A NOTE ON LEFSCHETZ FIBRATIONS ON COMPACT STEIN 4-MANIFOLDS

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Abstract. Loi-Piergallini and Akbulut-Ozbagci showed that every compact Stein surface admits a Lefschetz fibration over the disk D^2 with bounded fibers. In this note we give a more intrinsic alternative proof of this result.

1. Introduction

In [AO] (also [LP] and [P]) it was proven that every compact Stein surface admits a positive allowable Lefschetz fibration over D^2 with bounded fibers (PALF in short), and conversely in [AO] it was shown that every 4-dimensional positive Lefschetz fibration over D^2 with bounded fibers is a Stein surface. The proof of [AO] uses the fact that every torus link is fibered in S^3. Here we prove this by using a more intrinsic different approach, namely by an algorithmic use of positive stabilizations. This new approach is more closely related Giroux's proof of constructing open books to contact manifolds via "contact cell decomposition" [Gi]. The algorithm in [A] constructs compatible open books for contact structures on 3-manifolds using their surgery representations. The algorithm here is for 4-manifolds, it has a similar technique and constructs PALF's on compact Stein surfaces starting from their handle diagrams which are explained briefly in [Go]. We will give a different proof of the following result:

Theorem 1.1. *Any compact Stein surface W^4 admits infinitely many pairwise inequivalent PALF's. Moreover, the corresponding open books on ∂W supports the contact structure induced by the Stein structure on W.*

We refer the reader to [GS, K] for Lefschetz fibrations, to [E, Go] for Stein manifolds, to [Et1, Et2, Ge, Gi] for contact structures and open books.

2. An alternative proof of Theorem 1.1

Let W be a compact 4-manifold admitting a Stein structure. By [E], W has a handle decomposition which consists of a single 0-handle, 1-handles, and 2-handles attached to the union of the 0-handle and the 1-handles along some Legendrian knots $L_1, ..., L_n$ with framing $tb(L_i)-1$ where "tb" denotes the Thurston-Bennequin framing. So by using [Go] we can describe W by a standard handle diagram given in Figure 1 where Legendrian tangle contains all crossings of the Legendrian link $L = L_1 \cup ... \cup L_n$ and for each crossing we assume the convention that the part of L having more negative slope crosses over the

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one having less negative slope.

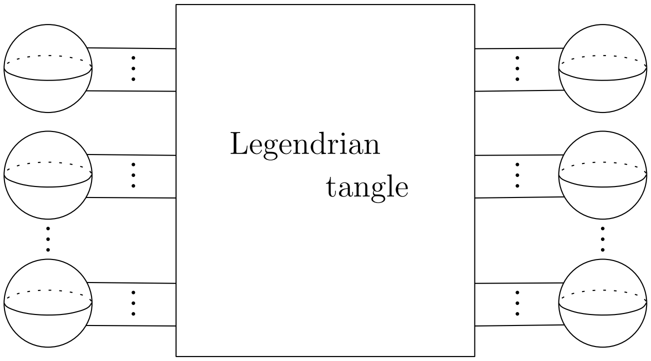


Figure 1. Handle diagram describing W

There are two cases that we need to consider:

Case 1: *If there are no 1-handles in* W.

Suppose that W is obtained from D^4 by attaching 2-handles $H_1, ..., H_m$ along a Legendrian link $L = L_1 \cup ... \cup L_m$ sitting in (S^3, ξ_{std}). We will modify the algorithm of [A] to construct a PALF on D^4 where we can realize each component of L on a page of the PALF. Note that the algorithm of [A] also guarantees that the page framing of L_i is equal to $tb(L_i)$ for each i. Therefore, once we realize L on pages of the PALF of D^4, attaching each H_i along L_i will extend the PALF structure, and we will be done.

For simplicity we'll take L to be Legendrian right trefoil in our pictures. Given $L \subset (S^3, \xi_{std})$ we consider its front projection onto the yz-plane in $(\mathbb{R}^3, \xi_0 = \text{Ker}(dz + xdy))$ as in Figure 2. We divide the interior of the projection into rectangles $\{R_i\}$ using the lines with slope $+1$ (see Figure 2-b). Note that i increases from down to up and right to left.

