Deforming associatives in G_2 manifolds

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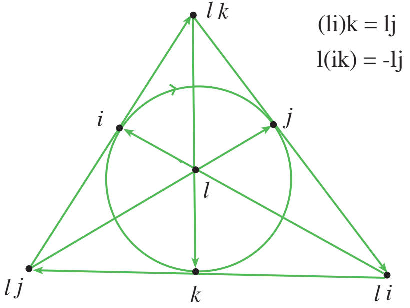
Octonions

Octonions: The division algebra $\mathbb{O} = \mathbb{H} \oplus \mathbb{H} = \mathbb{R}^8$ is generated by $\langle 1, i, j, k, l, li, lj, lk \rangle$ by the multiplication rule:

$$i^2 = j^2 = k^2 = l^2 = -1 \text{ , and :}$$



$$u = a + bi + cj + ... \Rightarrow \bar{u} = a - bj - cj - ..$$

(Foundamental works: Harvey, Lawson, Bryant, Salamon, Joyce..)

The group G_2

- The imaginary part $im\,(\mathbb{O}) = \mathbb{R}^7$ has the cross product operation $\times : \mathbb{R}^7 \times \mathbb{R}^7 \to \mathbb{R}^7$ defined by $u \times v = im(\bar{v}.u)$
- G_2 is the linear automorphisms of $im\,\mathbb{O}$ preserving $\times$.
- $G_2 = \{(u_1, u_2, u_3) \in (im\,\mathbb{O})^3 \mid \langle u_i, u_j \rangle = \delta_{ij},\ \langle u_1 \times u_2, u_3 \rangle = 0\ \}.$
- $G_2 = \{A \in GL(7, \mathbb{R}) \mid A^* \varphi_0 = \varphi_0\ \},$ with $e^{ijk} = dx^i \wedge dx^j \wedge dx^k$

  $\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$

  49 dim $GL(7, \mathbf{R})$ acting on 35 dim $\Lambda^3(\mathbf{R}^7)$ transitively
  with stabilizer $49 - 35 = 14$ dimensional subgroup G_2
  with open orbits $U_\pm$ (complement of "degenerate" 3-forms).

Topology of G_2

$$S^3 \subset SU(3) \subset G_2 \subset Spin(7)$$

$$\varphi = dx^1 \wedge A + B \in \Lambda^3(\mathbf{R} \times \mathbf{C}^3)$$

$$A = i/2 \sum dz^j \wedge d\bar{z}^j, \text{ and } B = Re\,(dz^1 \wedge dz^2 \wedge dz^3)$$

k	3	6	9	11	14
$H^k(G_2)$	$\mathbf{Z}$	$\mathbf{Z}_2$	$\mathbf{Z}_2$	$\mathbf{Z}$	$\mathbf{Z}$
$\pi_k(G_2)$	$\mathbf{Z}$	$\mathbf{Z}_3$	*	*	*

$Spin(7)$ acts on S^7 with stabilizer $G_2 \Rightarrow$ fibrations:

$$G_2 \to Spin(7) \to S^7 \to BG_2 \to BSpin(7)$$

Structure group of every *spin* manifold M^7 lifts to G_2

Manifold with G_2 structure (M, φ)

- M^7 is a **manifold with G_2 structure** if its tangent frame bundle reduces to a G_2 bundle. $\Leftrightarrow$ If $\exists\, \varphi \in \Omega^3(M)$ such that $\forall x \in M$ the pair $(T_x(M), \varphi(x))$ is isomorphic to $(T_0(\mathbb{R}^7), \varphi_0)$ (pointwise cond.) Such (M, φ) is called a manifold with a G_2 structure.
- (M, φ) has a canonical orientation $\mu_\varphi = \mu \in \Omega^7(M)$
- (M, φ) has a canonical metric $g = g_\varphi = \langle\ ,\ \rangle$

$$\langle u, v\rangle = [(u \lrcorner\, \varphi) \wedge (v \lrcorner\, \varphi) \wedge \varphi]/6\mu$$

