Dolgachev surface

Selman Akbulut

Michigan State University

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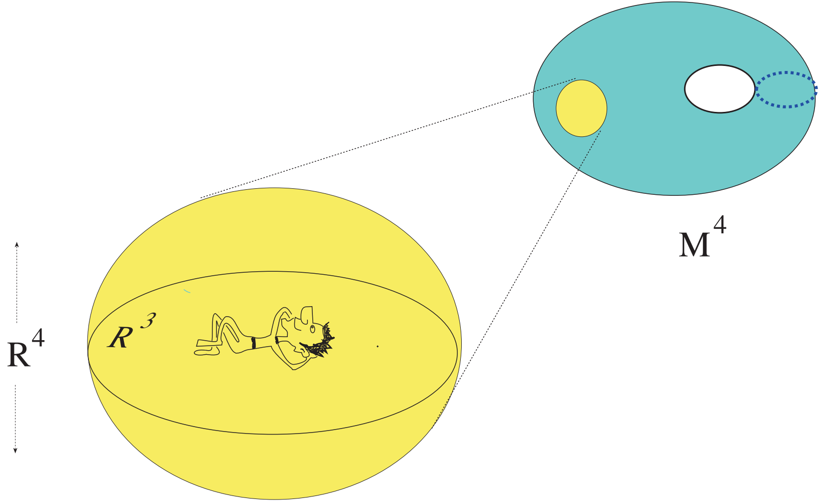
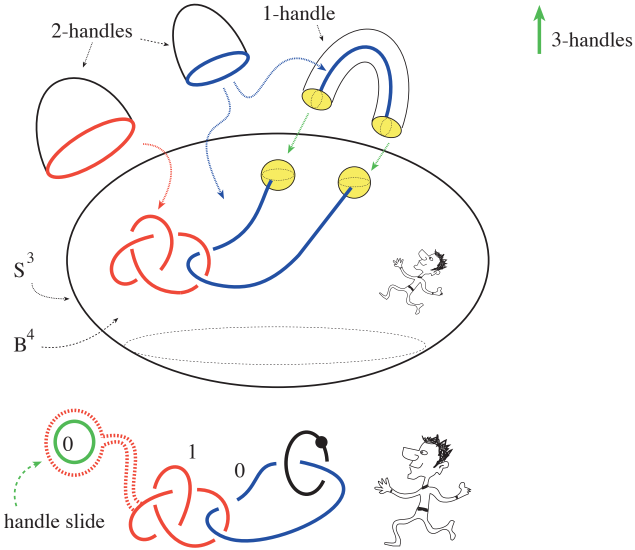
4-manifolds are spaces that locally look like R^4

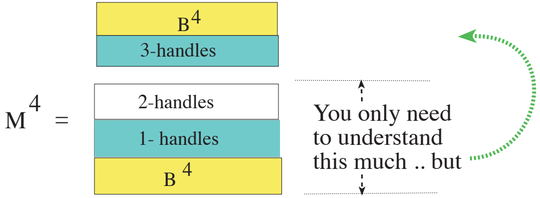


4-Manifolds, as you see it

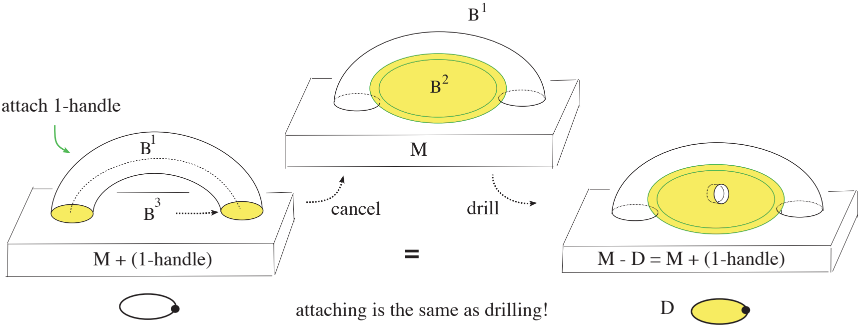


he lives in 4-manifold !

k-handle $B^4 = \mathbf{B}^k \times B^{4-k}$ attached along $\mathbf{S}^{k-1} \times B^{4-k}$

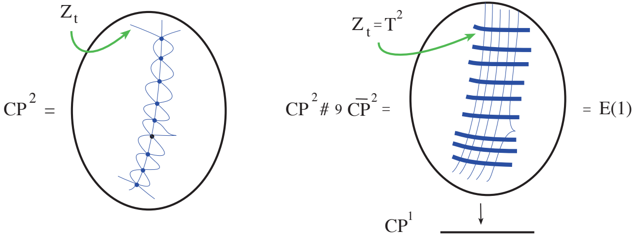


These 3-handles at top makes our understanding harder!



