Lectures on Seiberg-Witten Invariants

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In October 1994 Seiberg-Witten invariants entered in 4-manifold theory with a big bang. Not only did these invariants tidy up the Gauge Theory, but they also gave some exciting new results on topology of smooth 4-manifolds. These notes grew out of the lectures I have given in learning seminars at MPI in Bonn, and METU in Ankara on this subject. The main goal of these notes is not to survey the whole area, but rather establish conventions for novice topologist like myself, and go through some recent selected results. In these notes I avoided the general Clifford algebra constructions in favor of more direct representation theory of $Spin_c(4)$.

I have benefited greatly from stimulating papers [**KM**], [**W**], and [**T**], as well as unpublished lecture notes of C.Taubes. I also benefited seminar talks by R.Fintushel, D.Salamon, T.Parker and T.Draghici. I thank I.Hambelton and T.Onder for inviting me to MPI and METU, giving me the chance to work on these notes. I make no claim of originality in these notes, they are merely modest efforts to understand the results from the sources mentioned above.

1 Introduction

Every compact oriented smooth 4-manifold has a $Spin_c$ structure, i.e. the second Steifel-Whitney $w_2(X) \in H^2(X;\mathbb{Z}_2)$ has an integral lifting. This is because: The map $a \longmapsto a^2$ defines a homomorphisim of the integral homology $H_2(X) \longmapsto \mathbb{Z}_2$, hence there is a $a \in H_2(X)$ with $x.a = x^2 \pmod 2$ for all x. So if α is the Poincare dual of a, the class $b = w_2(X) - \rho(\alpha)$ lies in the kernel of the evaluation map h in the universal coefficient sequence:

$$\begin{array}{ccccccccc} 0 \to & Ext(H_1(X);\mathbb{Z}_2) & \longrightarrow & H^2(X;\mathbb{Z}_2) & \xrightarrow{h} & Hom(H_2(X);\mathbb{Z}_2) & \to 0 \\ & \uparrow & & \uparrow \rho & & \uparrow & \\ 0 \to & Ext(H_1(X);\mathbb{Z}) & \longrightarrow & H^2(X;\mathbb{Z}) & \longrightarrow & Hom(H_2(X);\mathbb{Z}) & \to 0 \end{array}$$

So b comes from $Ext(H_1(X);\mathbb{Z}_2)$. The universal coefficient sequence is natural with respect to coefficient homomorphisim and the first vertical arrow is onto, hence we get $b = \rho(\beta)$ for some β .

$$
\begin{aligned}
\text{Recal:} \qquad Spin(4) &= SU(2)\times SU(2)\\
Spin_c(4) &= (\ SU(2)\times SU(2)\times S^1\ )/\mathbb{Z}_2 = (\ Spin(4)\times S^1)/\mathbb{Z}_2\\
SO(4) &= (\ SU(2)\times SU(2)\ )/\mathbb{Z}_2\\
U(2) &= (\ SU(2)\times S^1\ )/\mathbb{Z}_2
\end{aligned}
$$

We have fibrations:

$$S^1 \longrightarrow Spin_c(4) \to SO(4)$$

$$\mathbb{Z}_2 \longrightarrow Spin_c(4) \to SO(4)\times S^1$$

We can also identify $Spin_c(4) = \{(A,B) \in U_2 \times U_2 \mid det(A) = det(B)\ \}$ by

$$(A,B) \rightsquigarrow (A.(detA)^{-1/2}\ I\ ,\ B.(detB)^{-1/2}\ I\ ,\ (detA)^{1/2})$$

We also have 2 fold cover $Spin_c(4) \to SO(4)\times S^1$ .The fibrations above extend to fibrations:

$$S^1 \to Spin_c(4) \to SO(4) \to K(\mathbb{Z},2) \to BSpin_c(4) \to BSO(4) \to K(\mathbb{Z},3)$$

The last map in the sequence is given by the Bokstein of the second Steifel-Whitney class $\delta(w_2)$ which explains why lifting of w_2 to an integral class corresponds to a $Spin_c(4)$-structure. We also have the fibration:

$$\mathbb{Z}_2 \to Spin_c(4) \to SO(4)\times S^1 \to K(\mathbb{Z}_2,1) \to BSpin_c(4) \to BSO(4)\times BS^1 \to K(\mathbb{Z}_2,2)$$

The last map in this sequence is given by $w_2 \times 1 + 1 \times \rho(c_1)$ which clearly vanishes exactly when $\delta(w_2) = 0$. Finally we have the fibration:

$$\mathbb{Z}_2 \to Spin(4)\times S^1 \to Spin_c(4) \to K(\mathbb{Z}_2,1) \to BSpin(4)\times BS^1 \to BSpin_c(4) \to K(\mathbb{Z}_2,2)$$

The last map is given by w_2. This sequence says that locally a $Spin_c(4)$ bundle consists a pair of a $Spin(4)$ bundle and a complex line bundle. Also recall $H^2(X;\mathbb{Z}) = [X, K(\mathbb{Z},2)] = [X, BS^1] = \{\text{complex line bundles on X}\}$

Definition: Let $L \longrightarrow X$ be a complex line bundle over a smooth oriented 4-manifold with $c_1(L) = w_2(TX)$ (i.e. L is a characteristic line bundle). A $Spin_c(4)$ structure on X, corresponding L, is a principal $Spin_c(4)$-bundle $P \longrightarrow X$ such that the associated framed bundles of TX and L satisfy:

$$P_{SO(4)}(TX) = P \times_{\rho_0} SO(4)$$

$$P_{S^1}(L) = P \times_{\rho_1} S^1$$

where $(\rho_0,\rho_1) : Spin_c(4) \to SO(4)\times S^1$ are the obvious projections

$$
\begin{array}{ccccc}
 & & Spin(4)\times S^1 & & \\
 & & \downarrow \pi & & \\
SO(4) & \overset{\rho_0}{\longleftarrow} & Spin_c(4) & \overset{\rho_1}{\longrightarrow} & S^1 \\
 & \swarrow \rho_+ & & \rho_- \searrow & \\
U(2) & & \downarrow \tilde{\rho}_+ \quad \tilde{\rho}_- \downarrow & & U(2) \\
 & Ad \searrow & & \swarrow Ad & \\
 & & SO(3) & &
\end{array}
$$

So $\tilde{\rho}_{\pm} = Ad \circ \rho_{\pm}$, also call $\bar{\rho}_{\pm} = \rho_{\pm} \circ \pi$. For $x \in \mathbb{H} = \mathbb{R}^4$ we have

$$
\begin{array}{rcll}
\rho_1[\, q_+, q_-, \lambda \,] & = & \lambda^2 & \\
\rho_0[\, q_+, q_-, \lambda \,] & = & [\, q_+, q_- \,] & \text{, i.e. } x \longmapsto q_+ x q_-^{-1} \\
\rho_{\pm}[\, q_+, q_-, \lambda \,] & = & [\, q_{\pm}, \lambda \,] & \text{, i.e. } x \longmapsto q_{\pm} x \lambda^{-1} \\
\tilde{\rho}_{\pm}[\, q_+, q_-, \lambda \,] & = & Ad \circ q_{\pm} & \text{, i.e. } x \longmapsto q_{\pm} x q_{\pm}^{-1} \\
\bar{\rho}_{\pm}(\, q_+, q_-, \lambda \,) & = & \lambda q_{\pm} &
\end{array}
$$