- (M, φ) has a cross product structure:

$$TM \times TM \mapsto TM\colon (u, v) \mapsto u \times v = u \times_\varphi v$$

$$\varphi(u, v, w) = \langle u \times v, w\rangle$$

Orientation $\mu_\varphi \in \Omega^7(M)$

- $\mu \in \Lambda^7(V^*)$?. Find linear $\Lambda^7(V) \ni x \to \mu(x) \in \mathbf{R} \ \ (V = TM)$.
-
$$Q_x : V \times V \xrightarrow{A_\varphi} \Lambda^7(V^*) \xrightarrow{\langle\ ,\, x\rangle} \mathbf{R}$$
$$A_\varphi(u,v) = (u \lrcorner\, \varphi) \wedge (v \lrcorner\, \varphi) \wedge \varphi$$
$$\Rightarrow \quad Q_x : V \to V^*$$
-
$$x \in \Lambda^7(V) \xrightarrow{det(Q_x)} \Lambda^7(V^*) \xrightarrow{\langle\ ,\, x\rangle} \mathbf{R}$$
$$\alpha(x) = \langle det(Q_x)(x), x\rangle$$
-
$$\alpha(\lambda x) = \lambda^9 \alpha(x) \quad \text{since}:$$
$$\langle det(Q_{\lambda x}(\lambda x), \lambda x\rangle = \lambda^2 \langle det(Q_x(\lambda x), x\rangle = \lambda^9 \langle det(Q_x(x), x\rangle$$
-
$$\mu(x) := \alpha(x)^{1/9}$$

G_2-manifold: (M, φ) with φ integrable

- A manifold with G_2 structure (M, φ) is called a G_2 **manifold** if the holonomy group of the Levi-Civita connection (of the metric g_φ) lies inside G_2. In this case φ is called integrable.

  $\Leftrightarrow \nabla_{g_\varphi}(\varphi) = 0 \Leftrightarrow d\varphi = 0,\ d(*_{g_\varphi}\varphi) = 0$ (i.e. φ harmonic)

  $\Leftrightarrow \forall\ x_0 \in M\ \ \exists$ chart $\lambda : (U, x_0) \to (\mathbb{R}^7, 0)$ such that:
  $\varphi(x) = \varphi_0 + O(|x|^2),\ \forall x \in \lambda(U)$.

- Examples: (1) T^7 (not full holonomy)

  (2) $\Lambda^2_+(S^4) \longrightarrow S^4$ (Bryant-Salamon)
  $S^3 \times \mathbf{R}^4 \longrightarrow S^3$

  (3) $Y^3 \times \mathbf{R}^4 \longrightarrow Y^3$ (Robles-Salur)

  (4) $M^7 \overset{\text{resolve}}{\longrightarrow} T^7/(\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2)$ (Joyce)

  (5) Gluing techniques (Kovalev..)

Associative submanifolds of (M, φ)

- $X^4 \subset (M^7, \varphi)$ is called **coassociative** if $\varphi|_Y \equiv 0$
- $Y^3 \subset (M^7, \varphi)$ is called **associative** if $\varphi|_Y \equiv vol(Y)$
  $\Leftrightarrow$ TY is closed under cross product operation "$\times$"
  $\Leftrightarrow \chi|_Y \equiv 0$, where $\chi \in \Omega^3(M, TM)$ is defined by:

$$\langle \chi(u, v, w), z \rangle = *\varphi(u, v, w, z)$$

$$\langle \chi, z \rangle = z \lrcorner * \varphi$$

$$\Rightarrow \chi\text{-flow} \; : \qquad \chi(u, v, w) \perp 3 \;\text{ plane} < u, v, w >$$

- χ can be expressed in terms of cross product and the metric:

$$\chi(u, v, w) = -u \times (v \times w) - \langle u, v \rangle w + \langle u, w \rangle v$$

$$2\chi(u, v, w) = (u.v)w - u.(v.w)$$

Associative grasmannians $G_3^\varphi(M) \subset G_3(M)$

- Any 3-plane $E \in G_3(M) :=$ Grassmannian of 3 planes in TM is associative if $\varphi|_E = vol(E)$. Let $G_3^\varphi(M) \subset G_3(M)$ be associative 3 planes in TM. So $Y^3 \subset M$ associative if $TY \in G_3^\varphi(M)$. We have the universal bundles $\mathbb{E} \to G_3(M)$ and $\mathbb{V} = \mathbb{E}^\perp \to G_3(M)$
- Bundles $G_3^\varphi(M) \to M$, $G_3(M) \to M$ with fibers $G_3^\varphi(\mathbf{R}^7) \subset G_3(\mathbf{R}^7)$

$$G_3^\varphi(\mathbf{R}^7) = G_2/SO(4) \subset SO(7)/SO(3) \times SO(4) = G_3(\mathbf{R}^7)$$