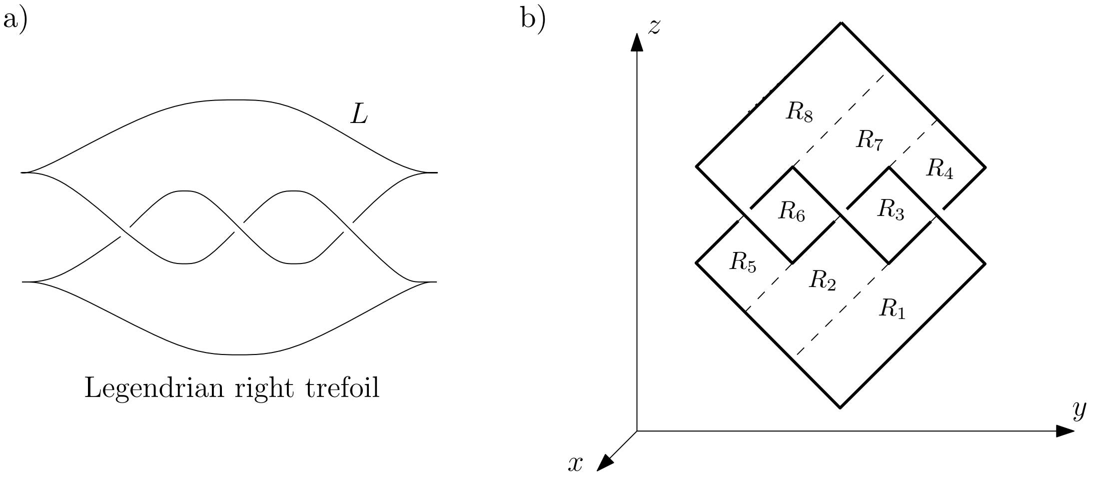


Figure 2. a) Legendrian $L \subset (S^3, \xi_{std})$, b) Front projection of L

Here the main difference is that the Legendrian link L (equipped with contact surgery coefficients) describes a contact surgery diagram (for some contact manifold) in [A], whereas here in our case it describes a compact Stein surface obtained from D^4.

As in [A], for each R_i, we construct the Hopf band F_i in $(\mathbb{R}^3, \xi_0) \subset (S^3, \xi_{std})$ by following the contact planes. Also we push the opposite sides of R_i along the positive and negative x-axis and glue them using cords along the x-axis. This gives us a Legendrian unknot γ_i sitting on F_i with page framing equal to $tb(\gamma_i) = -1$ (see Figure 3).

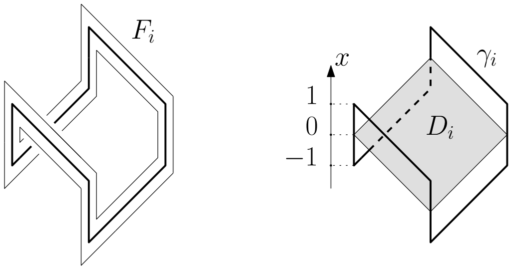


FIGURE 3. Constructing the Hopf band F_i and the Legendrian unknot γ_i

We first consider the trivial PALF on D^4 with fibers D^2 and the trivial monodromy as in Figure 4-a. Note that the unique Stein structure on D^4 induces the unique tight structure ξ_{std} on S^3, and this trivial PALF induces the compatible open book for ξ_{std} with the same pages (fibers) and the monodromy. Now consider R_1 and glue the missing part of F_1 to the fiber D^2 and compose the positive Dehn twist t_1 along γ_1 with the existing (trivial) monodromy. This gives a new PALF structure on D^4 with a regular fiber F_1 and the monodromy t_1. In this process, we are actually positively stabilizing D^4 (see Figure 4-b). Also the new corresponding open book on S^3 still supports ξ_{std} by [A].

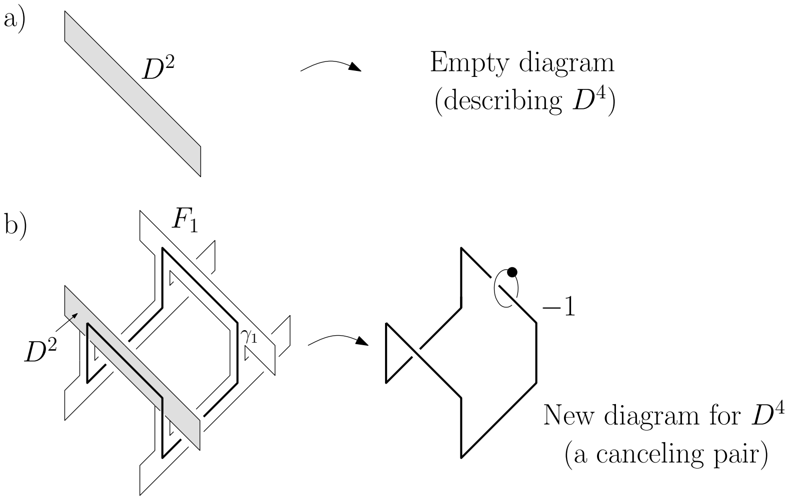


FIGURE 4. a) A regular fiber D^2 of the trivial PALF on D^4, b) a new PALF structure on D^4 with a regular fiber F_1

Next we introduce other rectangles one by one (in ascending order) to the projection. For each R_i introduced, we positively stabilize D^4 and extend the PALF structure as explained in the following remark.

Remark 2.1. To be able to extend the PALF structure, we must introduce the rectangles in a special order. Such order is provided by how we number the rectangles above. More precisely, the above ordering guaranties that attaching the missing part of the Hopf band F_i to the fiber, say S_{i-1}, of the PALF corresponding to $R_1, ...R_{i-1}$ is equivalent to plumbing a positive Hopf band to S_{i-1}, and so the new surface S_i is a fiber of a new PALF. We will explicitly show this equivalence only for Case 2 below (see Lemma 2.2). Showing the equivalence for Case 1 is straightforward (comparing to that for Case 2). Therefore, we leave it to the reader.

