\xi^{\perp}\subset TM$. So it is natural to study manifolds (M,φ,Λ), with a G_2 structure φ, and a non-vanishing oriented 2-plane field Λ inducing the splitting $TM=\mathbf{E}\oplus\mathbf{V}$, and $\mathbf{J}=J_{v\times u}$. Note that each of these terms depend on φ and Λ.

Definition 4. *We call $Y^3\subset(M,\Lambda)$ a Λ-spin submanifold if $\Lambda|_Y\subset TY$, and call $Y^3\subset(M,\varphi,\Lambda)$ a Λ-associative submanifold if $\mathbf{E}|_Y = TY$.*

Clearly Λ-associative submanifolds $Y\subset(M,\varphi,\Lambda)$ are Λ-spin. Also since Y has a natural metric induced from the metric of (M,φ), we can identify the set of $Spin^c$ structures $Spin^c(Y)\cong H^2(Y,\mathbb{Z})$ on Y by the homotopy classes of 2-plane fields on Y (as well as the homotopy classes of vector fields on Y). So, any Λ-spin submanifold Y inherits a natural $Spin^c$ structure $s=s(\Lambda)$ from Λ. How abundant are the Λ-associative (or Λ-spin) submanifolds? Some answers:

Lemma 2. *Let (M^7,φ) be a manifold with G_2 structure, then every $Spin^c$ submanifold $(Y^3,s)\subset M^7$ is Λ-spin for some Λ with $s=s(\Lambda)$, and every associative $Y\subset(M,\varphi)$ is Λ-associative for some Λ.*

Proof. Let $s=\langle u',v'\rangle$ be the $Spin^c$ structure generated by an orthonormal frame field on TY. By using [T] we choose a nonvanishing orthonormal 2-frame field $\{u,v\}$ on M. Let $V_2(\mathbb{R}^7)\to V_2(M)\to M$ be the Steifel bundle of 2-frames in $T(M)$. Now the restriction of this bundle to Y has two sections $\{u',v'\}$ and

$\{u,v\}|_Y$ which are homotopic, since the fiber $V_2(\mathbb{R}^7)$ is 4-connected. By the homotopy extension property $\{u',v'\}$ extends to orthonormal 2-frame field $\{u'',v''\}$ to M, then we let $\Lambda = \langle u'',v''\rangle$. Furthermore, when Y is associative, we can start with an orthonormal 3-frame of TY of the form $\{u',v',u'\times v'\}$, then we get the corresponding $\mathbf{E}_\Lambda = \langle u'',v'',u''\times v''\rangle$, which makes Y Λ-associative. □

More generally, for any manifold with a G_2 structure (M,φ) we can study the bundle of oriented 2-planes $G_2(M)\to M$ on M, and construct the corresponding universal bundles $\mathbb{E}\to G_2(M)$ and $\mathbb{V}\to G_2(M)$, and a complex structure $\mathbb{J}$ on $\mathbb{V}$, where $\mathbb{J}=J_\Lambda$ on the fiber over $\Lambda=\langle u,v\rangle$. Then each (M,φ,Λ) is a section of $G_2(M)\to M$, inducing $\mathbf{E},\mathbf{V},\mathbf{J}$. We can do the same construction on the bundle of oriented 2-frames $V_2(M)\to M$ and get the same quantities, in this case we get a *hyper-complex structure* on $\mathbb{V}$, i.e. we get three complex structures $\mathbb{J}=\mathbb{J}_1,\mathbb{J}_2,\mathbb{J}_3$ on $\mathbb{V}$ corresponding to $J_{u\times v}, J_u, J_v$, over each fiber $\{u,v\}$, and they anti-commute and cyclically commute e.g. $\mathbb{J}_1\mathbb{J}_2=\mathbb{J}_3$. Notice also that $\mathbb{J}_1$ depends only on the oriented 2-plane field, whereas $\mathbb{J}_2,\mathbb{J}_3$ depend on the 2-frame field.

