Double knot surgeries to S^4 and $S^2 \times S^2$

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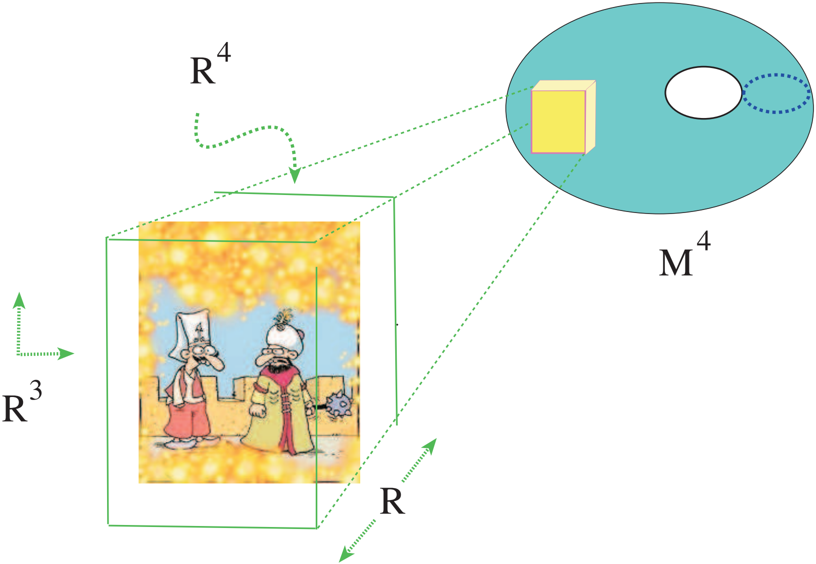
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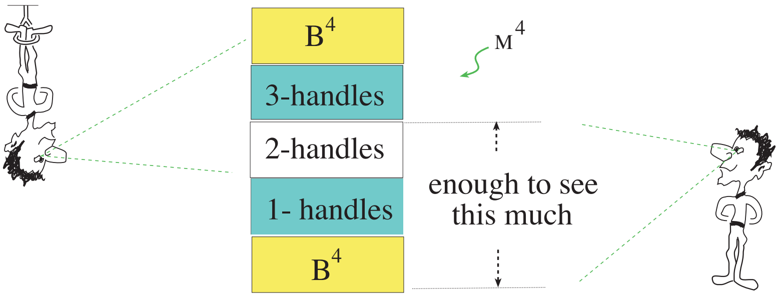
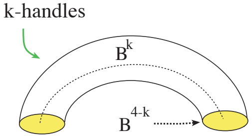
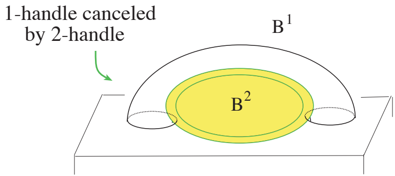
4-manifold as a handlebody (Morse function)

A k-handle is just a ball $B^4 = \mathbf{B}^k \times B^{4-k}$ ($k = 0, 1, 2, 3, 4$) attached along $\partial \mathbf{B}^k \times B^{4-k} = \mathbf{S}^{k-1} \times B^{4-k}$.