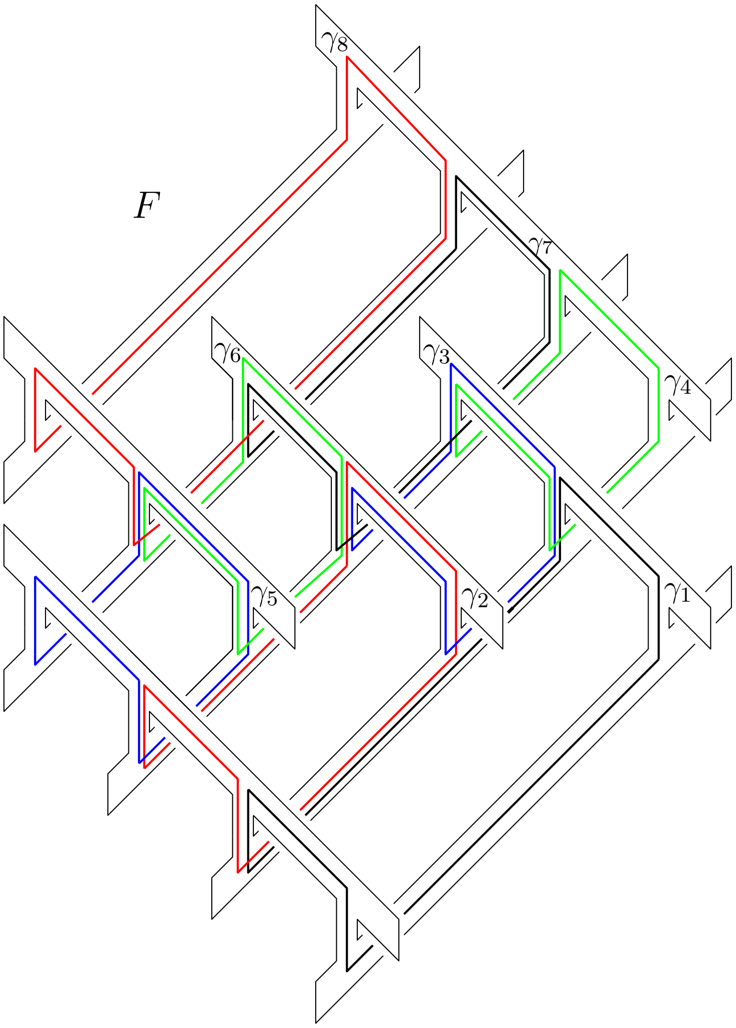


FIGURE 5. The fiber F of the PALF on D^4 containing L

When all rectangles are introduced, we get a new PALF structure on D^4 with bounded fiber F (union of Hopf bands $F_1, ..., F_n$) and monodromy $t_1 t_2 ... t_n$ where each t_i is the positive Dehn twist along the Legendrian unknot γ_i coming from the stabilization corresponding to R_i (see Figure 5). Note that this process does not change D^4 because in the corresponding handle diagram we have n canceling pairs of 1- and 2-handles as shown in Figure 6 where we use the convention that γ_i crosses over γ_j if $i > j$. Here we consider each γ-curve in a different page F of the corresponding open book supporting ξ_{std} by pushing them in the (pointing out) normal direction of F.

So far we have constructed a PALF structure on D^4 such that the Legendrian link $L = L_1 \cup ... \cup L_m$ is embedded on a page F of the open book supporting ξ_{std}. Also the framing of each L_i coming from F is equal to $tb(L_i)$. Therefore, when we attach 2-handles $H_1, ..., H_m$ along the Legendrian knots $L_1, ..., L_m$, we do not only get the compact Stein surface W but also extend the PALF on D^4 to a PALF on W. Note that a regular fiber of the resulting PALF on W is still F, but the monodromy is now equal to

$$t_1 t_2 ... t_n s_1 s_2 ... s_m$$

where s_i is the positive Dehn twist along L_i for each $i = 1, ..., m$.

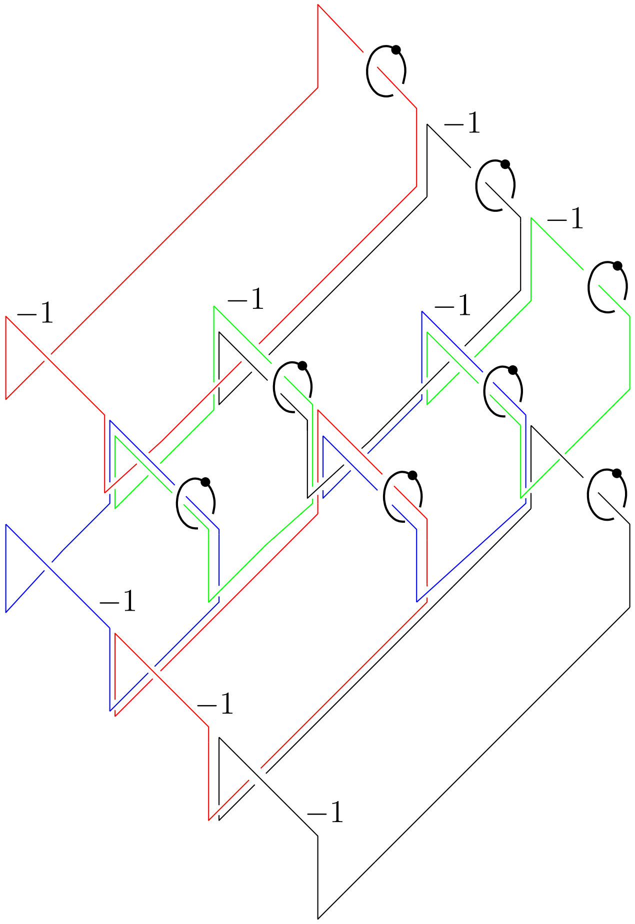


FIGURE 6. The diagram for D^4 corresponding to the PALF in Figure 5 (n canceling pairs; they are canceled from right to left)

Case 2: *If there are 1-handles in W.*

Suppose that W is obtained from D^4 by attaching r 1-handles and 2-handles $H_1, ..., H_m$ along a Legendrian link $L = L_1 \cup ... \cup L_m$ sitting in $(\#_r S^1 \times S^2, \eta_{std})$. A standard handle diagram for W is given in Figure 1. The union of D^4 and 1-handles gives $\natural_r S^1 \times D^3$ whose Stein structure induces the unique tight structure on η_{std} on $\#_r S^1 \times S^2$. First consider the trivial PALF (with trivial monodromy) on $\natural_r S^1 \times D^3$. A regular fiber of this PALF is given in Figure 7 where the reader should realize that the core circle of each Hopf band links the corresponding dotted circle once. Note that the corresponding open book supports η_{std} and in the standard handle diagram of W, if a knot passing over a particular 1-handle, then it must link to the core circle of the corresponding Hopf band.