By using $\mathbb{J}_1$ (or one of the other $\mathbb{J}_p$, $p=2,3$) we can split $\mathbb{V}_{\mathbb{C}}=\mathbb{W}\oplus\bar{\mathbb{W}}$, as a pair of conjugate $\mathbb{C}^2$-bundles ($\pm i$ eigenspaces of $\mathbb{J}_1$). This gives a complex line bundle $K=\Lambda^2(\bar{W})$ which corresponds to the 2-plane field Λ. Corresponding to K we get a canonical $Spin^c$ structure on $\mathbb{V}$. More specifically, recall that $U(2)=(S^1\times S^3)/\mathbb{Z}_2$, $SO(4)=(S^3\times S^3)/\mathbb{Z}_2$, $Spin^c(4)=(S^3\times S^3\times S^1)/\mathbb{Z}_2$,

$$\begin{array}{ccc} & & Spin^c(4) \\ & \nearrow & \downarrow \\ U(2) & \rightarrow & SO(4)\times S^1 \end{array} \tag{9}$$

where the horizontal map $[\lambda,A]\mapsto([\lambda,A],\lambda^2)$ canonically lifts to the map $[\lambda,A]\mapsto(\lambda,A,\lambda)$, where the transition functions λ^2 corresponds to K (see for example [A]). This means that there are pair of complex $\mathbb{C}^2$-bundles, $\mathbb{W}^\pm\to V_2(M)$ with $\mathbb{V}_{\mathbb{C}}=\mathbb{W}^+\otimes\mathbb{W}^-$. This fact can be checked directly by taking $\mathbb{W}^+=K^{-1}+\mathbb{C}$ and $\mathbb{W}^-=\bar{W}$ (note $\Lambda^2(\mathbb{W})\otimes\bar{W}\cong\Lambda^2(\mathbb{W})\otimes\mathbb{W}^*\cong\mathbb{W}$). This gives an action $\mathbb{E}=\Lambda^2_+(\mathbb{V}):\mathbb{W}^+\to\mathbb{W}^+$; in our case this action will come from cross product structure, Lemma 3 will do this by identifying $\mathbb{W}^+$ with S.

Note also that from (6) and (7) the cross product operation $\rho(a)(w)=a\times w$ induces a Clifford representation by $\rho(u\times v)=-\mathbb{J}_1$, $\rho(u)=-\mathbb{J}_2$, $\rho(v)=-\mathbb{J}_3$

$$\rho:\mathbb{E}\to End(\mathbb{V}). \tag{10}$$

2.1. G_2 frame fields on G_2 manifolds.

In the case of a manifold with G_2 structure (M,φ), Thomas's theorem can be strengthened to the conclusion that M admits a 2-frame field Λ, with the property that on the tubular neighborhood of the 3-skeleton of M, Λ is the restriction of a G_2 frame field. To see this, we start with an orthonormal 2-frame field $\{u_1,u_2\}$, then let $\Lambda = \langle u_1,u_2,u_1\times u_2\rangle$ and $\mathbb{V}\to M$ be the corresponding universal 4-plane bundle as in the last section. Then we pick a unit section u_3 of $\mathbb{V}\to M$ over the 3-skeleton $M^{(3)}$; there is no obstruction doing this since we are sectioning an S^3-bundle over the 3-skeleton of M. Now, from the definition of G_2 in (1) we see that $\{u_1,u_2,u_3\}$ is a G_2 frame on $M^{(3)}$.

Definition 5. *We call (M,φ,Λ) a framed G_2 manifold if Λ is the restriction of a G_2 frame field on M.*

The above discussion says that every (M,φ) admits a 2-frame field Λ such that $(M^{(3)},\varphi,\Lambda)$ is a G_2-framed manifold. From now on, the notation (M,φ,Λ) will refer to a manifold with a G_2 structure and a 2-frame field Λ, such that on $M^{(3)}$, Λ is the restriction of a G_2 frame as above. From the above discussion, the last condition is equivalent to picking a nonvanishing section of $\mathbb{V}\to M^{(3)}$ (called u_3 above). This will be useful when studying local deformations of associative submanifolds $Y^3\subset M$ (they live near $M^{(3)}$). Using the same notations of the last section we state:

Lemma 3. *Let (M,φ,Λ) be a framed G_2 manifold. Then we can decompose $\mathbb{V}_{\mathbb{C}} = S\oplus\bar{S}$ as a pair of bundles, each of which is isomorphic to $W^+ = K^{-1}+\mathbb{C}$, and the cross product ρ induces a representation $\rho_{\mathbb{C}}:\mathbb{E}_{\mathbb{C}}\to End(S)$ given by:*

$$u\times v\mapsto\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}\quad u\mapsto\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\quad v\mapsto\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}$$

Proof. We choose a local orthonormal frame $\{e_1,..,e_7\}$ where φ is in the form (2) with $\{u\times v,u,v\}=\{e_1,e_2,e_3\}$ (because of the canonical metric we will not distinguish the notations of local frames and coframes). From (2) and (3) we compute the cross product operation, $\mathbb{J}_1,\mathbb{J}_2,\mathbb{J}_3$, and $\mathbb{W}$ from the tables below

$\times$	e_4	e_5	e_6	e_7
e_1	e_5	$-e_4$	e_7	$-e_6$
e_2	e_6	$-e_7$	$-e_4$	e_5
e_3	$-e_7$	$-e_6$	e_5	e_4

$$\mathbb{J}_1 : \begin{pmatrix} e_4 \mapsto -e_5 \\ e_5 \mapsto e_4 \\ e_6 \mapsto -e_7 \\ e_7 \mapsto e_6 \end{pmatrix}, \quad \mathbb{J}_2 : \begin{pmatrix} e_4 \mapsto -e_6 \\ e_5 \mapsto e_7 \\ e_6 \mapsto e_4 \\ e_7 \mapsto -e_5 \end{pmatrix}, \quad \mathbb{J}_3 : \begin{pmatrix} e_4 \mapsto e_7 \\ e_5 \mapsto e_6 \\ e_6 \mapsto -e_5 \\ e_7 \mapsto -e_4 \end{pmatrix}$$

$\mathbb{W} = \langle e_p - i\,\mathbb{J}_1(e_p) \mid p = 4,..,7\rangle = \langle e_4 + ie_5, e_6 + ie_7\rangle_{\mathbb{C}} = \langle E_1, E_2\rangle_{\mathbb{C}}$.
$\bar{\mathbb{W}} = \langle e_p + i\,\mathbb{J}_1(e_p) \mid p = 4,..,7\rangle = \langle e_4 - ie_5, e_6 - ie_7\rangle_{\mathbb{C}} = \langle \bar{E}_1, \bar{E}_2\rangle_{\mathbb{C}}$.

$\mathbb{J}_2$ and $\mathbb{J}_3 : \mathbb{W} \to \bar{\mathbb{W}}$ are given by $(E_1, E_2) \mapsto (-\bar{E}_2, \bar{E}_1)$ and $(i\bar{E}_2, -i\bar{E}_1)$, respectively; by composing them with complex conjugation we can view them as complex structures on $\mathbb{W}$ (hence we get a quaternionic structure on $\mathbb{W}$). We can decompose $\mathbb{V}_{\mathbb{C}} = S \oplus \bar{S}$, where $S = \langle E_1, J_2E_1\rangle_{\mathbb{C}} = \langle E_1, -\bar{E}_2\rangle_{\mathbb{C}}$, and hence $\bar{S} = \langle E_2, J_2E_2\rangle_{\mathbb{C}} = \langle E_2, \bar{E}_1\rangle_{\mathbb{C}}$, then it is straightforward to check that, the maps $\mathbb{J}_p$ give complex structures on S and $\rho(e_p)$ are given by the matrices in the statement of this Lemma, for $p = 1, 2, 3$.