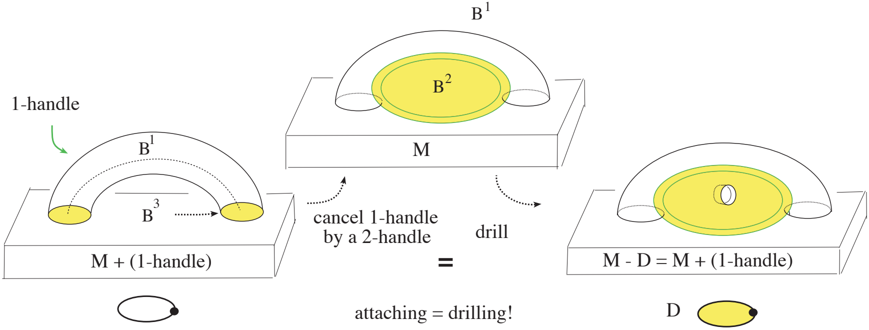




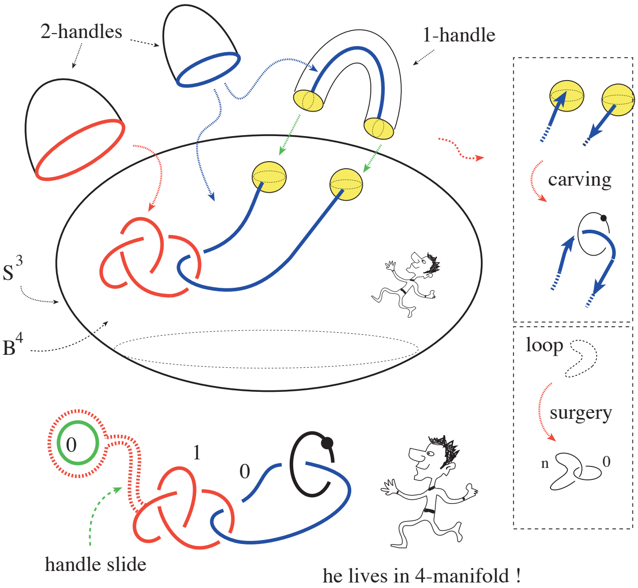


3 basic principles for 4-manifold handlebodies

(1) 0, 1, and 2-handles are sufficient (Laudenbach-Poenaru/1972)
(2) "**Carving**" : Adding 1-handle = Subtracting 2-handle (A/1977)
(3) When stuck don't despair, turn handlebody upside down!



Visualizing 4-manifolds

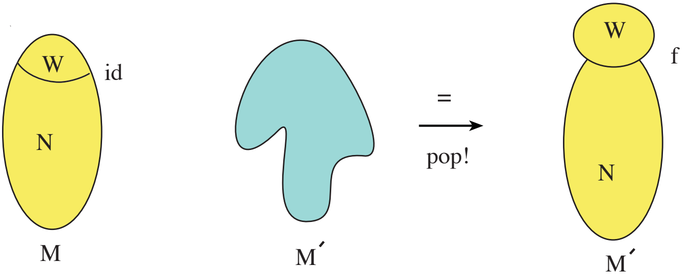


How does an exotic copy of a smooth M look like?

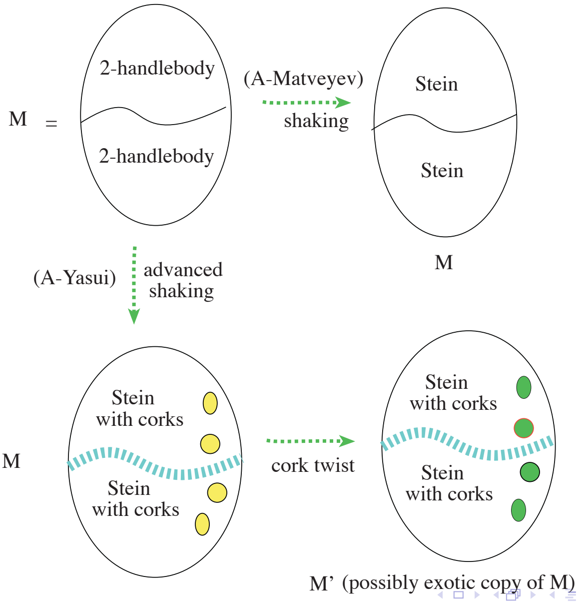
Let M be a smooth closed simply connected 4-manifold, and M' be an exotic copy of M. Then we can find a compact contractible codim zero submanifold (cork) $W \subset M$ with complement N, and an involution $f : \partial W \to \partial W$ giving decompositions: $M = N \cup_{id} W$, $M' = N \cup_f W$

Furthermore, we can make the each piece W and N Stein manifolds!