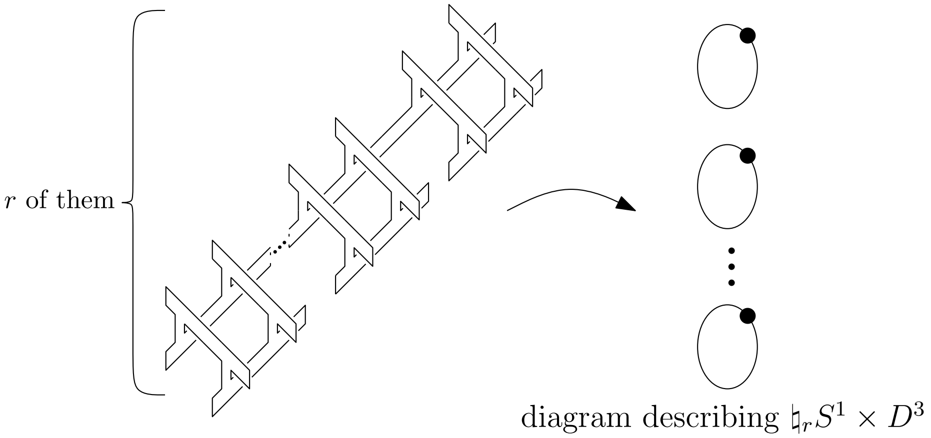


FIGURE 7. A fiber of the trivial PALF on $\natural_r S^1 \times D^3$ and the corresponding diagram

We first modify the handle diagram in Figure 1 by twisting the strands going through each 1-handle and replace 1-handles with dotted Legendrian unknots as illustrated in Figure 8 (compare with [AO]).

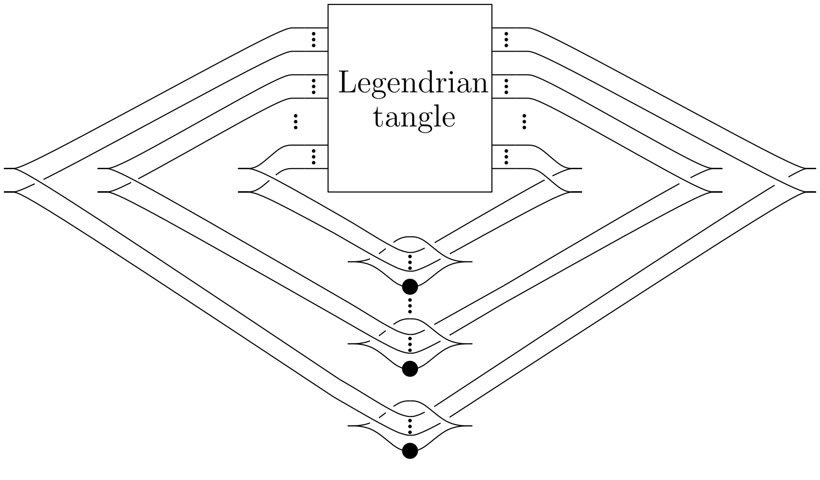


FIGURE 8. Converting the diagram

Then by pretending the resulting diagram sits in $(\mathbb{R}^3, \xi_0)$, we consider its projection onto the yz-plane as in Figure 9-a. Next we want to divide the interior of the projection into rectangles. Surely this can be done in many different ways. However, with a little care we can decrease the number n of rectangles as illustrated in Figure 9-b where we assume that the bold arcs are introduced first and they correspond the fiber of the trivial PALF on $\natural_r S^1 \times D^3$ given in Figure 7. Note that the region bounded by the bold arcs are divided into concentric rectangles, and that we add some additional arcs to the projection to be able to extend PALF structures (why we need these additional arcs will be explained below). We also number the rectangles as follows: The concentric rectangles inside bold squares come first. The squares in the colored regions $(1), (2), ..., (2r-2)$ come second in the ascending order. (Here, for each of these regions, we number the rectangles in the order indicated by the arrow. Also, for the regions $(r), ..., (2r-2)$, the boundary rectangles come first.) Finally, the rectangles in the yellow region come last in the order explained in Case 1.