By the identity $a(e_4+ie_5)+b(e_6-ie_7) = [a(e_4+ie_5)\wedge(e_6+ie_7)]+b]\otimes(e_6-ie_7)$, we can identify $S \cong K^{-1}+\mathbb{C}$, i.e. tensoring with the section $s := (e_6-ie_7)$ gives the isomorphism. Here s is a nonvanishing section of $\mathbb{V}_{\mathbb{C}}$ which is determined by the unit section u_3 coming from the G_2 framing (discussed above). This is because we can choose $\{e_4 = u_3, e_5 = J_1(u_3), e_6 = J_2(u_3), e_7 = J_3(u_3)\}$. □

There is also the useful bundle map $\sigma : S \to \mathbb{E}$ induced by

$$\sigma(z, w) = (\frac{|z|^2 - |w|^2}{2})u \times v + Re(z\bar{w})u + Im(w\bar{z})v. \tag{11}$$

This is the quadratic map which appears in Seiberg-Witten theory, after identifying $\mathbb{E}$ with the Lie algebra $su(2)$ (skew adjoint endomorphisms of $\mathbb{C}^2$ with the inner product given by the Killing form) we get

$$\sigma(x) = \sigma(z, w) = \begin{pmatrix} \frac{|z|^2-|w|^2}{2} & z\bar{w} \\ w\bar{z} & \frac{|w|^2-|z|^2}{2} \end{pmatrix}.$$

$$\langle \sigma(x), x\rangle = 2|\sigma(x)|^2 = \frac{1}{2}|x|^4. \tag{12}$$

These identifications are standard tools used Seiberg-Witten theory (c.f [A]).

2.2. **Deforming G_2 structures.**

For a 7-manifold with a G_2 structure (M,φ), the space of all G_2 structures on M is identified with an open subset of 3-forms $\Omega^3_+(M) \subset \Omega^3(M)$, which is the orbit of φ by the gauge transformations of TM. The orbit is open by the dimension reason (recall that the action of $GL(7,\mathbb{R})$ on $\Omega^3(X)$ has G_2 as the stabilizer). The structure of $\Omega^3_+(M)$ is nicely explained in [B2] as follows: By definition, $\Omega^3_+(M)$ is the space of sections of a bundle over M with fiber $GL(7,\mathbb{R})/G_2$ (which is homotopy equivalent to $\mathbb{RP}^7$). Furthermore, the subspace of the G_2 structures inducing the same metric can be parametrized with the space of sections of the bundle $\mathbb{RP}^7 \to P(T^*M \oplus \mathbb{R}) \to M$ with fibers $SO(7)/G_2 = \mathbb{RP}^7$, where $P(T^*M \oplus \mathbb{R})$ is the projectivization of $T^*(M) \oplus \mathbb{R}$. That is, if $\lambda = [r, \alpha]$ with $r^2 + \alpha^2 = 1$, then the corresponding $\varphi_\lambda \in \Omega^3_+(M)$ is

$$\varphi_\lambda = \varphi - 2\alpha^{\#} \lrcorner \, [\, r(*\varphi) + \alpha \wedge \varphi \,] \tag{13}$$

where $\alpha^{\#}$ is the metric dual of α. This expression is given in [B2], and written slightly differently.

It is a natural question whether a submanifold $Y^3 \subset (M,\varphi)$ is associative. The following says that any Y can be made associative in (M,φ_λ), after deforming φ to φ_λ.

Proposition 4. *Let (M^7,φ) be a manifold with a G_2 structure, then any $Spin^c$ submanifold $(Y^3,s) \subset M^7$ is a Λ-associative submanifold of $(M,\varphi_\lambda,\Lambda)$ for some choice of $\lambda = [r,\alpha]$ and a plane field Λ.*

Proof. By Lemma 2, we can assume Y is Λ-spin for some $\Lambda = \langle u, v\rangle$. Hence this gives an orthogonal splitting $TM = \mathbf{E} \oplus \mathbf{V}$, with $\mathbf{E} = \langle u, v, u \times v\rangle$. Choose a unit vector field w in Y orthogonal to $\langle u, v\rangle|_Y$, then take any extension of w to M. Now we want to choose $\lambda = [r,\alpha]$ so that if $(u\times v)_\lambda$ is the cross product corresponding to the G_2 structure φ_λ, then $(u \times v)_\lambda|_Y = w$.