(*This was first observed on an example by A, then the general result was proven by Matveyev and independently by Curtis-Freedman-Hsiang-Stong. The Stein part is due to A and Matveyev.*)



"Cork" approach to produce exotic copies of M^4 ?



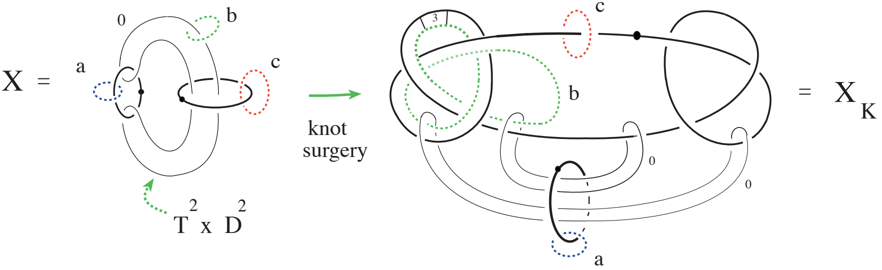
Fintushel-Stern knot surgery operation $X \rightsquigarrow X_K$:

Let X be a smooth 4-manifold, and $T^2 \times D^2 \subset X$ be an imbedded torus with trivial normal bundle, and $K \subset S^3$ be a knot, $N(K)$ be its tubular neighborhood. The Fintushel-Stern **knot surgery operation** is the operation of replacing $T^2 \times D^2$ with $(S^3 - N(K)) \times S^1$, so that the meridian $p \times \partial D^2$ of the torus coincides with the longtitude of K:

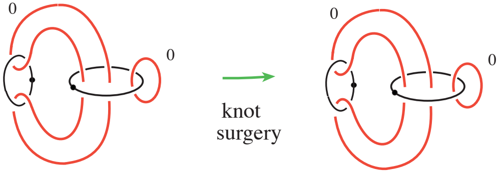
$$X \rightsquigarrow X_K = (X - T^2 \times D^2) \cup (S^3 - N(K)) \times S^1$$

- 1998 Fintushel-Stern introduced this operation, in particular they showed that if X has nonzero Seiberg-Witten invariant and $[T] \neq 0$ and $\Delta(K) \neq 1$, then X_K is an exotic copy of X.
- 1999 (A) gave a handlebody description of the operation $X \rightsquigarrow X_K$, (and used to construct exotic cusps and fishtails).

Handlebody picture of the operation $X \rightsquigarrow X_K$

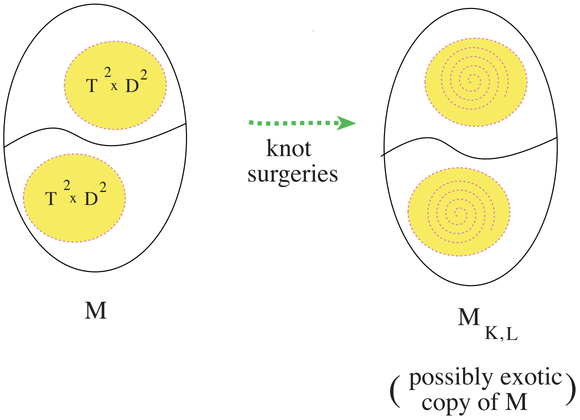


Fundamental Observtion :



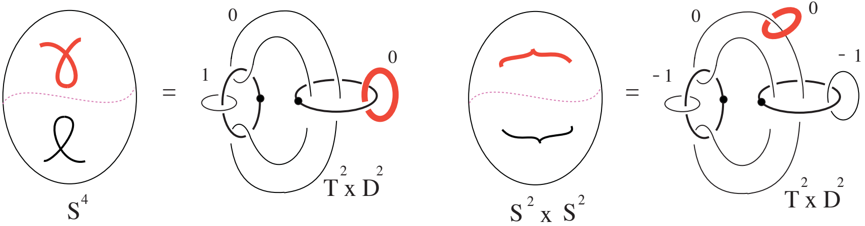
No change!