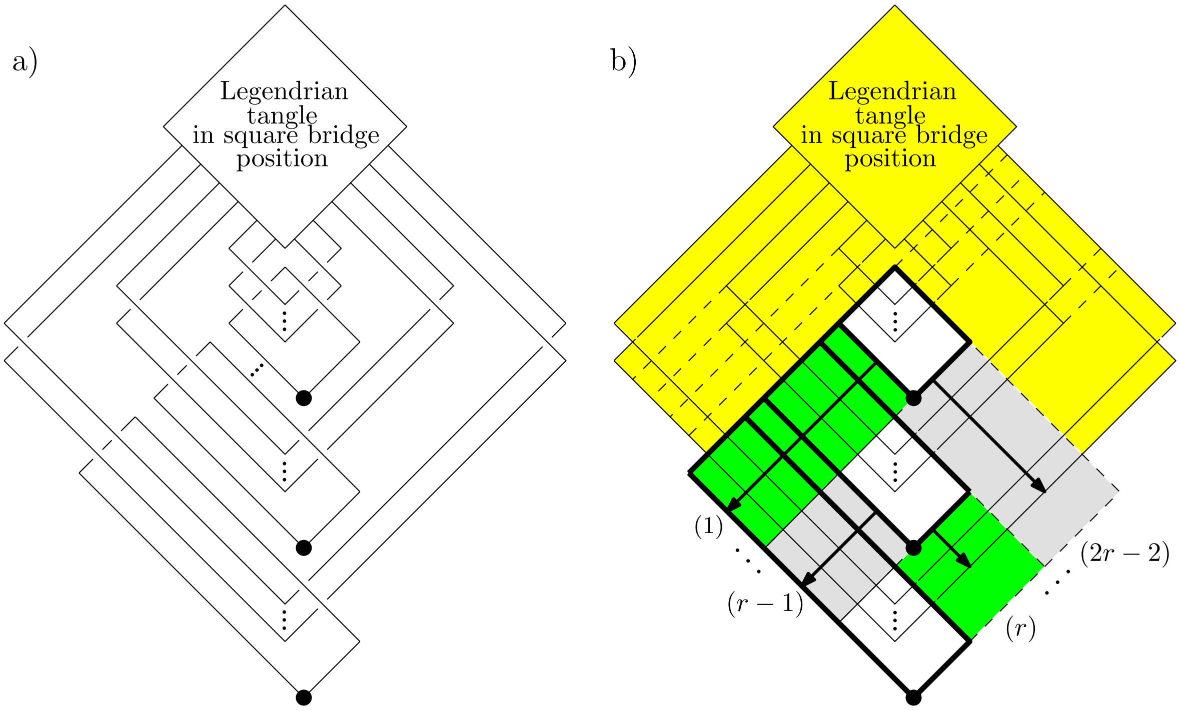
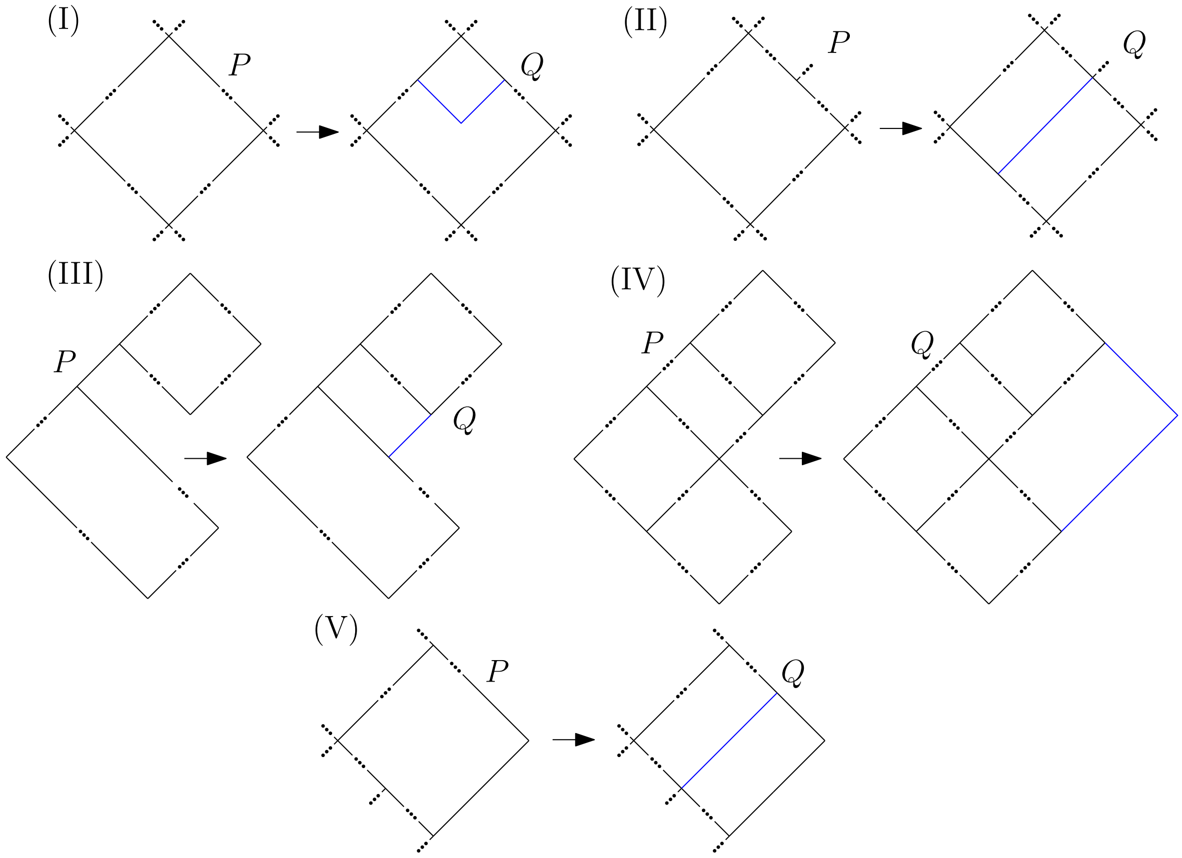


FIGURE 9. a) Projecting the diagram, b) Introducing and ordering the rectangles

As pointed out in Remark 2.1, we now explain how the ordering we choose in Figure 9-b extends PALF structures.