By (13), and the rules $(u\times v)^{\#} = v \lrcorner\, u \lrcorner\, \varphi$ and $(u\times v)^{\#}_\lambda = v \lrcorner\, u \lrcorner\, \varphi_\lambda$ we get

$$\begin{aligned}(u\times v)_\lambda/2 \;\; &= \;\; (u\times v)/2 - |\alpha|^2(u\times v) - r\chi(u,v,\alpha^{\#}) \\ &\quad + \alpha(v)(u\times\alpha^{\#}) - \alpha(u)(v\times\alpha^{\#}) + \alpha^{\#}\langle u\times v, \alpha^{\#}\rangle.\end{aligned}$$

This formula holds for any $u, v \in TM$. In our case $\{u, , v\}$ are orthonormal generators of Λ, so by (8) the third term on the right is $rJ(\alpha^{\#})$ where $J = J_{v\times u}$ is the complex structure of $\mathbf{V}$ given by Lemma 1 (and remarks following it).

Now if we call $w_0 = \frac{1}{2}[(u\times v) - w]$, and choose α among 1-forms whose $\mathbf{E}$ component zero (i.e. section of $\mathbf{V}$) with $|\alpha^{\#}| < 1$ (hence $r \neq 0$), the equation $(u\times v)_\lambda|_Y = w$ gives $w_0 = |\alpha|^2(u\times v) + rJ(\alpha^{\#})$. By taking inner products of both sides with basis elements of $\mathbf{E}$, we see that $w_0^{\perp} = rJ(\alpha^{\#})$ where $w_0^{\perp}$ is the $\mathbf{V}$-component of w_0. We can apply J to both sides and solve $\alpha^{\#} = -\frac{1}{r}J(w_0^{\perp})$. □

2.3. **Deforming associative submanifolds.**

Let $G(3,7) \cong SO(7)/SO(3) \times SO(4)$ be the Grassmannian manifold of oriented 3-planes in $\mathbb{R}^7$, and $G^{\varphi_0}(3,7) = \{L \in G(3,7) \mid \varphi_0|_L = vol(L)\}$ be the submanifold of associative 3-planes. Recall that G_2 acts on $G^{\varphi_0}(3,7)$ with stabilizer $SO(4)$ giving the identification $G^{\varphi_0}(3,7) = G_2/SO(4)$ [HL]. Recall also that if $\mathbb{E} \to G(3,7)$ and $\mathbb{V} = \mathbb{E}^{\perp} \to G(3,7)$ are the canonical 3-plane bundle and the complementary 4-plane bundle, then we can identify the tangent bundle by $TG(3,7) = \mathbb{E}^* \otimes \mathbb{V}$. How does the tangent bundle of $G^{\varphi_0}(3,7)$ sit inside of this? The answer is given by the following Lemma. By (7) the cross product operation maps $\mathbb{E} \times \mathbb{V} \to \mathbb{V}$, and the metric gives an identification $\mathbb{E}^* \cong \mathbb{E}$, now if $L = \langle e_1, e_2, e_3 \rangle \in G^{\varphi_0}(3,7)$ with $\{e_1, e_2, e_3 = e_1 \times e_2\}$ orthonormal, then

Lemma 5. $T_L G^{\varphi_0}(3,7) = \{ \sum_{j=1}^{3} e^j \otimes v_j \in \mathbb{E}^* \otimes \mathbb{V} \mid \sum e_j \times v_j = 0 \}$.

Proof. A tangent vector of $G(3,7)$ at L is a path of planes generated by three orthonormal vectors $L(t) = \langle e_1(t), e_2(t), e_3(t) \rangle$, such that $L(0) = L$, in other words $\dot{L} = \sum e_j \otimes \dot{e}_j$. Clearly this tangent vector lies in $G^{\varphi_0}(3,7)$ if $e_3(t) = e_1(t) \times e_2(t)$. So $\dot{e}_3 = \dot{e}_1 \times e_2 + e_1 \times \dot{e}_2$. By taking cross product of both sides with e_3 and then using the identity (6) we get

$$\chi(\dot{e}_1, e_2, e_3) + \chi(e_1, \dot{e}_2, e_3) + \chi(e_1, e_2, \dot{e}_3) = 0.$$

Now by using (8) and the fact that the cross product of two of the vectors in $\{e_1, e_2, e_3\}$ is equal to the third (in cyclic ordering), we get the result. □.

