Knot surgery and Scharlemann manifolds

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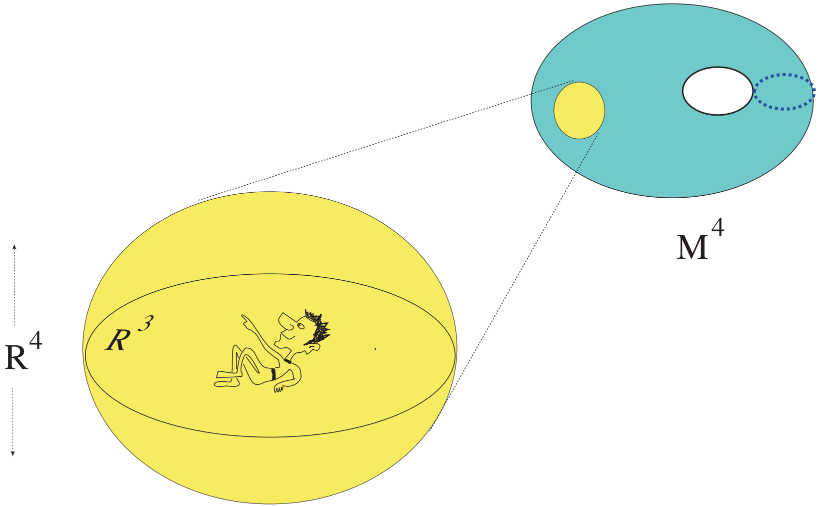
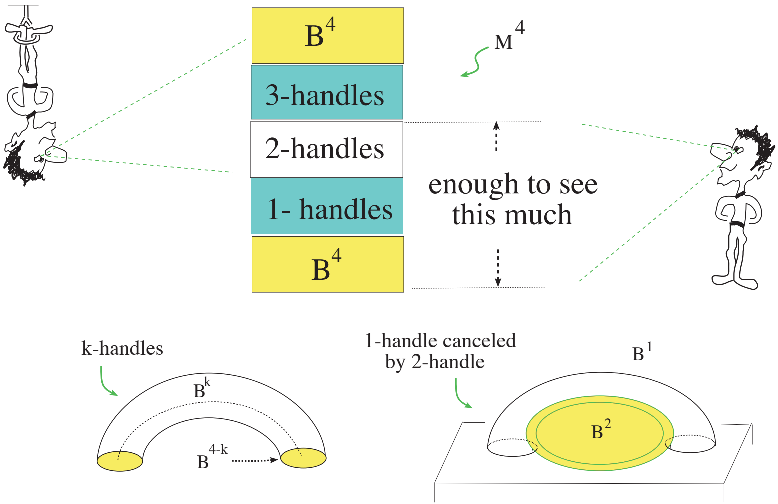
4-manifolds are spaces that are locally R^4



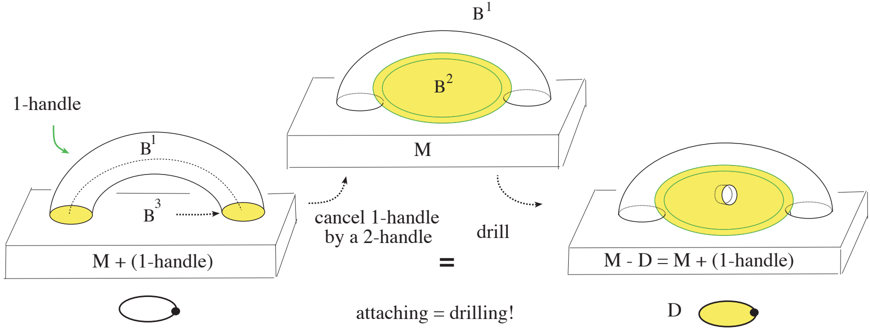
Handles of 4-manifolds:

A k-handle is just a ball $B^4 = \mathbf{B}^k \times B^{4-k}$ $(k = 0, 1, 2, 3, 4)$ attached along $\partial \mathbf{B}^k \times B^{4-k} = \mathbf{S}^{k-1} \times B^{4-k}$.

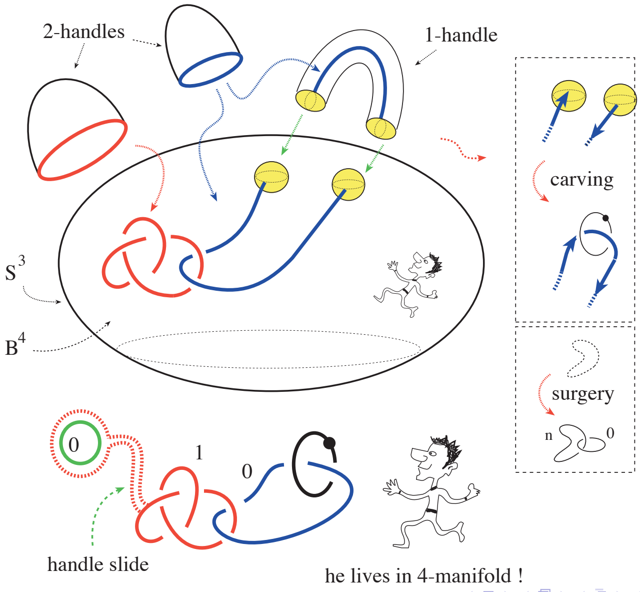


3 basic principles for 4-manifold handlebodies:

(1) 0, 1, and 2-handles are sufficient (Laudenbach-Poenaru/1972)
(2) "**Carving**" : Adding 1-handle = Subtracting 2-handle (A/1977)
(3) When stuck don't despair, turn handlebody upside down!