The Dolgachev surface

- Dolgachev surface $E(1)_{2,3}$ is an elliptic complex surface, which is obtained from the standard elliptic surface $E(1) = \mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2$, by the log transform operation of orders 2 and 3 on two disjoint fibers.
- $E(1)$: Let $P_0(z)$ and $P_1(z)$ be a generic pair of homogenious cubic polynomials in $\mathbf{C}^3$. For each $t = [t_0, t_1] \in \mathbf{CP}^1$ the following sets $Z_t = \{z \in \mathbf{CP}^2 \mid t_0 P_0(z) + t_1 P_1(z) = 0\}$ fill $\mathbf{CP}^2$ (generically each Z_t is a torus).

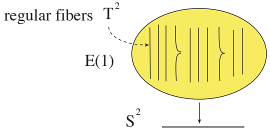


$E(1)_{2,3}$

- $E(1)_{2,3}$ is obtained from the standard elliptic surface $E(1)$, by the log transform operation of orders $p = 2$ and $p = 3$ on two disjoint fibers. Remove two disjoint fibers $T^2 \times D^2$ and glue back by the diffeomorphism of T^3 given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & p \end{pmatrix}$$

- Equivalently $E(1)_{2,3}$ is obtained by replacing one $T^2 \times D^2$ by $(S^3 - N(K)) \times S^1$, where $N(K)$ is the tublar neighborhood of the trefoil knot in S^3 (so called "Knot Surgery" operation of Fint-Stern)

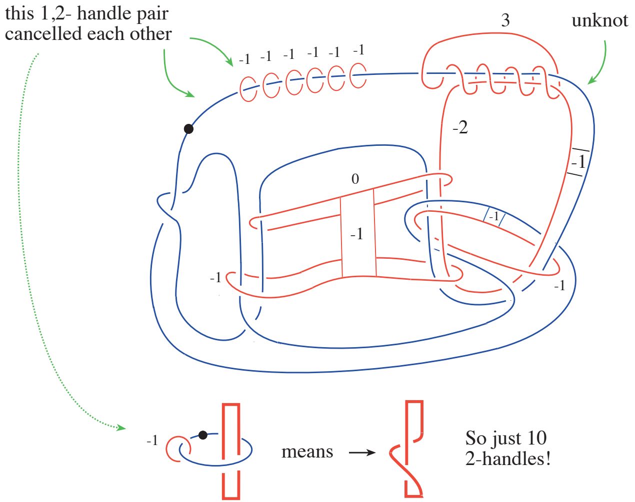


Is there an exotic copy of S^4 or $\mathbf{CP}^2$?

- If an exotic copy of S^4 or $\mathbf{CP}^2$ exits it must have 1- or 3- handles!
- 24 years ago Donaldson gave the first example of an oriented exotic smooth 4-manifold. he proved that Dolgachev surface $E(1)_{2,3}$ is an exotic copy of $E(1)$. About the same time, Harer, Kas and Kirby wrote a book about $E(1)_{2,3}$ where they conjectured that it must contain 1- or 3- handles.
- (A) In 2008 Kirby-Kas-Harer conjecture was disproved: $E(1)_{2,3}$ admits an handlebody without 1- and 3-handles.To prove this we start with an handle picture of $E(1)_{2,3}$, and turn this handlebody upside down twice while canceling its handles! What does this say about the exotic smooth structures on 4 manifolds?

First cancel 1-handles, in order to cancel 3-handles turn it upside down and cancel 1-handles. Finally to make it look pretty turn it upside down again.. Keep turning it upside down as you cancel its handles..

Here is the picture of $E(1)_{2,3}$ without 1- or 3-handles



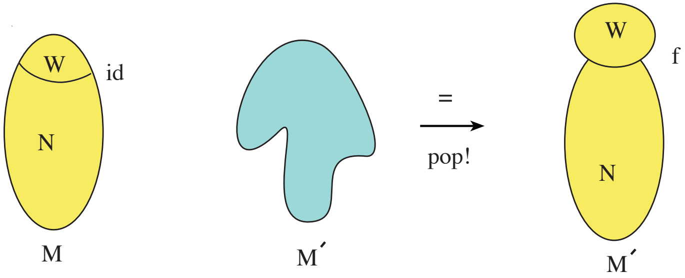
- How does this exotic copy of $E(1)$ compere with its original?