Apart from TX and L, $Spin_c(4)$ bundle $P \to X$ induces a pair of $U(2)$ bundles:

$$W^{\pm} = P \times_{\rho_{\pm}} \mathbb{C}^2 \longrightarrow X$$

Let $\Lambda^p(X) = \Lambda^p T^*(X)$ be the bundle of exterior p forms. If X is a Riemanian manifold (i.e. with metric), we can construct the bundle of self(antiself)-dual 2-forms $\Lambda^2_{\pm}(X)$ which we abbreviate by $\Lambda^{\pm}(X)$. We can identify $\Lambda^2(X)$ by the Lie algebra $so(4)$-bundle

$$\Lambda^2(X) = P(T^*X) \times_{ad} so(4) \qquad \text{by} \quad \Sigma\, a_{ij}\; dx^i \wedge dx^j \;\longleftrightarrow\; (a_{ij})$$

where $ad : SO(4) \to so(4)$ is the adjoint representation. The adjoint action preserves the two summands of $so(4) = spin(4) = so(3) \times so(3) = \mathbb{R}^3 \oplus \mathbb{R}^3$. By above identification it is easy to see that the ± 1 eigenspaces $\Lambda^{\pm}(X)$ of the star operator $* : \Lambda(X) \to \Lambda(X)$ corresponds to these two $\mathbb{R}^3$-bundles; this gives:

$$\Lambda^{\pm}(X) = P \times_{\tilde{\rho}_{\pm}} \mathbb{R}^3$$

If the $Spin_c(4)$ bundle $P \to X$ lifts to $Spin(4)$ bundle $\bar{P} \to X$ (i.e. when $w_2(X) = 0$), corresponding to the obvious projections $p_{\pm} : Spin(4) \to SU(2)$, $p_{\pm}(q_-, q_+) = q_{\pm}$ we get a pair of $SU(2)$ bundles:

$$V^{\pm} = P \times_{p_{\pm}} \mathbb{C}^2$$

Clearly since $x \longmapsto q_{\pm} x \lambda^{-1} = q_{\pm} x\, (\lambda^2)^{-1/2}$ in this case we have:

$$W^{\pm} = V^{\pm} \otimes L^{-1/2}$$

1.1 Action of $\Lambda^*(X)$ on $W_\pm$

From the definition of $Spin_c(4)$ structure above we see that

$$T^*(X) = P \times \mathbb{H}/(p,v) \sim (\tilde{p}, q_+ v\, q_-^{-1}) \ \text{, where} \quad \tilde{p} = p[\, q_+, q_-, \lambda \,]$$

We define left actions (Clifford multiplications), which is well defined by

$$T^*(X) \otimes W^+ \longrightarrow W^- \ \text{, by} \quad [\, p, v \,] \otimes [\, p, x \,] \longmapsto [\, p, -\bar{v}x \,]$$

$$T^*(X) \otimes W^- \longrightarrow W^+ \ \text{, by} \quad [\, p, v \,] \otimes [\, p, x \,] \longmapsto [\, p, vx \,]$$

¿From identifications, we can check the well definededness of these actions, e.g.:

$$[p, v\,] \otimes [\, p, x \,] \sim [\, \tilde{p}, q_+ v\, q_-^{-1} \,] \otimes [\, \tilde{p}, q_+ x \lambda^{-1} \,] \longmapsto [\, \tilde{p},\ q_-(-\bar{v}x)\lambda^{-1} \,] \sim [\, p, -\bar{v}x \,]$$

By dimension reason complexification of these representation give

$$\rho : T^*(X)_{\mathbb{C}} \xrightarrow{\cong} Hom(W^\pm, W^\mp) \equiv W^\pm \otimes W^\mp$$

We can put them together as a single representation (which we still call ρ)

$$\rho : T^*(X) \longrightarrow Hom(W^+ \oplus W^-) \ \text{,} \ \text{by} \quad v \ \longmapsto \ \rho(v) = \begin{pmatrix} 0 & v \\ -\bar{v} & 0 \end{pmatrix}$$

We have $\rho(v) \circ \rho(v) = -|v|^2 I$. By universal property of the Clifford algebra this representation extends to the Clifford algebra $C(X) = \Lambda^*(X)$ (exterior algebra)

$$\begin{array}{ccc} \Lambda^*(X) & & \\ \downarrow & \searrow & \\ T^*(X) & \longrightarrow & Hom\ (W^+ \oplus W^-) \end{array}$$

One can construct this extension without the aid of the universal property of the Clifford algebra, for example since

$$\Lambda^2(X) = \left\{ \ v_1 \wedge v_2 = \frac{1}{2}(v_1 \otimes v_2 - v_2 \otimes v_1) \mid v_1, v_2 \in T^*(X) \ \right\}$$

The action of $T^*(X)$ on $W^\pm$ determines the action of $\Lambda^2(X) = \Lambda^+(X) \otimes \Lambda^-(X)$, and since $2Im\ (v_2\bar{v}_1) = -v_1\bar{v}_2 + v_2\bar{v}_1$ we have the action ρ with property:

$$\Lambda^+(X) \otimes W^+ \longrightarrow W^+ \quad \text{to be} \quad [\, p, v_1 \wedge v_2 \,] \otimes [\, p, x \,] \longrightarrow [\, p, Im\ (v_2\bar{v}_1)x \,]$$

$$\rho : \Lambda^+ \longrightarrow Hom(W^+, W^+)$$

$$\rho(v_1 \wedge v_2) \quad = \quad \frac{1}{2}\, [\, \rho(v_1), \rho(v_2) \,] \tag{1}$$

Let us write the local descriptions of these representations: We first pick a local orthonormal basis $\{e^1, e^2, e^3, e^4\}$ for $T^*(X)$, then we can take

$$\{\ f_1 = \frac{1}{2}(e^1 \wedge e^2 \pm e^3 \wedge e^4),\ f_2 = \frac{1}{2}(e^1 \wedge e^3 \pm e^4 \wedge e^2),\ f_3 = \frac{1}{2}(e^1 \wedge e^4 \pm e^2 \wedge e^3)\ \}$$

to be a basis for $\Lambda^{\pm}(X)$. After the local identification $T^*(X) = \mathbb{H}$ we can take $e^1 = 1,\ e^2 = i,\ e^3 = j,\ e^4 = k$. Let us identify $W^{\pm} = \mathbb{C}^2 = \{z + jw \mid z, w \in \mathbb{C}\ \}$, then the multiplication by $1, i, j, k$ (action on $\mathbb{C}^2$ as multiplication on left) induce the representations $\rho(e^i)$, $i = 1, 2, 3, 4$. From this we see that $\Lambda^+(X)$ acts trivially on W^-; and the basis f_1, f_2, f_3 of $\Lambda^+(X)$ acts on W^+ as multiplication by i, j, k, respectively (these are called Pauli matrices).