All the tools you need to study 4-manifolds:



Scharlemann manifolds $M(K)$

Let $K \subset S^3$ be a knot, and S^3_K be the 3-manifold obtained from S^3 by ± 1 surgery to K. **Scharlemann manifold** $M(K)$ is the manifold obtained by surgering the meridianal circle $C \subset S^1 \times S^3_K(\pm 1)$ of K.

- 1974 (Scharlemann) studied $M(K)$ when K is the trefoil knot, and showed that there is an exotic homotopy equivalence:

$$f : M(K) \to S^1 \times S^3 \# (S^2 \times S^2)$$

  He asked if $M(K) = S^1 \times S^3 \# (S^2 \times S^2)$? or if $M(K)$ itself exotic.

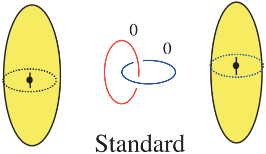
- 1999 (A) Scharlemann problem was solved affirmatively, i.e.

$$M(K) = S^1 \times S^3 \# (S^2 \times S^2) \qquad (*)$$

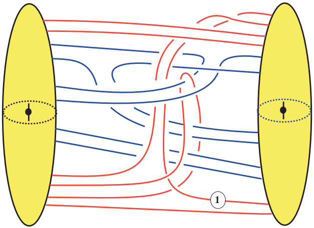
- 2010 (A,T) I will show a quick proof of the equality (*) for any knot K via the handlebody description of the Fintushel-Stern knot surgery which we gave previously (Tange proposed another longer proof via a result of Ohyama which I haven't verified).

An interesting exotic cousin of $M(K)$

1985 (A) In an attempt to show $M(K)$ exotic, we ended up constructing an exotic copy of the nonorientable manifold $S^1 \tilde{\times} S^3 \# (S^2 \times S^2)$. This is the only exotic manifold we know, which is obtained by the **Gluck twisting** operation from its standard copy!



Standard



Exotic

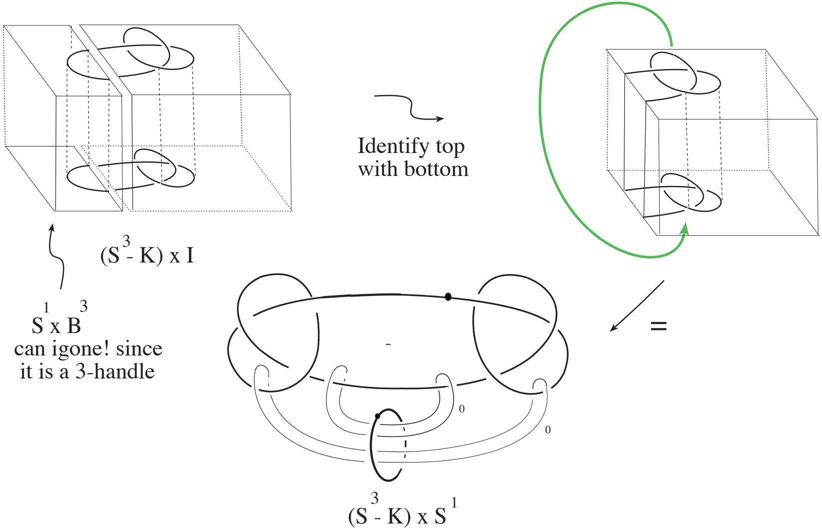
Fintushel-Stern knot surgery operation:

Let X be a smooth 4-manifold, and $T^2 \times D^2 \subset X$ be an imbedded torus with trivial normal bundle, and $K \subset S^3$ be a knot, $N(K)$ be its tubular neighborhood. The Fintushel-Stern **knot surgery operation** is the operation of replacing $T^2 \times D^2$ with $(S^3 - N(K)) \times S^1$, so that the meridian $p \times \partial D^2$ of the torus coincides with the longtitude of K:

$$X \rightsquigarrow X_K = (X - T^2 \times D^2) \cup (S^3 - N(K)) \times S^1$$

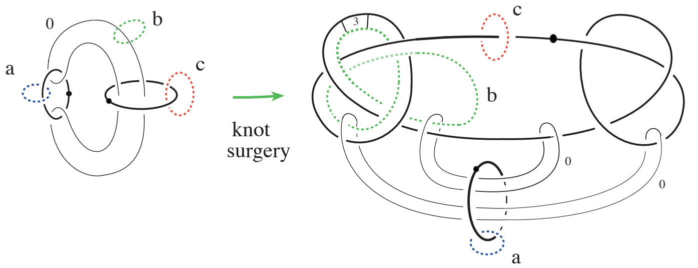
- 1998 Fintushel-Stern introduced this operation, in particular they showed that if X has nonzero Seiberg-Witten invariant and $[T] \neq 0$ and $\Delta(K) \neq 1$, then X_K is an exotic copy of X.
- 1999 (A) gave a handlebody description of the operation $X \rightsquigarrow X_K$, (and used to construct exotic cusps and fishtails). Picture shows: **Corollary**: X_K and $M(K)$ are closely realated to each other!