"Knot surgery" approach to find exotic copies of M^4 ?



Seeking candidates for exotic S^4 and $S^2 \times S^2$

- S^4 = Union of two *Fishtails* glued along their common boundaries,
- $S^2 \times S^2$ = Union of two Cusps glued along common boundaries
- Let $X_K = X$ "knot surgered" copy of X by using a knot $K \subset S^3$.



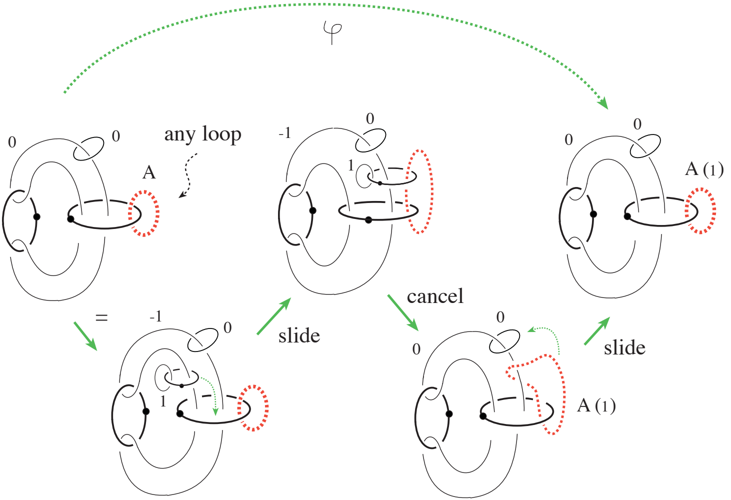
Theorem

For any choices of knots $K, L \subset S^3$ we have

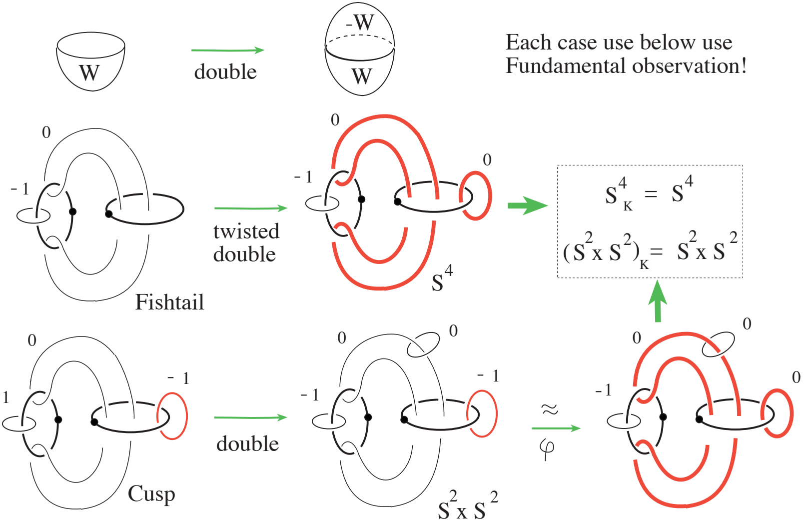
(a) $S^4_{K,L} = S^4$

(b) $(S^2 \times S^2)_{K,L} = S^2 \times S^2$

The diffeomorphism φ

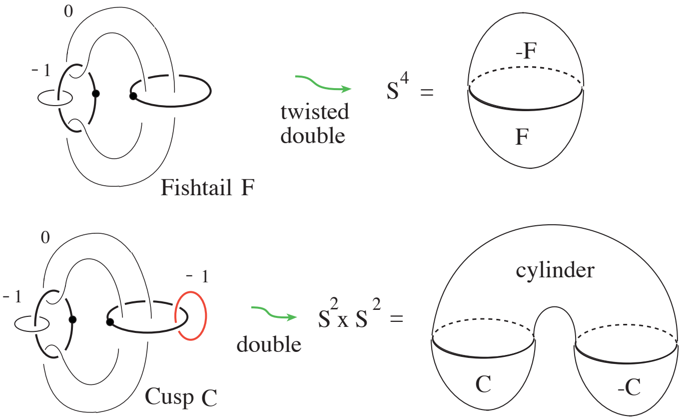


Poof (special case): $S^4_K = S^4$, $(S^2 \times S^2)_K = S^2 \times S^2$



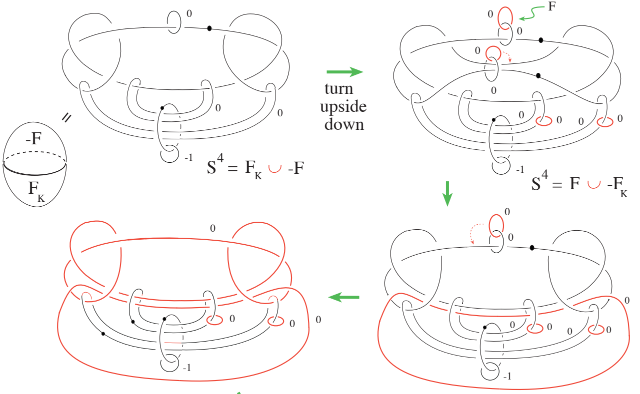