It is easy to see that the normal bundle of $G^{\varphi_0}(3,7)$ in $G(3,7)$ is isomorphic to $\mathbb{V}$ giving the exact sequence of the bundles over $G^{\varphi_0}(3,7)$:

$$0 \to TG^{\varphi_0}(3,7) \to TG(3,7) \xrightarrow{\times} \mathbb{V} \to 0$$

From (7) we know that, if $Y^3 \subset (M, \varphi)$ associative and ν is its normal bundle, then the cross product operation maps: $TY \times \nu \to \nu$, $\nu \times \nu \to TY$, and $TY \times TY \to TY$. Let $\{e_1, e_2, e_3\}$ and $\{e^1, e^2, e^3\}$ be local frames and the dual coframes on TY and $\mathbf{A_0}$ be the background Levi-Civita connection on ν induced from the metric on M (there is also the identification $TY \cong T^*Y$ by induced metric). Then we can define a Dirac operator $\not{D}_{\mathbf{A_0}} : \Omega^0(\nu) \to \Omega^0(\nu)$ as the covariant derivative $\nabla_{\mathbf{A_0}} = \sum e^j \otimes \nabla_{e_j}$ followed by the cross product:

$$\not{D}_{\mathbf{A_0}} = \sum e^j \times \nabla_{e_j}. \tag{14}$$

So the cross product plays the role of the Clifford multiplication in defining the Dirac operator in the normal bundle. We can extend this multiplication to 2-forms: $(a \wedge b) \times x = \frac{1}{2}[\ a \times (b \times x) - b \times (a \times x)\ ]$ then by using (6) we get:

$$(a \wedge b) \times x = \frac{1}{2}[\ x \lrcorner\ (a \wedge b)\ ] - \chi(a, b, x).$$

In particular, when $a, b \in TY$ and $x \in \nu$ then $(a \wedge b) \times x = -\chi(a, b, x)$. As usual we can twist this Dirac operator by connections on ν, by replacing $\mathbf{A_0}$ with $\mathbf{A_0} + a$, where $a \in \Omega^1(Y, ad\nu)$ is an endomorphism of ν valued 1-form. The following from [AS1], is a generalized version of McLean's theorem [M].

Theorem 6. *The tangent space to associative submanifolds of a manifold with a G_2 structure (M, φ) at an associative submanifold Y is given by the kernel of the the twisted Dirac operator $\not{D}_{\mathbf{A}} : \Omega^0(\nu) \to \Omega^0(\nu)$, where $\mathbf{A} = \mathbf{A_0} + a$ for some $a \in \Omega^1(Y, ad(\nu))$. The term $a = 0$ when φ is integrable.*

Proof. Recall the notations of Lemma 5. Let $L = \langle e_1, e_2, e_3 \rangle$ be a tangent plane to $Y \subset M$. Any normal vector field v to Y moves L by one parameter group of diffeomorphisms giving a path of 3-planes in M, hence it gives a vertical tangent vector $\dot{L} = \sum e_j \otimes \mathcal{L}_v(e_j) \in T_L G_3(M)$ of the Grassmannian bundle of 3-planes $G_3(M) \to M$ (where $\mathcal{L}_v$ is the Lie derivative along v). By Lemma 5 this path of planes remain associative if $\sum e_j \times \mathcal{L}_v(e_j) = 0$. Since $\mathcal{L}_v(e_j) = \bar{\nabla}_{e_j}(v) - \bar{\nabla}_v(e_j)$, where $\bar{\nabla}$ is the (torsion free) metric connection of M; then the result follows by letting $a(v) = \sum e^j \times \nabla_v(e_j)$ where ∇ is the normal component of $\bar{\nabla}$. If φ is integrable, then on a local chart it coincides with φ_0 up to quadratic terms, so $0 = \nabla_v(\varphi)|_Y = \nabla_v(e^1 \wedge e^2 \wedge e^3) = \sum (*e^j) \wedge \nabla_v(e^j)$, which implies $a = 0$. Also, by using the fact that the cross product operation preserves the tangent space of the associative manifold Y, the expression of $a(v)$ is independent of the choice of the orthonormal basis of L, because if we choose another such basis $\bar{e}_j = a^{jp} e_p$, $\bar{e}^j = a_{jq} e^q$ with $(a^{jk}) = (a_{kj})^{-1}$, then