Let P be a diagram in the yz-plane divided into rectangles whose sides are on the lines of slopes ± 1. Let F_P denote the surface obtained by following the contact planes in $(\mathbb{R}^3, \xi_0)$ with the front projection P (here we just generalize the construction of the positive Hopf band from a rectangle, see Figure 3).

Lemma 2.2. *In each case below, suppose that the surface F_P is a fiber of a PALF. Then so is F_Q where Q is obtained from P by adding the blue arc.*



Proof. In each case, the surface F_Q is obtained from F_P by adding the strip which projects onto the blue arc in the diagram Q. For each case, we draw a picture below where this strip is given in blue. In each picture, we show that adding the blue strip to the fiber F_P is equivalent to plumbing a positive Hopf band H^+ to F_P along the red arc in F_P. (Dashed arrows indicate the isotopies taking H^+ to the blue strip.) Thus, the surface F_Q is a fiber of a PALF. □

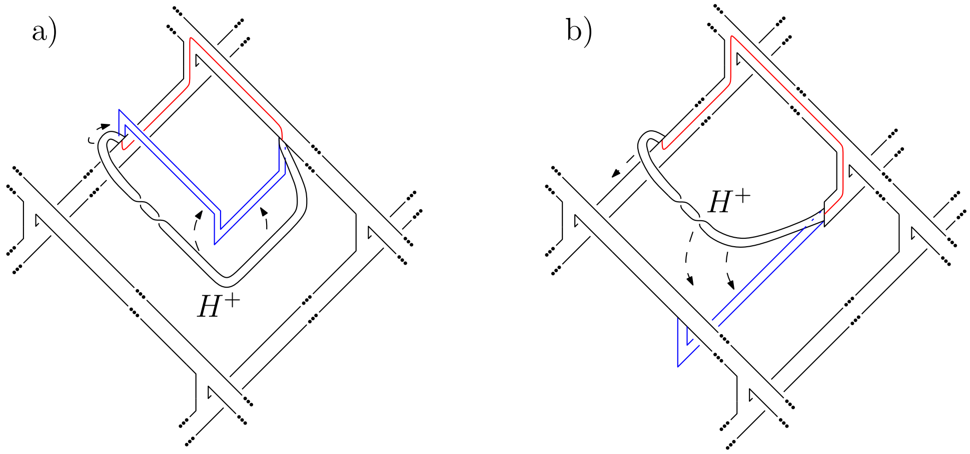


FIGURE 10. Proof of a) Lemma 2.2-(I), b) Lemma 2.2-(II)

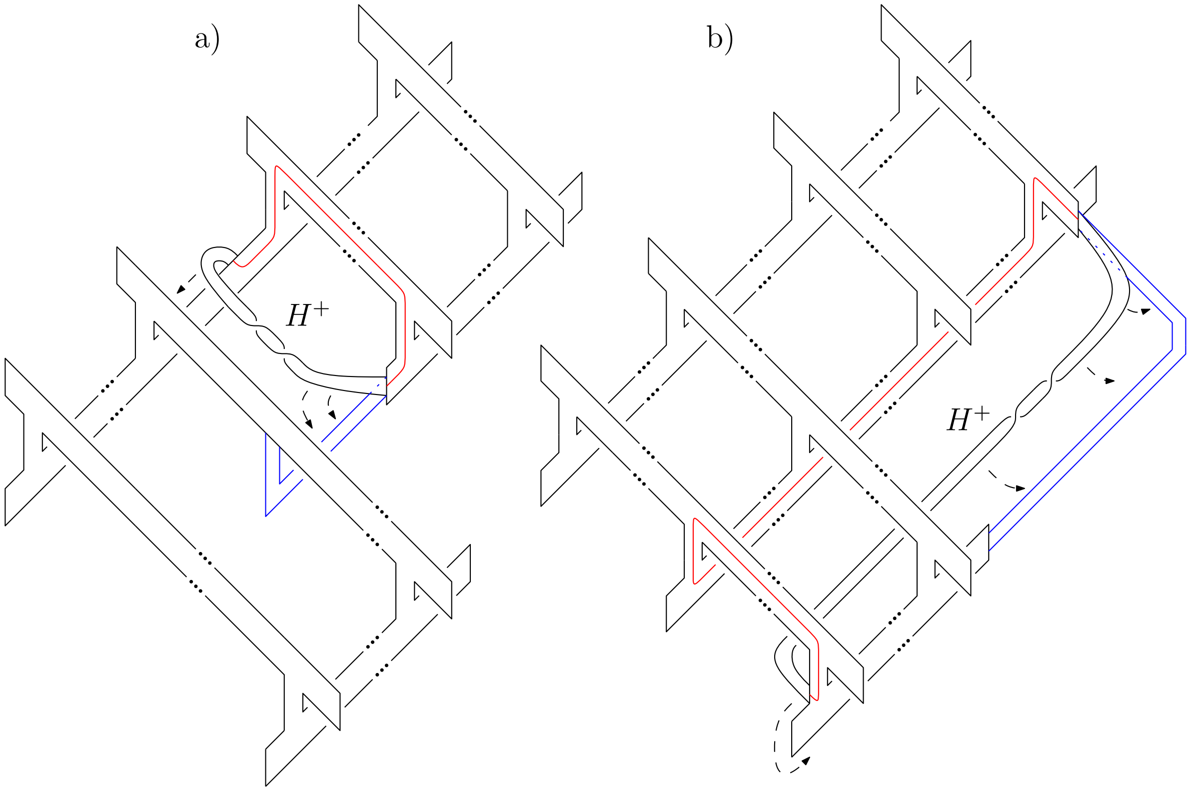


FIGURE 11. Proof of a) Lemma 2.2-(III), b) Lemma 2.2-(IV)