$$\rho(e^1) = \begin{pmatrix} & & 1 & 0 \\ & & 0 & 1 \\ -1 & 0 & & \\ 0 & -1 & & \end{pmatrix} \qquad \rho(e^2) = \begin{pmatrix} & & i & 0 \\ & & 0 & -i \\ i & 0 & & \\ 0 & -i & & \end{pmatrix}$$

$$\rho(e^3) = \begin{pmatrix} & & 0 & -1 \\ & & 1 & 0 \\ 0 & -1 & & \\ 1 & 0 & & \end{pmatrix} \qquad \rho(e^4) = \begin{pmatrix} & & 0 & -i \\ & & -i & 0 \\ 0 & -i & & \\ -i & 0 & & \end{pmatrix}$$

$$\rho(f_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \rho(f_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \rho(f_3) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

In particular we get an isomorphism $\Lambda^+(X) \longrightarrow su\ (W^+)$ (traceless skew adjoint endemorphism of W^+); which after complexifying extends to an isomorphism $\rho : \Lambda^+(X)_{\mathbb{C}} \cong sl\ (W^+)$ (traceless endemorphism of W^+)

$$\begin{array}{ccc} \Lambda^+(X) & \xrightarrow{\cong} & su\ (W^+) \\ \bigcap & & \bigcap \\ \Lambda^+(X)_{\mathbb{C}} & \xrightarrow{\rho} & sl\ (W^+) \end{array}$$

Recall $Hom(W^+, W^+) \cong W^+ \otimes (W^+)^*$; we identify the dual space $(W^+)^*$ naturally with $\bar{W}^+$ ($= W^+$ with scalar multiplication $c.v = \bar{c}v$) by the pairing

$$W^+ \otimes \bar{W} + \longrightarrow \mathbb{C}$$

given by $z \otimes w \rightarrow z\bar{w}$. Usually $sl\ (W^+)$ is denoted by $(W^+ \otimes \bar{W}^+)_0$ and the trace map gives the identification:

$$W^+ \otimes \bar{W}^+ = (W^+ \otimes \bar{W}^+)_0 \oplus \mathbb{C} = \Lambda^+(X)_{\mathbb{C}} \oplus \mathbb{C}$$

Let $\sigma : W^+ \longrightarrow \Lambda^+(X)$ be the map $[\ p, x\ ] \longmapsto [\ p,\ \frac{1}{2}(xi\ \bar{x})\ ]$. By local identification as above $W^+ = \mathbb{C}^2$ and $\Lambda^+(X) = \mathbb{R} \oplus \mathbb{C}$, we see σ corresponds to

$$(z,w) \longmapsto i\ \left(\frac{|z|^2 - |w|^2}{2} \right) - k\ Re(z\bar{w}) + j\ Im(z\bar{w}) = \left(\frac{|w|^2 - |z|^2}{2} \right) + z\bar{w}$$

We identify this by the element $i\sigma(z,w)$ of $su\ (W^+)$ (by Pauli matrices) where:

$$(z,w)\ \longmapsto \sigma(z,w)\ = \begin{pmatrix} (\ |z|^2 - |w|^2)/2 & z\bar{w} \\ \bar{z}w & (\ |w|^2 - |z|^2)/2 \end{pmatrix} \qquad (2)$$

σ is the projection of the diagonal elements of $W^+ \otimes \bar{W}^+$ onto $(W^+ \otimes \bar{W}^+)_0$
We can check:

$$i\ \sigma(z,w) = \rho\ [\ \frac{|z|^2 - |w|^2}{2}\ f_1 + Im(z\bar{w})\ f_2 - Re(z\bar{w})\ f_3\ ] \qquad (3)$$

¿From these identifications we see:

$$|\ \sigma(\psi)\ |^2 \ = \ \frac{1}{4}|\ \psi\ |^4 \qquad (4)$$

$$<\ \sigma(\psi)\ \psi, \psi\ > \ = \ \frac{1}{2}|\ \psi\ |^4 \qquad (5)$$

$$<\rho(\omega)\ \psi\ ,\ \psi> \ = \ 2i\ <\rho(\omega)\ , i\ \sigma(\psi)> \qquad (6)$$

Here the norm in $su(2)$ is induced by the inner product $< A, B >= \frac{1}{2}trace(AB)$.

By calling $\sigma(\psi,\psi) = \sigma(\psi)$ we extend the definition of σ to $W^+ \otimes \bar{W}^+$ by

$$<\rho(\omega)\ \psi\ ,\ \varphi>= 2i\ <\rho(\omega)\ , i\ \sigma(\psi,\varphi)>$$

$$\begin{array}{ccccc} \Lambda^+(X) & = & su(W^+) & \overset{i\ \sigma}{\longleftarrow} & W^+ \\ & & \cap & & \cap \\ (W^+ \otimes \bar{W}^+)_0 & = & sl\ (W^+) & \overset{i\ \sigma}{\longleftarrow} & W^+ \otimes \bar{W}^+ \end{array}$$

Remark: A $Spin_c(4)$ structure can also be defined as a pair of $U(2)$ bundles:

$W^{\pm} \longrightarrow X$ with $det(W^+) = det(W^-) \longrightarrow X$ (a complex line bundle),

and an action $c_{\pm} : T^*(X) \longrightarrow Hom(W^{\pm}, W^{\mp})$ with $c_{\pm}(v)c_{\mp}(v) = -|v|^2 I$

The first definition clearly implies this, and conversely we can obtain the first definition by letting the principal $Spin_c(4)$ bundle to be:

$$P = \{\ (p_+, p_-) \in P(W^+) \times P(W^-)\ |\ det(p_+) = det(p_-)\ \}$$

Clearly, $Spin_c(4) = \{(A,B) \in U_2 \times U_2 \mid det(A) = det(B)\ \}$ acts on P freely.

This definition generalizes and gives way to the following definition:

Definition: A Dirac bundle $W \longrightarrow X$ is a Riemanian vector bundle with an action $\rho : T^*(X) \longrightarrow Hom(W,W)$ satisfying $\rho(v) \circ \rho(v) = -|v|^2 I$. W is also equipped with a connection D satisfying:

$$< \rho(v)x, \rho(v)y > = < x, y >$$

$$D_Y(\rho(v)s) = \rho(\nabla_X v)s + \rho(v) D_Y(s)$$

where ∇ is the Levi-Civita connection on $T^*(X)$, and Y is a vector filed on X

An example of a Dirac bundle is $W = W^+ \oplus W^- \longrightarrow X$ and $D = d + d^*$ with $W^+ = \oplus \Lambda^{2k}(X)$ and $W^- = \oplus \Lambda^{2k+1}(X)$ where $\rho(v) = v \wedge . + v \perp ..$ (exterior + interior product with v). In this case $\rho : W^{\pm} \to W^{\mp}$. In the next section we will discuss the natural connections D for $Spin_c$ structures $W^{\pm}$.