How does an exotic copy of any smooth M look like?

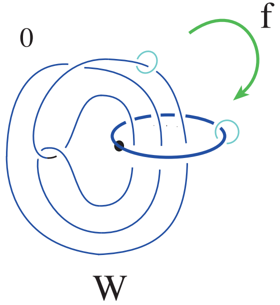
Let M be a smooth closed simply connected 4-manifold, and M' be an exotic copy of M. Then we can find a compact contractible codim zero submanifold $W \subset M$ with complement N, and an involution $f : \partial W \to \partial W$ giving decompositions: $M = N \cup_{id} W \ , \quad M' = N \cup_f W$

Furthermore, we can make the each piece W and N Stein manifolds!

(*This was first observed on an example by A, then the general result was proven by Matveyev and independently by Curtis-Freedman-Hsiang-Stong. The Stein part is due to A and Matveyev.)*

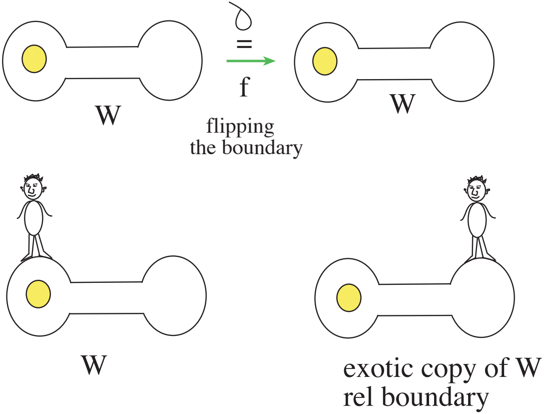


- A **Cork** is a pair (W, f), where W is a compact contractible Stein manifold, and $f : \partial W \to \partial W$ is an involution, which extends to a self-homemorphism of W, but it does not extend to a self-diffeomorphism of W. We say (W, f) is a cork of M, if we have the above decomposition for some exotic copy M' of M.

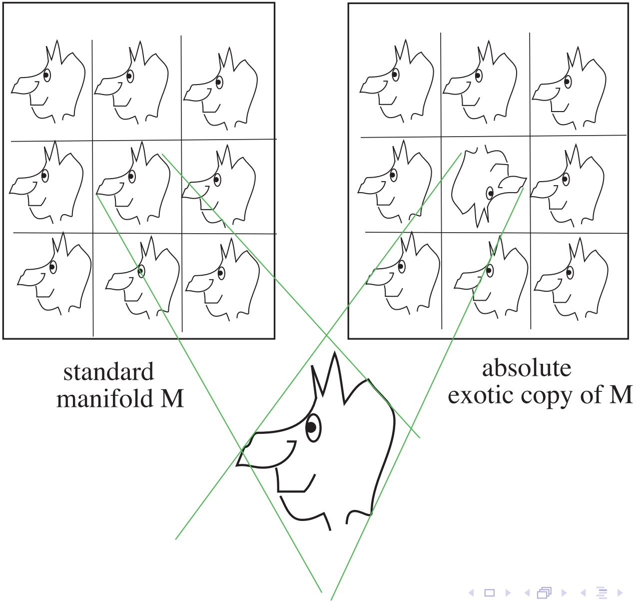


Cork! smallest exotic manifold relative to its boundary

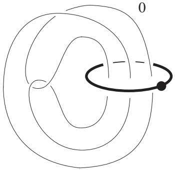
Only when I am standing on it, this space looks exotic!



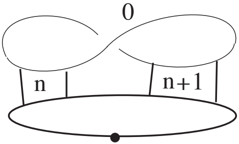
Wallpaper depiction of a manifold and its exotic copy