$$\bar{\nabla}_v(\bar{e}_j) = v(a^{jp}) e_p + a^{jp} \bar{\nabla}_v(e_p)\ ,\ \text{hence}\ \ \nabla_v(\bar{e}_j) = a^{jp} \nabla_v(e_p).$$

$$\sum \bar{e}^j \times \nabla_v(\bar{e}_j) = \sum a_{jq} a^{jp}\ e^q \times \nabla_v(e_p) = \sum e^p \times \nabla_v(e_p). \qquad \square$$

Notice that at any point by choosing normal coordinates we can make $a = 0$. This reflects the fact that φ coincides only pointwise with φ_0, not on a chart. To make the Dirac operator onto, we can twist it by 1-forms $a \in \Omega^1(Y)$, i.e.

Lemma 7. *For associative $Y \subset (M, \varphi)$ the map $\Omega^1 \times \Omega^0(\nu) \to \Omega^0(\nu)$ defined by $(a, x) \mapsto D_{\mathbf{A}}(x) + a \times x$ is onto, (by using appropriate Sobolev norms)*

Proof. It suffices to show that the orthogonal complement of the image of this map is zero: Assume $\langle D_{\mathbf{A}}(x), y\rangle + \langle a \times x, y\rangle = 0$ for all x and a, then by taking $a = 0$ and using the self adjointness of the Dirac operator we get $D_{\mathbf{A}}(y) = 0$. Hence $\langle a \times x, y\rangle = 0$, then the fact that the map $(x, a) \mapsto a \times x$ is surjective gives the result. Note that by (6) $a \times (a \times x) = -|a|^2 x$. $\square$.

So for a generic choice of a this twisted Dirac operator is onto, but what does this mean in terms of the deformation space of the associative submanifolds? The next Proposition ([AS1]) gives an answer. It says that if we perturb the deformation space with the gauge group (i.e. allowing a slight rotation of TY by the gauge group of TM during deformation) then it becomes smooth.

Note that Theorem 6 may be explained by a Gromov-Witten set-up: Let $G_3^{\varphi}(M) \subset G_3(M) \to M$ be the subbundle of associative 3-planes with fiber $G_3^{\varphi} \cong G_2/SO(4)$ ([HL]). We can form a bundle $G_3(Y, M) \to Im(Y^3, M)$ over the space of imbeddings, whose fiber over $f : Y \hookrightarrow M$ are the liftings F:

$$\begin{array}{ccc} & & G_3(M) \supset G_3^{\varphi}(M) \\ & F \nearrow & \downarrow \\ Y & \xrightarrow{f} & M \end{array}$$

The Gauss map $f \mapsto \mu(f)$ gives a natural section to this bundle, and Y is associative if and only if this section maps into $G_3^{\varphi}(M)$. Theorem 6 gives the condition that the derivative of μ maps into the tangent space of $G_3^{\varphi}(Y, M)$, which is the subbundle of $G_3(Y, M)$ consisting of F's mapping into $G_3^{\varphi}(M)$ (in [AS1] the elements of this bundle are called "pseudo-associatives"). Recall that if $P \to M$ denotes the tangent frame bundle of M, then the gauge group $\mathcal{G}(M)$ of M is defined to be the sections of the $SO(7)$-bundle $Ad(P) \to M$, where $Ad(P) = P \times SO(7)/(p, h) \sim (pg, g^{-1}hg)$. By perturbing the Gauss map with the gauge group (i.e. by composing μ with the gauge group action $G_3(M) \to G_3(M)$ we can make it transversal to $G_3^{\varphi}(Y, M)$.