2 Dirac Operator

Let $\mathcal{A}(L)$ denote the space of connections on a $U(1)$ bundle $L \longrightarrow X$. Any $A \in \mathcal{A}(L)$ and the Levi-Civita connection A_0 on the tangent bundle coming from Riemanian metric of X defines a product connection on $P_{SO(4)} \times P_{S^1}$. Since $Spin_c(4)$ is the two fold covering of $SO(4) \times S^1$, they have the same Lie algebras $spin_c(4) = so(4) \oplus i\,\mathbb{R}$. Hence we get a connection $\tilde{A}$ on the $Spin_c(4)$ principle bundle $P \longrightarrow X$. In particular the connection $\tilde{A}$ defines connections to all the associated bundles of P, giving back A, A_0 on L, $T(X)$ respectively, and two new connections $A^{\pm}$ on bundles $W^{\pm}$. We denote the corresponding covariant derivatives by ∇_A.

$$\nabla_A : \Gamma(W^+) \to \Gamma(\ T^*X \otimes W^+)$$

Composing this with the Clifford multiplication $\Gamma(\ T^*X \otimes W^+) \to \Gamma(W^-)$ gives the Dirac operator

$$\not{D}_A : \Gamma(W^+) \to \Gamma(W^-)$$

Locally, by choosing orthonormal tangent vector field $e = \{e_i\ \}_{i=1}^4$ and the dual basis of 1-forms $\{e^i\ \}_{i=1}^4$ in a neighborhood U of a point $x \in X$ we can write

$$\not{D}_A = \sum \rho(e^i) \nabla_{e_i}$$

where $\nabla_{e_i} : \Gamma(W^+) \to \Gamma(W^+)$ is the covariant derivative ∇_A along e_i. Also locally $W^{\pm} = V^{\pm} \otimes L^{1/2}$, hence by Leibnitz rule, the connection A and the untwisted Dirac operator

$$\not{\partial} : \Gamma(V^+) \to \Gamma(V^-)$$

determines $\not{D}_A$. Notice that as in $W^{\pm}$, forms $\Lambda^*(X)$ act on $V^{\pm}$. Now let $\omega = (\omega_{ij})$ be the Levi-Civita connection 1-form, i.e. $so(4)$-valued "equivariant" 1-form on $P_{SO(4)}(X)$ and $\tilde{\omega} = (\tilde{\omega}_{ij}) = e^*(\omega)$ be the pull-back 1-form on U. Since $P_{SO(4)}(U) = P_{Spin(4)}(U)$ the orthonormal basis $e \in P_{SO(4)}(U)$ determines an orthonormal basis $\sigma = \{\sigma^k\} \in P_{SU_2}(V^+)$, then (e.g. [**LM**])

$$\not{\partial}\,(\sigma^k) = \frac{1}{2}\sum_{i<j} \rho(\tilde{\omega}_{ji})\,\rho(e^i)\rho(e^j)\,\sigma^k$$

Metrics on $T(X)$ and L give metrics on $W^{\pm}$ and $T^*(X) \otimes W^{\pm}$, hence we can define the adjoint $\nabla_A^* : \Gamma(T^*X \otimes W^-) \to \Gamma(W^+)$. Similarly we can define $\not{D}_A : \Gamma(W^-) \to \Gamma(W^+)$ which turns out to be the adjoint of the previous $\not{D}_A$ and makes the following commute (vertical maps are Clifford multiplications):

$$\begin{array}{ccccc}
\Gamma(W^+) & \xrightarrow{\nabla_A} & \Gamma(\,T^*X \otimes W^+) & \xrightarrow{\nabla_A} & \Gamma(\,T^*X \otimes T^*X \otimes W^+) \\
\| & & \downarrow & & \downarrow \\
\Gamma(W^+) & \xrightarrow{\not{D}_A} & \Gamma(W^-) & \xrightarrow{\not{D}_A} & \Gamma(W^+)
\end{array}$$

Let $F_A \in \Lambda^2(X)$ be the curvature of the connection A on L, and $F_A^+ \in \Lambda^+(X)$ be the self dual part of this curvature, and s be the scalar curvature of X. Weitzenbock formula says that:

$$\not{D}_A^2(\psi) \quad = \quad \nabla_A^*\nabla_A\psi + \frac{s}{4}\psi + \frac{1}{4}\rho(F_A^+)\psi \tag{7}$$

To see this we we can assume $\nabla_{e_i}(e^j) = 0$ at the point x

$$\begin{aligned}
\not{D}_A^2\psi &= \sum \rho(e^i).\,\nabla_{e_i}\,[\,\sum \rho(e^j).\,\nabla_{e_j}\psi\,] \\
&= \nabla_A^*\nabla_A\psi + \frac{1}{2}\sum_{i,j}\rho(e^i)\,\rho(e^j)\,(\nabla_{e_i}\nabla_{e_j} - \nabla_{e_j}\nabla_{e_i})\,\psi \\
&= \nabla_A^*\nabla_A\psi + \frac{1}{2}\sum_{i,j}\rho(e^i)\,\rho(e^j)\,\Omega_{ij}^A\,\psi
\end{aligned}$$

$\Omega_{ij}^A = R_{ij} + \frac{1}{2}F_{ij}$ is curvature on $V^+ \otimes L^{1/2}$, i.e. R_{ij} is the Riemanian curvature and the imaginary valued 2-form F_{ij} is the curvature of A for the line bundle L (endemorphisims of W^+). So if $\psi = \sigma \otimes \alpha \in \Gamma(V^+ \otimes L^{1/2})$, then

$$\begin{aligned}
\frac{1}{2}\sum_{i,j}\rho(e^i)\,\rho(e^j)\,\Omega_{ij}^A\,(\sigma\otimes\alpha) &= \frac{1}{2}(\sum \rho(e^i)\,\rho(e^j)\,R_{ij}\,\sigma)\otimes\alpha \\
&\quad + \frac{1}{4}\sum \rho(e^i)\,\rho(e^j)\sigma\otimes(F_{ij}\alpha) \\
&= \frac{1}{8}\,\sum \rho(e^i)\,\rho(e^j)\,\rho(e^k)\,\rho(e^l)\,R_{ijkl}\,(\psi) \\
&\quad + \frac{1}{4}\,\rho\,(\sum F_{ij}\,e^i\wedge e^j)\,(\psi)
\end{aligned}$$

The last identity follows from (1). It is a standard calculation that the first term is $s/4$ (e.g.[**LM**], pp. 161), and since $\Lambda^-(X)$ act as zero on W^+, the second term can be replaced by

$$\frac{1}{4}\,\rho(F_A^+)\,\psi = \frac{1}{4}\,\rho\,(\sum F_{ij}^+\, e^i \wedge e^j)\,\psi$$