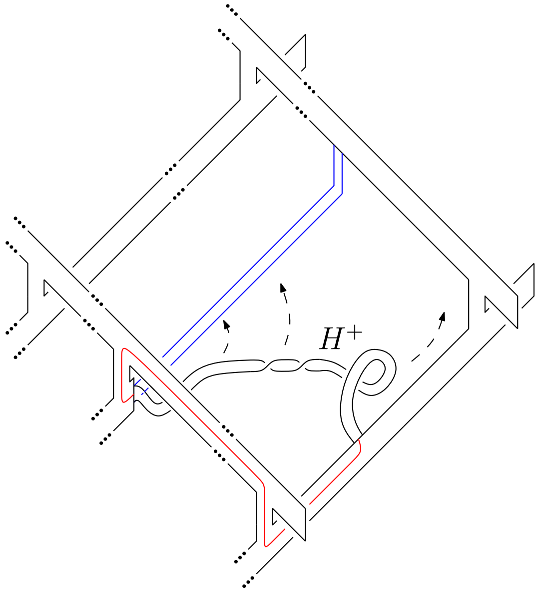


FIGURE 12. Proof of Lemma 2.2-(V)

Now we introduce the rectangles in the described order to the diagram in Figure 9-b, and accordingly positively stabilize $\natural_r S^1 \times D^3$. By Lemma 2.2, we know that PALF structure extends for the rectangles in the regions $(1), ..., (2r-2)$ and the bold rectangles. For the remaining rectangles (the ones in the yellow region in Figure 9-b), the PALF structures extend as in Case 1. When all rectangles are introduced, we get a new PALF structure on $\natural_r S^1 \times D^3$ with bounded fiber F and monodromy T which is a product of (+)-Dehn twists $t_1, t_2, ..., t_n$ in a certain order determined by the plumbings (here each t_i is as before). Also the Legendrian link $L = L_1 \cup ... \cup L_m$ is embedded on a page F of the open book supporting η_{std}, and the framing of each L_i coming from F is equal to $tb(L_i)$. Therefore, as in Case 1 attaching 2-handles $H_1, ..., H_m$ along $L_1, ..., L_m$ extends the PALF on $\natural_r S^1 \times D^3$ to a PALF on W. The resulting PALF on W has a bounded regular fiber F, and its monodromy is

$$T \cdot s_1 s_2 ... s_m$$

where s_i is the positive Dehn twist along L_i for each $i = 1, ..., m$.

We remark that in both of the above cases the final open book corresponding to the final PALF supports the contact structure on ∂W induced by the Stein structure on W. The reader is referred to [Et1] for details on compatibility. Also observe that once we have a PALF structure on W, we can get infinitely many pairwise inequivalent PALF's on W by positively stabilizing the original one. □

3. Example

As an example, we will apply our algorithm to construct a PALF structure on the compact Stein surface W given in Figure 13-a.

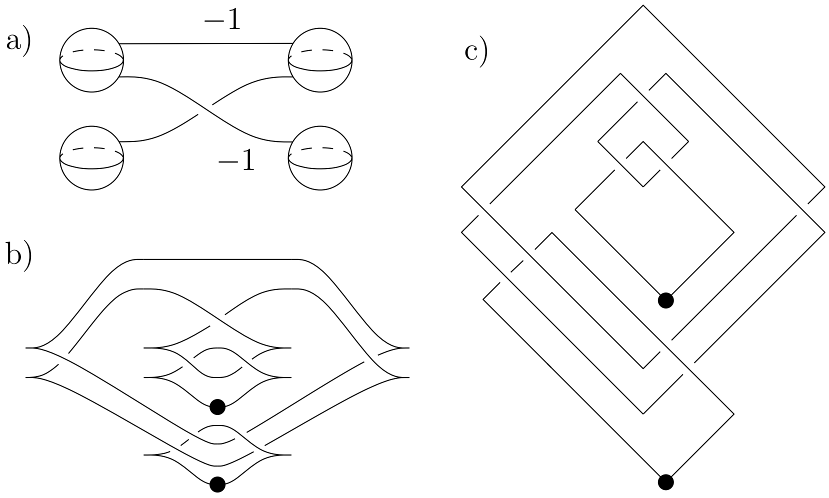


Figure 13. a) A Stein surface W ($r = m = 2$, coefficients are relative to tb-framing), b) converting the diagram, c) projecting onto the yz-plane

We first convert the 1-handles into dotted circles and obtain the diagram in Figure 13-b. Then we consider its projection onto the yz-plane as in Figure 13-c. Next we introduce the rectangles to the projection in the order depicted in Figure 14, and accordingly positively stabilize $\natural_r S^1 \times D^3$. By Lemma 2.2, we know that PALF structure extends for the rectangles $R_1, ..., R_{10}$. For $R_{11}, ...R_{28}$, we extend the PALF structures as in Case 1.