- χ is a section of $\mathbb{V} \to G_3(M)$ with $\chi^{-1}(0) = G_3^\varphi(M)$
- The canonical metric identifies $E^* \cong E$, when E is associative then $V = E^\perp \subset TM$ ia coassiative 4-plane. The cross product operation induces: $\mathbb{E} \times \mathbb{V} \to \mathbb{V},\ \mathbb{V} \times \mathbb{V} \to \mathbb{E},\ \mathbb{E} \times \mathbb{E} \to \mathbb{E}$.

Deforming associative submanifolds of (M, φ)

- Let (M^7, φ_0) be manifold with G_2 structure, $Y^3 \subset M$ be associative, and $\{e_j\}$ and $\{e^j\}$ be ON tangent and cotangent frame field on Y. Let ν be the normal bundle of Y, and A_0 be the connection on ν induced by the metric g_{φ_0}. The Dirac operator is the composition:

$$D_{A_0} = \sum e^j \times \nabla_j : \quad \Omega^0(\nu) \xrightarrow{\nabla_{A_0}} \Omega^0(T^*Y \otimes \nu) \xrightarrow{\times} \Omega^0(\nu)$$

$$\mathcal{A}(M) = \mathcal{A}(M, \varphi_0) := \{\text{All associative submanifods}\, Y^3 \subset (M, \varphi_0)\}$$

- *Theorem* (McLean): If (M, φ_0) is a G_2 manifold, then the tangent space of $\mathcal{A}(M)$ at Y has the identification $T_Y\mathcal{A}(M) \cong ker(D_{A_0})$.

- *Theorem* (A-Salur): If (M, φ_0) is a manifold with a G_2 structure, then the tangent space has the identification $T_Y\mathcal{A}(M) \cong ker(D_A)$, where $A = A_0 + a(\varphi_0)$ is a deformarion of A_0, $a \in \Omega^1(Y, ad(\nu))$. ($a = 0$ if φ_0 integrable)

How to make $\mathcal{A}(M)$ smooth?

- Keeping a deformation $f_t : Y \hookrightarrow (M, \varphi)$ associative $\Leftrightarrow$ Image of the associated Gauss map $F_t : Y \to G_3(M)$ lies in $G_3^{\varphi_0}(M)$

$$\begin{array}{ccc} & & G_3(M) \supset G_3^{\varphi_0}(M) \\ & F \nearrow & \downarrow \\ Y & \overset{f}{\hookrightarrow} & M \end{array}$$

- $\mathcal{A}(M)$ smooth at Y means that the twisted Dirac operator D_A is surjective, where $A = A_0 + a$. and $a = a(\varphi_0) \in \Omega^1(Y, ad(\nu))$

- **Problem :** By perturbing either the Gauss map F, or the G_2 structure φ_0 can we make $\mathcal{A}(M)$ smooth?

  Solution: Gauge group $\mathcal{G}(P)$ of the (principal) tangent bundle $P \to M$ can be used to perform both perturbations, i.e. $s \in \mathcal{G}(P)$

$$\left\{ \begin{array}{ccc} F & \mapsto & s \circ F \\ \varphi_0 & \mapsto & s^*(\varphi_0) \end{array} \right\}$$

Perturbed smooth moduli spaces

- *Theorem* (A-Salur): If (M, φ_0) is a manifold with a G_2 structure, and $Y \in \mathcal{A}(M)$, then for generic choice of $s \in \mathcal{G}(P)$ the perturbed moduli space $\mathcal{A}(M, s^*(\varphi_0))$ is smooth.