2.1 A Special Calculation

In Section 4 we need some a special case (7). For this, suppose

$$V^+ = L^{1/2} \oplus L^{-1/2}$$

where $L^{1/2} \longrightarrow X$ is some complex line bundle with $L^{1/2} \otimes L^{1/2} = L$. Hence $W^+ = (L^{1/2} \oplus L^{-1/2}) \otimes L^{-1/2} = L^{-1} \oplus \mathbb{C}$. In this case there is a unique connection $\frac{1}{2}A_0$ in $L^{-1/2} \to X$ such that the induced Dirac operator D_{A_0} on W^+ restricted to the trivial summand $\underline{\mathbb{C}} \to X$ is the exterior derivative d. This is because for $\sigma_\pm \in \Gamma(L^{\pm 1/2})$, the following determines $\nabla_{\frac{A_0}{2}}(\sigma_-)$:

$$\begin{aligned} \nabla_{A_0}(\sigma_+ + 0) \otimes \sigma_- &= \partial\!\!\!/\,(\sigma_+ + 0) \otimes \sigma_- + (\sigma_+ + 0) \otimes \nabla_{\frac{A_0}{2}}(\sigma_-) \\ = \nabla_{A_0}(\sigma_+ \otimes \sigma_-) &= d(\sigma_+ \otimes \sigma_-) \end{aligned}$$

The following is essentially the Leibnitz formula for Laplacian applied to Weitzenbock formula (7)

Proposition: Let $A, A_0 \in \mathcal{A}(L^{-1})$ and $i\,a = A - A_0$. Let $\nabla_a = d + i\,a$ be the covariant derivative of the trivial bundle $\underline{\mathbb{C}} \longrightarrow X$, and $\alpha : X \to \mathbb{C}$. Let u_0 be a section of $W^+ = L^{-1} \oplus \mathbb{C}$ with a constant $\mathbb{C}$ component and $D\!\!\!/_{A_0}(u_0) = 0$ then:

$$D\!\!\!/^2_A(\alpha u_0) = (\nabla_a^*\nabla_a\alpha)u_0 + \frac{1}{2}\rho(F_a)\;\alpha\;u_0 - 2 < \nabla_a\alpha\;, \nabla_{A_0}(u_0) > \tag{8}$$

Proof: By writing $\nabla_A = \nabla^A$ for the sake of not cluttering notations, and abbreviating $\nabla_{e_j} = \nabla_j$ and $\nabla_j^a(\alpha) = \nabla_j(\alpha) + i\;a_j\alpha$, and leaving out summation signs for repeated indices (Einstein convention) we calculate:

$$\begin{aligned} \nabla^A(\alpha u_0) &= \nabla^A(\alpha)u_0 + \alpha\nabla^A(u_0) \\ &= e^j \otimes \nabla_j(\alpha)u_0 + \alpha(\nabla^{A_0}(u_0) + i\;e^j \otimes a_j\;u_0) \\ &= e^j \otimes (\nabla_j(\alpha) + i\;a_j\alpha\;)u_0 + \alpha\nabla^{A_0}(u_0) \\ D\!\!\!/_A(\alpha u_0) &= \rho(e^j)\;\nabla_j^a(\alpha)\;u_0 + \alpha\;D\!\!\!/_{A_0}(u_0) = \rho(e^j)\;\nabla_j^a(\alpha)\;u_0 \end{aligned} \tag{9}$$

By abbreviating $\mu = \nabla_j^a(\alpha)$ we calculate:

$$
\begin{aligned}
\nabla^A(\rho(e^j)\ \mu\ u_0) &= e^k\otimes\rho(e^j)\ \nabla_k(\mu)u_0 + e^k\otimes\ \rho(e^j)\ \mu\ (\nabla_k^{A_0}(u_0)+i\ a_k\ u_0) \\
&= e^k\otimes\rho(e^j)\ \nabla_k^a(\mu)\ u_0\ + e^k\otimes\rho(e^j)\ \mu\ \nabla_k^{A_0}(u_0) \\
D\!\!\!/_A(\rho(e^j)\ \mu\ u_0) &= \rho(e^k)\rho(e^j)\ \nabla_k^a(\mu)\ u_0\ +\rho(e^k)\rho(e^j)\ \mu\ \nabla_k^{A_0}(u_0) \\
&= -\nabla_j^a(\mu)\ u_0\ +\frac{1}{2}\sum_{k,j}\rho(e^k)\rho(e^j)(\nabla_k^a(\mu)-\nabla_j^a(\mu))u_0 \\
&\quad -\mu\ \nabla_j^{A_0}(u_0)-\mu\ \rho(e^j)\sum_{k\neq j}\rho(e^k)\nabla_k^{A_0}(u_0) \qquad (10)
\end{aligned}
$$

Since $0 = D\!\!\!/_{A_0}(u_0)=\sum\rho(e^k)\nabla_k^{A_0}(u_0)$ the last term of (3) is $-\mu\ \nabla_j^{A_0}(u_0)$.

By plugging $\mu=\nabla_j^a(\alpha)$ in (10) and summing over j, from (2) we see

$$
D\!\!\!/_A^2(\alpha u_0) = -\nabla_j^a\nabla_j^a(\alpha)u_0+\frac{1}{2}\rho(\sum F_{k,j}^a\ e^k\wedge e^j)\ \alpha\ u_0 - 2\sum\nabla_j^a(\alpha)\nabla_j^{A_0}(u_0) \quad \square
$$

Remark: Notice that since u_0 has a constant $\mathbb{C}$ component and ∇_{A_0} restricts to the usual d the $\mathbb{C}$ component, the term $<\nabla_a\alpha\ ,\nabla_{A_0}(u_0)>$ lies entirely in L component of W^+

3 Seiberg-Witten invariants

Let X be a closed oriented Riemanian manifold, and $L\longrightarrow X$ a characteristic line bundle. Seiberg -Witten equations are defined for $(A,\psi)\in\mathcal{A}(L)\times\Gamma(W^+)$,

$$
\begin{aligned}
D\!\!\!/_A(\psi) &= 0 \qquad (11)\\
\rho(F_A^+) &= \sigma(\psi) \qquad (12)
\end{aligned}
$$

Gauge group $\mathcal{G}(L)=Map(X,S^1)$ acts on $\tilde{\mathcal{B}}(L)=\mathcal{A}(L)\times\Gamma(W^+)$ as follows: for $s=e^{if}\in\mathcal{G}(L)$

$$
s^*(A,\psi)=(s^*A,s^{-1}\psi)=(A+s^{-1}ds\ ,\ s^{-1}\psi)=(A+i\ df\ ,\ s^{-1}\psi)
$$

By locally writing $W^\pm=V^\pm\otimes L^{1/2}$, and $\psi=\varphi\otimes\lambda\in\Gamma(V^\pm\otimes L^{1/2})$ and from:

$$
D\!\!\!/_{s^*A}(\varphi\otimes\lambda) = \partial\!\!\!/(\varphi)\otimes\lambda+[\ \varphi\otimes\nabla_A(\lambda)+i\ df\ (\varphi\otimes\lambda)\ ]
$$

we see that $D\!\!\!/_{s^*A}\ (s^{-1}\psi)=s^{-1}D\!\!\!/_A(\psi)$, and from definitions

$$
\rho(F_{s^*A}^+)=s^{-1}\rho(F_A^+)\ s=\rho(F_A^+)=\sigma(\psi)=\sigma(s^{-1}\psi)
$$