Proposition 8. *The map* $\tilde{\mu} : \mathcal{G}(M) \times Im(Y, M) \to G_3(Y, M)$ *is transversal to* $G_3^{\varphi}(Y, M)$*, where* $\tilde{\mu}(s, f) = s(f)\mu(f)$*.*

Proof. We start with the local calculation of the proof of Theorem 6, except in this setting we need to take $\dot{L} = \sum e_j \otimes \mathcal{L}_v(se_j)$, where $s \in SO(7)$ is the gauge group in the chart. Then the resulting equation is $\not{D}_{\mathbf{A}}(v) + \sum e_j \times \mathbf{v}(s)e_j = 0$, where $\mathbf{v}(s)e_j$ denotes the normal component of $v(s)e_j$ (here we are doing the calculation in normal coordinates where $\nabla_v(e_k) = 0$ pointwise). Then the argument as in the proof of Lemma 7 (by showing the second term is surjective) gives the proof. $\square$.

The kernel of this operator gives the deformations of $Y \subset (M, \varphi)$, defined as the submanifolds where the perturbed Gauss map $\tilde{\mu}$ (perturbed by a generic element of the Gauge group) maps Y into $G_3^{\varphi}(M)$.

For Λ-associative submanifolds Y there is a special way to deform them to obtain smoothness in their moduli space, this is because their normal bundles come with a $U(2)$ structure, which gives a canonical $Spin^c(4)$ structure: i.e. a complex line bundle K and a complex version of the normal bundle W^+. The background connection on M with the help of connections on K allows us to deform Y in W^+. So we get new parameters, namely the "connections on K"; by constraining them in a natural way gives us Seiberg-Witten type equations.

More specifically, let $Y \subset (M, \varphi, \Lambda)$ be a Λ-associative submanifold, we deform Y in the complex bundle $S \cong K^{-1} + \mathbb{C} = \mathbb{W}^+$ defined in Lemma 3. From projections (9), the background $SO(4)$ metric connection on the normal bundle $\nu(Y)$ along with a choice of a connection A on the line bundle $K \to Y$ gives a connection of the $Spin^c$ bundle, which in turn induces a connection on the associated $U(2)$ bundle $W^+ \to Y$. By using the Clifford multiplication $TY_{\mathbb{C}} \otimes W^+ \to W^+$ coming from the cross product (Lemma 3) we can form the Dirac operator $\not{D}_A : \Omega^0(W^+) \to \Omega^0(W^+)$, whose kernel identifies locally the deformations of Y in the bundle W^+. Then if we constraint the new variable A by (11) we obtain deformations resembling to the Seiberg-Witten equations.

$$\begin{aligned} \not{D}_{\mathbf{A}}(x) &= 0 \\ *F_A &= \sigma(x). \end{aligned} \tag{15}$$

where $*$ is the star operator of Y induced from the background metric of M, and $(x, A) \in \Omega^0(W^+) \times \mathcal{A}(K)$, and $\mathcal{A}(K)$ is the space of connections on K. From Weitzenböck formula and (12), the above equations give compactness to this type of local deformation space (and an orientation on them), hence allow us to assign Seiberg-Witten invariant to Y.

Now the natural question is how easy to produce Λ-associative submanifolds $Y^3 \subset (M, \varphi, \Lambda)$? This might be a too restricted class, however there is a more general class of submanifolds which we can write the equations (15) and define the same invariant: They are the zero set Y^3 of a transverse section of the bundle $\mathbf{V} \to M$, let us call them *almost* Λ-*associative* submanifolds. Transversality gives canonical identifications $TY \cong \mathbf{E}|_Y$ and $\nu(X) \cong \mathbf{V}|_Y$, hence we have the induced $U(2)$ structure on $\nu(Y)$ and the cross product action $T^*Y \otimes \nu(Y) \to \nu(Y)$, and the rest of the above discussions hold for these manifolds. Clearly this integer valued Seiberg-Witten invariant induced from (15) is invariant under small isotopies of almost Λ-associative submanifolds.

Note that in the usual Seiberg-Witten equations on Y, we use an action $T^*Y \otimes W^+ \to W^-$ coming from the $Spin^c$ structure, which then extends to an action $\Lambda^2(Y) \otimes W^+ \to W^+$. Here the action of T^*Y on $W^+ \to W^+$ is given by Lemma 3. On the other hand by the background metric we have the identification $T^*Y \cong \Lambda^2(Y)$.

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