  We can perform the perturbatios staying in the set of closed G_2 structures, if we are willing to move Y by a small isotopy:

- *Theorem* (Gayet): If (M, φ_0) is a manifold with a closed G_2 structure i.e. $d(\varphi_0) = 0$, and $Y \in \mathcal{A}(M)$, then for any generic closed G_2 structure φ close enough to φ_0, there is a smooth point $Y' \in \mathcal{A}(M, \varphi)$ which is a nearby isotopic copy of Y.

2-frame fileds of Emery Thomas

The following could be used to constrain deformation equation of an associative submanifold $Y^3 \subset (M, \varphi)$ (A-Salur/Adv. Math 2008).

- *Theorem* (Thomas): Every closed oriented M^7 admits a nonvanishing 2-frame field $\Lambda = \{u, v\}$, in particular it admits an oriented 2-plane field $\lambda = \langle u, v \rangle$.
- Analogies with dimension 3:

$$\{\text{nonvanishing 2}-\text{frame fields on } M^7\} \longleftrightarrow \{\text{spin structures on} Y^3\}$$

$$\{\text{nonvanishing 2}-\text{plane fields on } M^7\} \longleftrightarrow \{\text{spin}^c \text{ structures on} Y^3\}$$

Starting with (M, φ, λ), we get a splitting $TM \cong \mathbb{E}_\lambda \oplus \mathbb{V}_\lambda$, where $\mathbb{E}_\lambda = \langle u, v, u \times v \rangle$ and $\mathbb{V} = \mathbb{E}^\perp$. Furthermore, we get a complex structure on $\mathbb{V}$ given by $\mathbb{J}_\lambda : \mathbb{V} \to \mathbb{V}$

$$x \mapsto (u \times v) \times x$$

λ-associative submanifolds of (M, φ, λ)

- **Definition:** We call $Y^3 \subset (M^7, \varphi, \lambda)$ λ-spin if $\lambda|_Y \subset TY$,
  and call λ- associative if $\mathbb{E}_\lambda|_Y = TY$.
- $\forall$ *spin*^{c} $(Y^3, s) \subset (M, \varphi)$ is λ-spin for some λ, with $\lambda|_Y = s$.
- $\forall$ associative $Y^3 \subset (M, \varphi)$ is λ-associative for some λ.
- $\forall$ *spin*^{c} $(Y^3, s) \subset (M, \varphi)$ is λ-associative in $(M, \varphi_\lambda, \lambda)$, where φ_λ is a deformation of the G_2 structure φ.
- Given $(M^7, \varphi, \lambda) \Rightarrow TM \cong \mathbb{E} \oplus \mathbb{V} \to M$. The transverse sections of $\mathbb{V} \to M^7$ provides λ-associative submanifolds $Y^3 \subset M$.

Seiberg-Witten like deformations of associatives

- Given (M, φ, λ), $\lambda = \langle u, v \rangle$, we can choose a unit section w of $\mathbb{V} \to M^7$ in the tubular eneighborhood of the 3-skeleton $M^{(3)} \Rightarrow$ Generically, every λ-associative submanifold $Y^3 \subset M$ admits a G_2-**frame field** $\langle u, v, w \rangle$, i.e. $TY = \langle u, v, u \times v \rangle$ and $u \times v \perp w$.

- To make deformation space of a generic assoc $Y^3 \subset (M, \varphi, \lambda)$ compact, we can consider restricted deformations: The complex structure $\mathbb{J}_\lambda : \mathbb{V} \to \mathbb{V}$ decomposes $\mathbb{V}_{\mathbb{C}} = S \oplus \bar{S} \to M^{(3)}$, where S is a $U(2)$-bundle. Cross product operation $\mathbb{E} \times \mathbb{V} \to \mathbb{V}$ induces Clifford action $T^*Y_{\mathbb{C}} \times S \to S$, $\sigma : (S \otimes \bar{S})_0 \cong su(S) \to T^*Y_{\mathbb{C}}$. Then, we can write the corresponding Seiberg-Witten equations:

$$D_{\mathbf{A}}(x) = 0$$
$$*F_A = \sigma(x, \bar{x}).$$

- Question: What do these invariants measure?