Hence the solution set $\tilde{\mathcal{M}}(L)\subset\tilde{\mathcal{B}}(L)$ of Seiberg-Witten equations is preserved by the action $(A,\psi)\longmapsto s^*(A,\psi)$ of $\mathcal{G}(L)$ on $\tilde{\mathcal{M}}(L)$. Define

$$
\mathcal{M}(L)=\tilde{\mathcal{M}}(L)/\mathcal{G}(L)\quad\subset\quad\mathcal{B}(L)=\tilde{\mathcal{B}}(L)/\mathcal{G}(L)
$$

We call a solution (A,ψ) of (11) and (12) an irreducible solution if $\psi \neq 0$. $\mathcal{G}(L)$ acts on the subset $\tilde{\mathcal{M}}^*(L)$ of the irreducible solutions freely, we denote

$$\mathcal{M}^*(L) = \tilde{\mathcal{M}}^*(L)/\mathcal{G}(L)$$

Any solution (A,ψ) of Seiberg -Witten equations satisfies the C^0 bound

$$|\psi|^2 \leq \max(0,-2s) \tag{13}$$

where s is the scalar curvature function of X. This follows by plugging (12) in the Weitzenbock formula (7).

$$\not{D}_A^2(\psi) \quad = \quad \nabla_A^* \nabla_A \psi + \frac{s}{4}\psi + \frac{1}{4}\sigma(\psi)\psi \tag{14}$$

Then at the points where where $|\psi|^2$ is maximum, we calculate

$$\begin{aligned} 0 \leq \frac{1}{2}\Delta|\psi|^2 &= \frac{1}{2}d^*d <\ \psi,\psi\ >= \frac{1}{2}d^*(<\nabla_A\psi,\psi> + <\psi,\nabla_A\psi>) \\ &= \frac{1}{2}d^*(<\psi,\bar{\nabla}_A\psi>+<\psi,\nabla_A\psi>) = d^* <\psi\ ,\nabla_A\psi>_{\mathbb{R}} \\ &= <\psi,\nabla_A^*\nabla_A\psi> -|\nabla_A\psi|^2\ \leq <\psi,\nabla_A^*\nabla_A\psi> \\ &\leq -\frac{s}{4}|\ \psi\ |^2 - \frac{1}{8}|\ \psi\ |^4 \end{aligned}$$

The last step follows from (14), (11) and (5), and the last inequality gives (13)

Proposition 3.1 $\mathcal{M}(L)$ is compact

Proof: Given a sequence $[\ A_n,\psi_n\ ] \in \mathcal{M}(L)$ we claim that there is a convergent subsequence (which we will denote by the same index), i.e. there is a sequence of gauge transformations $g_n \in \mathcal{G}(L)$ such that $g_n^*(A_n,\psi_n)$ converges in C^∞. Let A_0 be a base connection. By Hodge theory of the elliptic complex:

$$\Omega^0(X) \xrightarrow{d^0} \Omega^1(X) \xrightarrow{d^+} \Omega^2_+(X)$$

$$A - A_0 = h_n + a_n + b_n \in \mathcal{H} \oplus im(d^+)^* \oplus im(d)$$

where $\mathcal{H}$ are the harmonic 1-forms. After applying gauge transformation g_n we can assume that $b_n = 0$, i.e. if $b_n = i\ df_n$ we can let $g_n = e^{if}$. Also

$$h_n \in \mathcal{H} = H^1(X;\mathbb{R}) \quad \text{and a component of } \ \mathcal{G}(L) \ \text{ is } \ H^1(X;\mathbb{Z})$$

Hence after a gauge transformation we can assume $h_n \in H^1(X;\mathbb{R})/H^1(X;\mathbb{Z})$ so h_n has convergent subsequence. Consider the first order elliptic operator:

$$D = d^* \oplus d^+ : \Omega^1(X)_{L^p_k} \longrightarrow \Omega^0(X)_{L^p_{k-1}} \oplus \Omega^2_+(X)_{L^p_{k-1}}$$

The kernel of D consists of harmonic 1-forms, hence by Poincare inequality if a is a 1-form orthogonal to the harmonic forms, then for some constant C

$$||a||_{L^p_k} \leq C||D(a)||_{L^p_{k-1}}$$

Now $a_n = (d^+)^*\alpha_n$ implies $d^*(a_n) = 0$. Since a_n is orthogonal to harmonic forms, and by calling $A_n = A_0 + a_n$ we see :

$$||a_n||_{L^p_1} \leq C\ ||D(a_n)||_{L^p} \leq C||d^+ a_n||_{L^p} = C\ ||F^+_{A_n} - F^+_{A_0}||_{L^p}$$

Here we use C for a generic constant. By (12), (4) and (13) there is a C depending only on the scalar curvature s with

$$||a_n||_{L^p_1} \leq C \tag{15}$$

By iterating this process we get $||a_n||_{L^p_k} \leq C$ for all k , hence $||a_n||_\infty \leq C$. From the elliptic estimate and $\not{D}_{A_n}(\psi_n) = 0$:

$$\begin{aligned} ||\psi_n||_{L^p_1} &\leq C(\ ||\not{D}_{A_0}\psi_n\ ||_{L^p} + ||\psi_n||_{L^p}) = C(\ ||a_n\psi_n\ ||_{L^p} + ||\psi_n||_{L^p}) \\ ||\psi_n||_{L^p_1} &\leq C(\ ||a_n||_\infty ||\psi_n||_{L^p} + ||\psi_n||_{L^p}) \leq C \qquad (16) \end{aligned}$$

By repeating this (boothstrapping) process we get $||\psi_n||_{L^p_k} \leq C$, for all k, where C depends only on the scalar curvature s and A_0. By Rallich theorem we get convergent subsequence of (a_n, ψ_n) in L^p_{k-1} norm for all k. So we get this convergence to be C^∞ convergence. □

It is not clear that the solution set of Seiberg-Witten equations is a smooth manifold. However we can perturb the Seiberg-Witten equations (11), (12) by any self dual 2-form $\delta \in \Omega^+(X)$, in a gauge invariant way, to obtain a new set of equations whose solutions set is a smooth manifold:

$$\not{D}_A(\psi) = 0 \tag{17}$$
$$\rho(F^+_A + i\ \delta) = \sigma(\psi) \tag{18}$$

Denote this solution space by $\tilde{\mathcal{M}}_\delta(L)$, and parametrized solution space by

$$\tilde{\mathcal{M}} = \bigcup_{\delta \in \Omega^+} \tilde{\mathcal{M}}_\delta(L) \times \{\ \delta\ \} \subset \mathcal{A}(L) \times \Gamma(W^+) \times \Omega^+(X)$$

$$\mathcal{M}_\delta(L) = \tilde{\mathcal{M}}_\delta(L)\ /\mathcal{G}(L) \quad \subset \quad \mathcal{M} = \tilde{\mathcal{M}}\ /\mathcal{G}(L)$$

Let $\tilde{\mathcal{M}}_\delta(L)^* \subset \tilde{\mathcal{M}}^*$ be the corresponding irreducible solutions, and also let $\mathcal{M}_\delta(L)^* \subset \mathcal{M}^*$ be their quotients by Gauge group. The following theorem says that for a generic choice of δ the set $\mathcal{M}_\delta(L)^*$ is a closed smooth manifold.