Handlebody picture of $(S^3 - K) \times S^1$



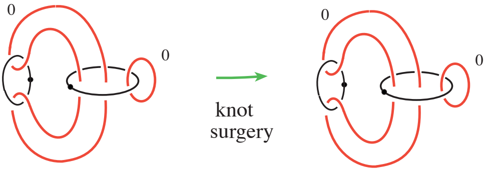
Handlebody picture of the operation $X \rightsquigarrow X_K$

X =



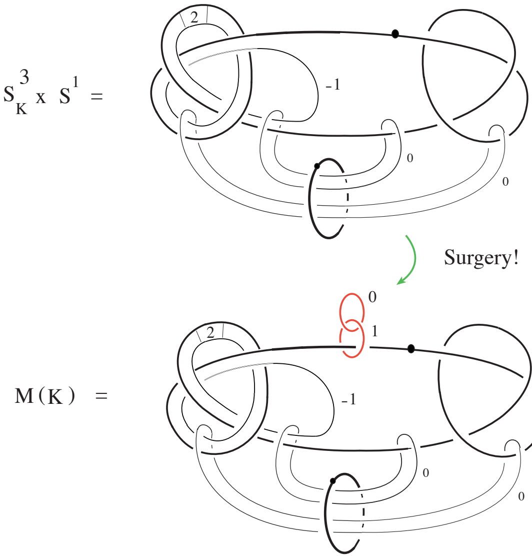
= X_K

Therefore:

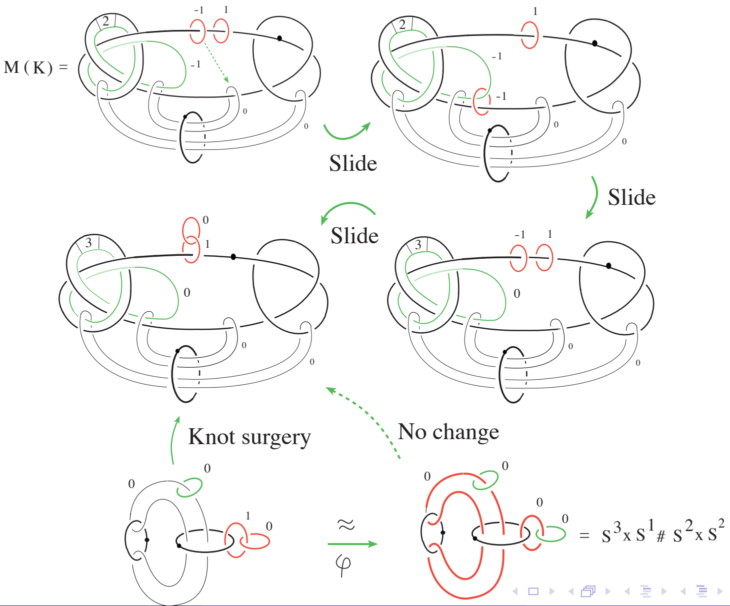


No change!

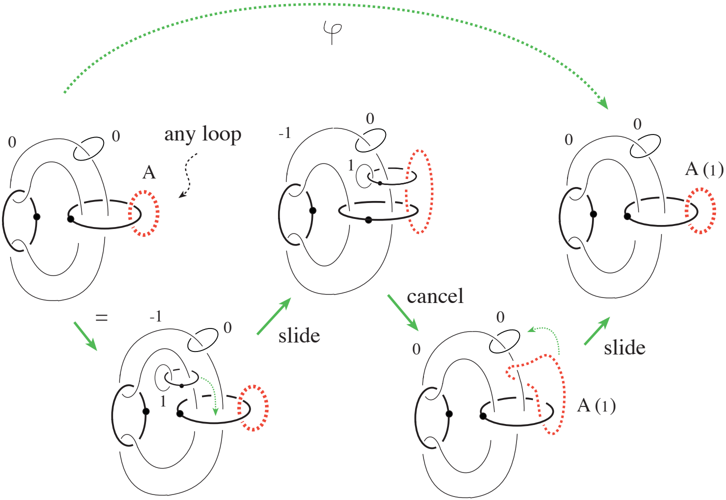
Handlebody picture of $M(K)$



Proving $M(K) = S^3 \times S^1 \# (S^2 \times S^2)$



The diffeomorphism φ



Showing $S^4_K = S^4$, and $(S^2 \times S^2)_K = S^2 \times S^2$