Leung map:

- Let $(Y, f_0) \in (M, \varphi)$ be an associative manifold, and $\mathcal{I}_0(Y^3, M)$ be the space of immersions in the path component of f_0. Define

$$\mathcal{L} : \mathcal{I}_0(Y^3, M) \to \mathbf{R}$$

For $(Y, f) \in (M, \varphi)$, pick $F : Y \times [0,1] \to M$ with $F_0 = f_0$ and $F_1 = f$

$$\mathcal{L}(Y, f) = \int_{Y \times [0,1]} F^*(*\varphi)$$

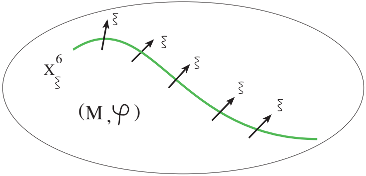
- Gradient flow $d\mathcal{L}(v) =<< \chi, v >>$, $v = \{F_t\}$ vector field on $\mathcal{I}_0$

$$d\mathcal{L}(Y, f_t) = \int_{Y \times [0,1]} F_t^*(*\varphi) = \int_{Y \times [0,1]} d(v \lrcorner * \varphi) = \int_Y < f^*\chi, v >$$

- What is the Morse homology of $\mathcal{L}$ with critical points associative 3-manifolds and the gradient flow χ-flow?

A mirror duality induced by (M, φ)

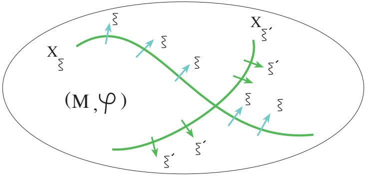
- (X^6, ω, Ω, J) is *almost Calabi-Yau*, if X is a Riemannian manifold with a non-deg 2-form ω ($\omega^3 = 6vol(X)$) which is co-closed, and J is a metric invariant almost complex structure which is compatible with ω, and Ω is a non-vanishing $(3,0)$ form with $Re\,\Omega$ closed. (when ω and $Im\,\Omega$ are closed it is a Calabi-Yau manifold).



- Thm (A-Salur): Given a G_2 manifold (M, φ) and a unit vector field ξ which comes from a codimension one foliation. Then $(X_\xi, \omega_\xi, \Omega_\xi, J_\xi)$ is an almost Calabi-Yau manifold with $\varphi|_{X_\xi} = Re\,\Omega_\xi$ and $*\varphi|_{X_\xi} = \star\omega_\xi$. Furthermore if $\mathcal{L}_\xi(\varphi)|_{X_\xi} = 0$ then $d\omega_\xi = 0$, and if $\mathcal{L}_\xi(*\varphi)|_{X_\xi} = 0$ then J_ξ is integrable, and when both of these conditions are satisfied then $(X_\xi, \omega_\xi, \Omega_\xi, J_\xi)$ is a Calabi-Yau.

Dual Calabi-Yau's in (M, φ)

- Let (M, φ, λ) be a G_2 manifold with a 2-plane field $\lambda \Rightarrow TM = \mathbb{E} \oplus \mathbb{V}$. Let $\xi \in \mathbb{E}$ and $\xi' \in \mathbb{V}$ be vector fields. We call the induced Calabi-Yau's X_ξ and $X_{\xi'}$ *mirror dual*.
- Question: Mirrror dual $\Rightarrow$ Mirror symmetric ? (Checked in the trivial case of T^7, with Salur)



- Theorem (A-Efe-Salur) In a Joyce manifold (M, φ) Mirror duality $\Rightarrow$ Mirror symmetry, where X_ξ and $X_{\xi'}$ are pair of Borcea-Voisins obtained by resolving $(E \times N^4)/k$ where E is an alliptic surface with involution i, and N is $K3$ surface with involution j and $k = i \times j$ with $b^{1,1}(X_\xi) = b^{2,1}(X_{\xi'})$=19.

The example $\wedge^2_+(S^4)$

- Let $X_1 = T^*(S^3)$, and $X_2 = \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ (bundle over S^2). By this method they occur as X_ξ and $X_{\xi'}$ inside of the Bryant-Salamon G_2 manifold $M = \wedge^2_+(S^4)$