General case: $S^4_{K,L} = S^4$, $(S^2 \times S^2)_{K,L} = S^2 \times S^2$

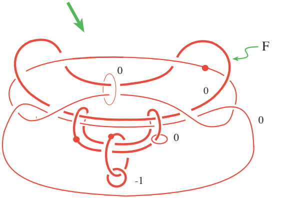
Lesson learned: Doubling different way results big difference in the handlebody, and each different case makes the each proof possible.



Then do Knot surgery
to each side!

$S^4_{K,L} = S^4$





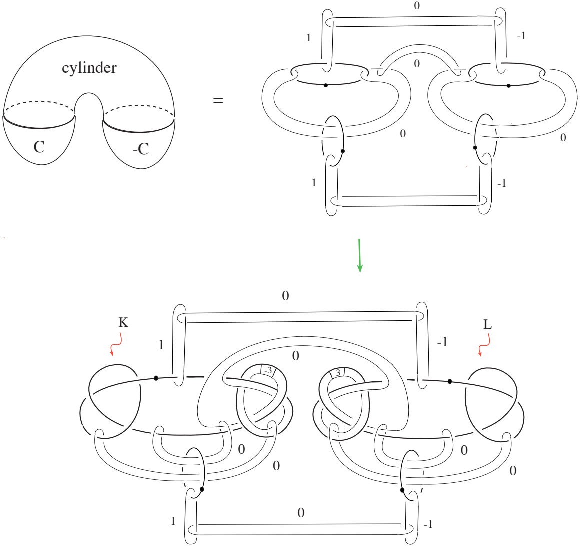
Now use fundamental observation for!

$S^4_K = F \cup -F_K$

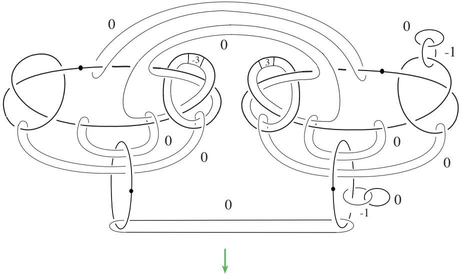
↓

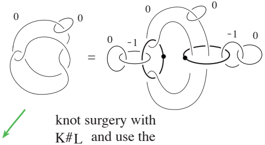
$S^4_{K,L} = F_L \cup -F_K$

$(S^2 \times S^2)_{K,L}$



$(S^2 \times S^2)_{K,L} = (S^2 \times S^2)_{K\#L} = S^2 \times S^2$





knot surgery with
K#L and use the
fundamental
observation