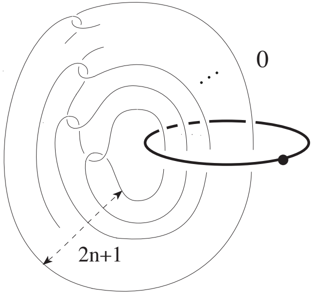
Various corks



W_1



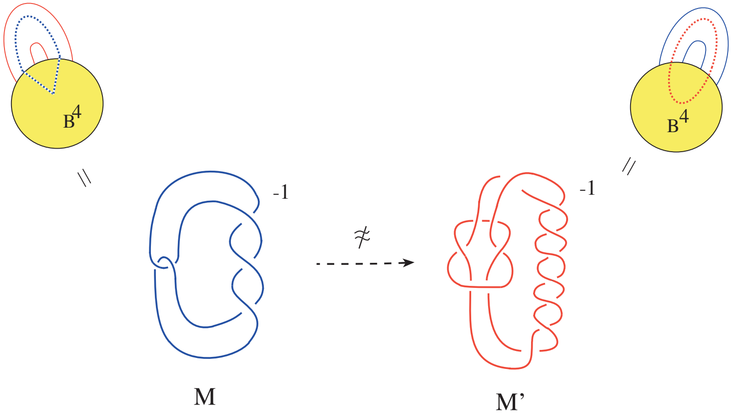
W_n



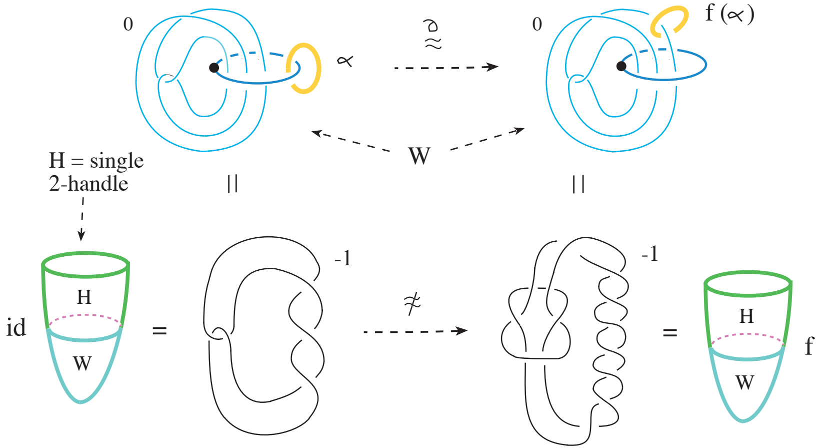
$\bar{W}_n$

The smallest absolutely exotic 4-manifold pairs I know

- (A) 1991 (where the first “cork” was introduced)

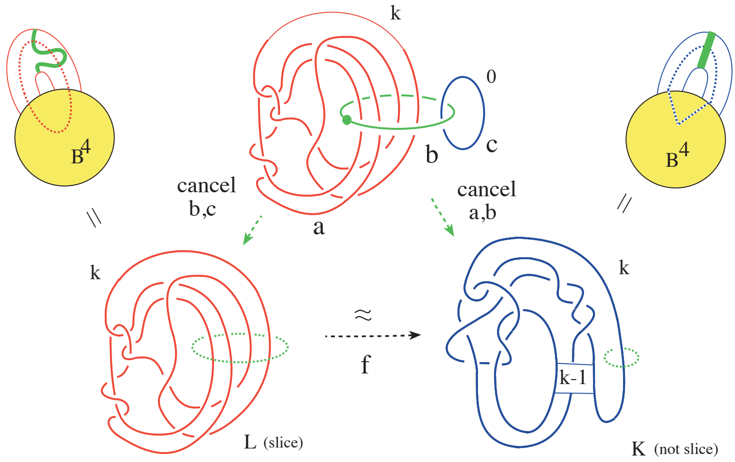


Locating corks in the small exotic 4-manifold pairs



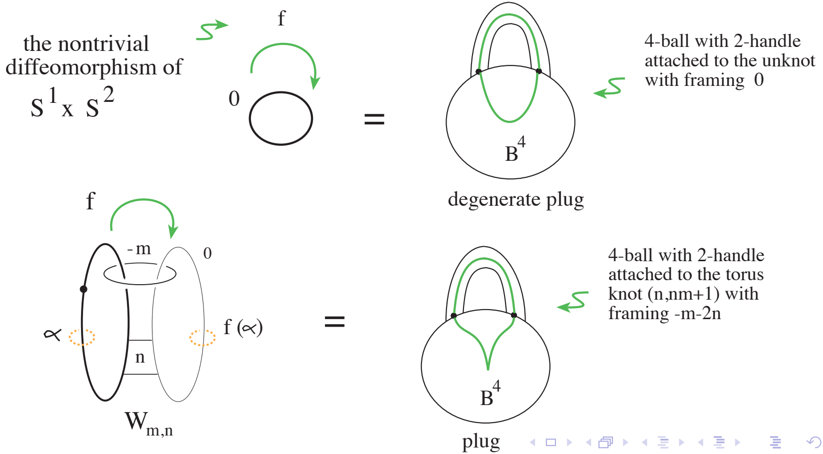
Amusing diffeomorphic 4-manifold pairs (not everything which looks exotic is exotic!)