Proposition 3.2 $\mathcal{M}^*$ is a smooth manifold. Projection $\pi : \mathcal{M}^* \longrightarrow \Omega^+(X)$ is a proper surjection of Fredholm index:

$$d(L) = \frac{1}{4}[\ c_1(L)^2 - (2\chi + 3\sigma)\ ]$$

where χ and σ are Euler characteristic and the signature of X.

Proof: The linearization of the map $(A, \psi, \delta) \longmapsto (\rho(F_A^+ + i\ \delta) - \sigma(\psi), \not{D}_A(\psi)\ )$ at (A_0, ψ_0, δ_0) is given by:

$$P : \Omega^1(X) \oplus \Gamma(W^+) \oplus \Omega^+(X) \longrightarrow su(W^+) \oplus \Gamma(W^-)$$

$$P(a, \psi, \epsilon) = (\rho(d^+a + i\ \epsilon) - 2\ \sigma(\psi, \psi_0)\ ,\ \not{D}_{A_0}\psi + \rho(a)\psi_0)$$

To see that this is onto we pick $(\kappa, \theta) \in su(W^+) \oplus \Gamma(W^-)$, by varying ϵ we can see that $(\kappa, 0)$ is in the image of P. To see $(0, \theta)$ is in the image of P, we prove that if it is orthogonal to $image(P)$ then it is $(0, 0)$; i.e. assume

$$< \not{D}_{A_0}\psi, \theta > + < \rho(a)\psi_0, \theta >= 0$$

for all a and ψ. By choosing $a = 0$ and by the self adjointness of Dirac operator we get $\not{D}_{A_0}\theta = 0$. Also since $\psi_0 \neq 0$ and $\not{D}_{A_0}\psi_0 = 0$, by analytic continuation $\psi_0 \neq 0$ on some open set U. By choosing $\psi = 0$ and a to be a "bump" form on U, we see that $< \rho(a)\psi_0, \theta >= 0$ for all a on an arbitrarily small open set U. This implies $< \rho(a)\psi_0, \theta >= 0$ pointwise on U, for all a and $\psi_0 \neq 0$, hence $\theta = 0$ on U hence by analytic continuation $\theta = 0$.

By implicit function theorem $\tilde{\mathcal{M}}$ is a smooth manifold, and by Sard's theorem $\tilde{\mathcal{M}}_\delta(L)$ are smooth manifolds, for generic choice of δ's. Hence their free quotients $\mathcal{M}^*$ and $\mathcal{M}_\delta(L)^*$ are smooth manifolds.

After taking "gauge fixig" account, the dimension of $\mathcal{M}_\delta(L)$ is given by the index of $P + d^*$ (c.f. [**DK**]). $P + d^*$ is the compact perturbation of

$$S : \Omega^1(X) \oplus \Gamma(W^+) \longrightarrow [\ \Omega^0(X) \oplus \Omega^2_+(X)\ ] \oplus \Gamma(W^-)$$

$$S = \begin{pmatrix} d^* \oplus d^+ & 0 \\ 0 & \not{D}_{A_0} \end{pmatrix}$$

By Atiyah-Singer index theorem

$$\begin{aligned} \dim \mathcal{M}_\delta(L) = \mathrm{ind}(S) &= \mathrm{index}(d^* \oplus d^+) + \mathrm{index}_{\mathbb{R}}\ \not{D}_{A_0} \\ &= -\frac{1}{2}(\chi + \sigma) + \frac{1}{4}(c_1(L)^2 - \sigma) \\ &= \frac{1}{4}\ [\ c_1(L)^2 - (2\chi + 3\sigma)\ ] \\ &= \frac{c_1(L)^2 - \sigma}{4}\ -\ (1 + b^+) \qquad (19) \end{aligned}$$

where b^+ is the dimension of positive define part H^2_+ of $H^2(X;\mathbb{Z})$. Notice that when b^+ is odd this expression is even, since L being a characteristic line bundle we have $c_1(L)^2 = \sigma \bmod 8$ $\square$

Now assume that $H^1(X) = 0$, then $\mathcal{G}(L) = K(\mathbb{Z}, 1)$. Than being a free quotient of a contractible space by $\mathcal{G}(L)$ we have

$$\mathcal{B}^*(L) = K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$$

The orientation of H^2_+ gives an orientation to $\mathcal{M}_\delta(L)$. Now By (19) if b^+ is odd $\mathcal{M}_\delta(L) \subset \mathcal{B}^*(L)$ is an even dimensional $2d$ smooth closed oriented submanifold, then we can define Seiberg-Witten invariants as:

$$SW_L(X) = < \mathcal{M}_\delta(L) \ , \ [\ \mathbb{CP}^d\ ] >$$

As in the case of Donaldson invariants ([**DK**]), even though $\mathcal{M}_\delta(L)$ depends on metric (and on the perturbation δ) the invariant $SW_L(X)$ is independent of these choices, provided $b^+ \geq 2$, i.e. there is a generic metric theorem.

Also by (13) if X has nonnegative scalar curvature then all the solutions are reducible, i.e. $\psi = 0$. This implies that A is anti-self-dual, i.e. $F^+_A = 0$; but just as in [**DK**] , If $b^+ \geq 2$ for a generic metric L can not admit such connections. Hence $\tilde{\mathcal{M}} = \emptyset$ which implies $SW_L(X) = 0$.

Similar to Donaldson invariants there is a "connected sum theorem" for Seiberg-Witten invariants: If X_i $i = 1, 2$ are oriented compact smooth manifolds such that $b^+(X_i) > 0$ and with common boundary, which is a 3-manifold with a positive scalar curvature; then gluing these manifolds together along their boundaries produces a manifold $X = X_1 \smile X_2$ with vanishing Seiberg-Witten invariants (cf [**F**],[**FS**]). There is also conjecture that only 0-dimensional moduli spaces $\mathcal{M}_\delta(L)$ give nonzero invariants $SW_L(X)$.

4 Almost Complex and Symplectic Structures

Now assume that X has an almost complex structure. This means that there is a principal $GL(2,\mathbb{C})$-bundle $Q \longrightarrow X$ such that

$$T(X) \cong Q \times_{GL(2,\mathbb{C})} \mathbb{C}^2$$

By choosing Hermitian metric on $T(X)$ we can assume $Q \longrightarrow X$ is a $U(2)$ bundle, and the tangent frame bundle $P_{SO(4)}(TX)$ comes from Q by the reduction map

$$U(2) = (\ S^1 \times SU(2)\ )/\mathbb{Z}_2 \hookrightarrow (\ SU(2) \times SU(2)\ )/\mathbb{Z}_2 = SO(4)$$

Equivalently there is an endemorphism $I \in \Gamma(End(TX))$ with $I^2 = -Id$

$$\begin{array}{ccc} T(X) & \xrightarrow{I} & T(X) \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

The $\pm i$ eigenspaces of I splits the complexified tangent space $T(X)_{\mathbb{C}}$

$$T(X)_{\mathbb{C}} \cong T^{1,0}(X) \oplus T^{0,1}(X) = \Lambda^{1,0}(X) \oplus \Lambda^{0,1}(X)$$