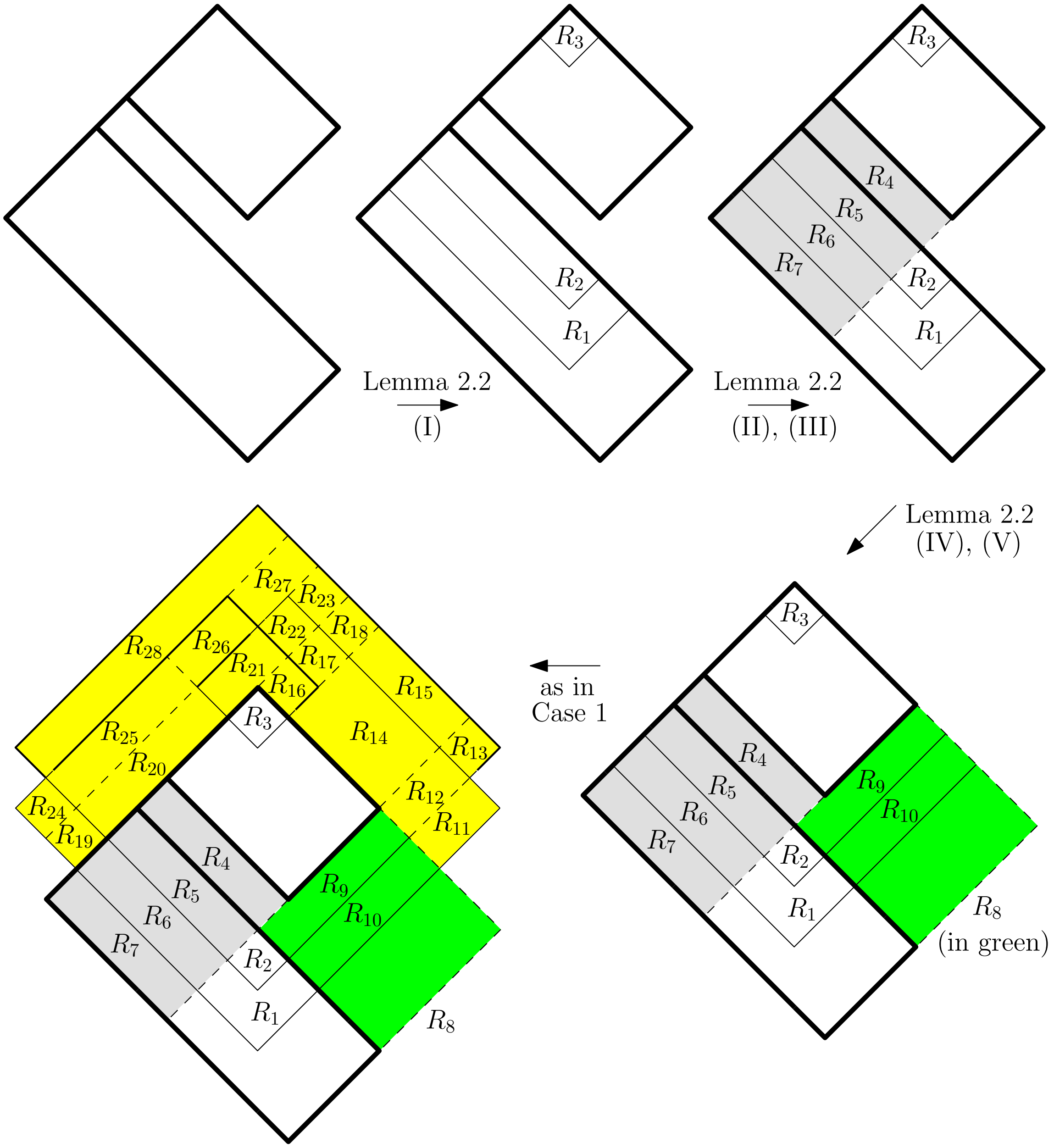


FIGURE 14. Introducing the rectangles in the order given by the algorithm.

The fiber of the resulting PALF on $\natural_2 S^1 \times D^3$ is shown in Figure 15. We remark that in the corresponding handle diagram we have 28 canceling pairs of 1- and 2-handles.

Finally, note that the Legendrian link describing W is now embedded on a page of the PALF on $\natural_2 S^1 \times D^3$ with page framing equal to tb-framing. Therefore, by attaching the 2-handles, we obtain a PALF structure on W.

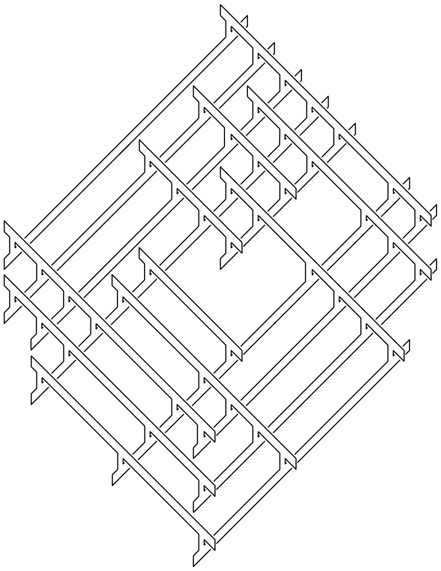


Figure 15. The fiber F of the PALF on $\natural_2 S^1 \times D^3$

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