- (A) 1977 (where “circle with dot ”notation for 1-handle introduced)

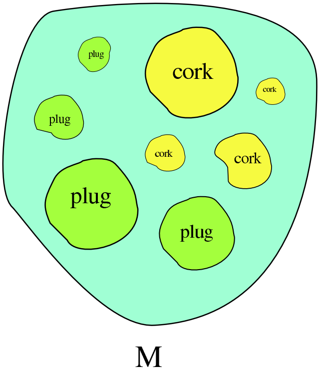


Plugs

- A **Plug** is a pair (W, f), where W is a compact Stein manifold, and $f : \partial W \to \partial W$ is an involution, which does **not** extend to a self homeomorphism of W, and there is the above decomposition for some exotic copy M' of M *(Plugs were discovered by A and K. Yasui)*.



Atomic picture of 4-manifolds



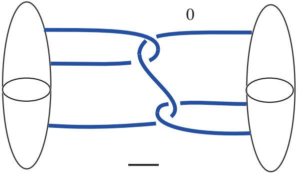
- Corks ans plugs should be considered as freely moving basic particles in a 4-manifold M relating it to its exotic copies.
- Corks and Plugs can knot in M infinitely many different ways (A-Yasui)
- Well then, where are the corks/plugs of $E(1)_{2,3}$ relating it to $E(1)$?

Cork and Plug decompositions of $E(1)_{2,3}$

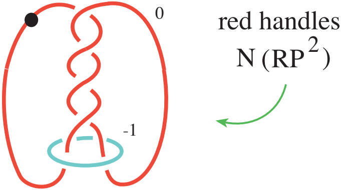
(1) $E(1)_{2,3}$ is obtained by cork twisting of $E(1)$ along the cork $\bar{W}_1$, i.e. we can decompose $E(1)_{2,3} = N \cup_{id} \bar{W}_1$, so that $E(1) = N \cup_f \bar{W}_1$.

(2) $E(1)_{2,3}$ is obtained by plug twisting of $E(1)$ along the plug $W_{1,2}$, we can decompose $E(1)_{2,3} = N \cup_{id} W_{1,2}$, with $E(1) = N \cup_f W_{1,2}$.

(3) $E(1)_{2,3}$ is obtained from $E(1)$ by twisting along an $\mathbf{RP}^2$.



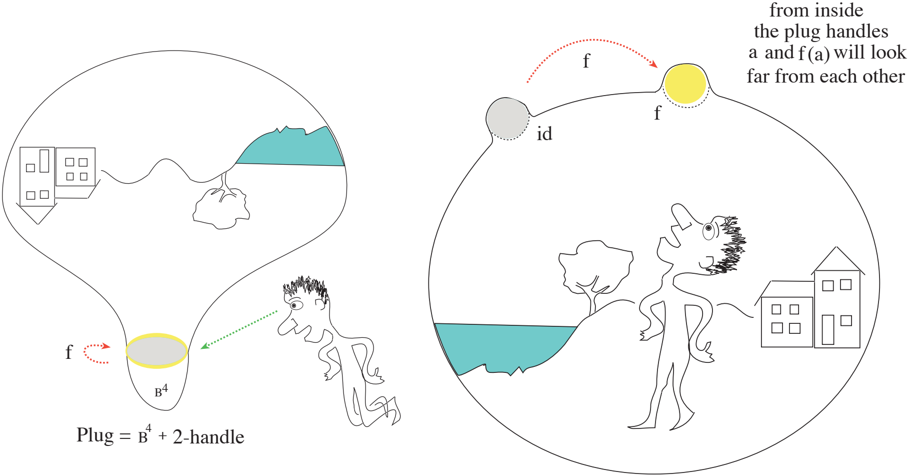
$\overline{W}_1$



$W_{1,2}$

Remarks: (1) is proven by inspection, (2) uses upside down turning technique, plus the result "Scharlemann manifold is standard". (3) follows from (2) since $W_{1,2}$ contains an $\mathbf{RP}^2$.

From inside 4-manifold watching a cork/plug in the sky



Locating the $W_{1,2}$ plug in $E(1)_{2,3}$ (a road map)