This gives us a complex line bundle which is called the canonical line bundle:

$$K = K_X = \Lambda^{2,0}(X) = \Lambda^2(T^{1,0}) \longrightarrow X$$

Both $K^{\pm}$ are characteristic; corresponding to line bundle $K \longrightarrow X$ there is a canonical $Spin_c(4)$ structure on X, given by the lifting of $f[\lambda, A] = ([\lambda, A], \lambda^2)$

$$\begin{array}{ccc} & & Spin_c(4) \\ & F \nearrow & l \downarrow \\ U(2) & \xrightarrow{f} & SO(4) \times S^1 \end{array}$$

$F[\lambda, A] = [\lambda, A, \lambda]$. Transition function λ^2 gives K, and the corresponding $\mathbb{C}^2$-bundles are given by:

$$\begin{aligned} W^+ &= \Lambda^{0,2}(X) \oplus \Lambda^{0,0}(X) = K^{-1} \oplus \mathbb{C} \\ W^- &= \Lambda^{0,1}(X) \end{aligned}$$

We can check this from the transition functions, e.g. for W^+, $x = z + jw \in \mathbb{H}$

$$x \longmapsto \lambda x \lambda^{-1} = \lambda(z + jw)\bar{\lambda} = z + jw\bar{\lambda}\bar{\lambda} = z + jw\lambda^{-2}$$

Since we can identify $\bar{\Lambda}^{0,1}(X) \cong \Lambda^{1,0}(X)$, and $\Lambda^{0,2}(X) \otimes \Lambda^{1,0}(X) \cong \Lambda^{0,1}(X)$ we readily see the decomposition $T(X)_{\mathbb{C}} \cong W^+ \otimes \bar{W}^-$. As real bundles we have

$$\Lambda^+(X) \cong K \oplus \mathbb{R}$$

We can verify this by taking $\{e^1, e^2 = I(e^1), e^3, e^4 = I(e^3)\}$ to be a local orthonormal basis for $T^*(X)$, then

$$\Lambda^{1,0}(X) = < e^1 - ie^2, e^3 - ie^4 > \quad , and \quad \Lambda^{0,1}(X) = < e^1 + ie^2, e^3 + ie^4 >$$

$$K = < f = (e^1 - ie^2) \wedge (e^3 - ie^4) >$$

$$\Lambda^+(X) = < \omega = \frac{1}{2}(e^1 \wedge e^2 + e^3 \wedge e^4),\ f_2 = \frac{1}{2}(e^1 \wedge e^3 + e^4 \wedge e^2),\ f_3 = \frac{1}{2}(e^1 \wedge e^4 + e^2 \wedge e^3) >$$

ω is the global form $\omega(X, Y) = g(X, IY)$ where g is the hermitian metric (which makes the basis $\{e^1, e^2, e^3, e^4\}$ orthogonal). Also since $f = 2(f_2 - if_3)$, we see as $\mathbb{R}^3$-bundles $\Lambda^+(X) \cong K \oplus \mathbb{R}(\omega)$. We can check:

$$W^+ \otimes \bar{W}^+ \cong \mathbb{C} \oplus \mathbb{C} \oplus K \oplus \bar{K} = (K \oplus \mathbb{R})_{\mathbb{C}} \oplus \mathbb{C}$$

As before by writing the sections of W^+ by $z + jw \in \Gamma(\mathbb{C} \oplus K^{-1})$ we see that ω, f_2, f_3 act as Pauli matrices; in particular

$$\begin{aligned}
\omega &\longmapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \\
f &\longmapsto 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} - 2i\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix} \\
\bar{f} &\longmapsto 2\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + 2i\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 4 & 0 \end{pmatrix}
\end{aligned}$$

So in particular, if we write $\psi \in \Gamma(W^+) = \Gamma(\mathbb{C} \oplus K^{-1})$ by $\psi = \alpha u_0 + \beta$, where β is a section of K^{-1}, and $\alpha : X \to \mathbb{C}$ and u_0 is a fixed section of the trivial bundle $\underline{\mathbb{C}} \to X$ with $||u_0|| = 1$, then

$$\begin{array}{ll}
\rho(\omega)\; u_0 = i u_0 & \rho(\omega)\; \beta = -i\beta \\
\rho(\beta)\; u_0 = 4\beta & \rho(\beta)\; \beta = 0 \qquad\qquad (*) \\
\rho(\bar{\beta})\; u_0 = 0 & \rho(\bar{\beta})\; \beta = -4\;|\beta|^2 u_0
\end{array}$$

We see these by locally writing ψ in terms of basis $\psi = \alpha u_0 + \lambda \bar{f}$, where $\beta = \lambda \bar{f}$ with $||\bar{f}|| = 1$. Writing Formula (3) in terms of the basis $\{\omega, f, \bar{f}\}$ we get:

$$\begin{aligned}
i\,\sigma(\alpha, \lambda) &= \rho\,[\, \frac{|\alpha|^2 - |\lambda|^2}{2}\,\omega - \frac{i}{4}\,\alpha\bar{\lambda} f + \frac{i}{4}\,\bar{\alpha}\lambda\bar{f}\,] \\
\sigma(\psi) &= \rho\,[\, \frac{|\beta|^2 - |\alpha|^2}{2}\, i\,\omega - \frac{1}{4}\,\alpha\,\bar{\beta} + \frac{1}{4}\,\bar{\alpha}\,\beta\,] \qquad (20)
\end{aligned}$$

If we consider the decomposition $F_A^+ = F_A^{2,0} + F_A^{0,2} + F_A^{1,1}$ the equation $\rho(F_A) = \sigma(\psi)$ gives Witten's formulas:

$$F_A^{2,0} = -\frac{1}{4}\alpha\,\bar{\beta} \qquad (21)$$

$$F_A^{0,2} = \frac{1}{4}\bar{\alpha}\,\beta \qquad (22)$$

$$F_A^{1,1} = \frac{|\beta|^2 - |\alpha|^2}{2}\, i\,\omega \qquad (23)$$

In case X is a Kahler surface the Dirac operator is given by (c.f.[**LM**])

$$\not{D}_A = \bar{\partial}^*_A + \bar{\partial}_A : \Gamma(W^+) \to \Gamma(W^-)$$

Hence from the Dirac part of the Seiberg-Witten equation (17) we have

$$\begin{aligned}
\bar{\partial}^*_A(\beta) + \bar{\partial}_A(\alpha u_0) &= 0 \\
\bar{\partial}_A\bar{\partial}^*_A(\beta) + \bar{\partial}_A\bar{\partial}_A(\alpha u_0) &= 0 \qquad (24)
\end{aligned}$$

The second term is $\bar{\partial\!\!\!/}_A \bar{\partial\!\!\!/}_A(\alpha u_0) = F_A^{0,2}\alpha u_0 = \frac{1}{4}|\alpha|^2\beta$. By taking inner product both sides of (24) by β and integrating over X we get the L^2 norms satisfy

$$||\alpha||^2||\beta||^2 = 0 \quad \Longrightarrow \quad \alpha = 0 \text{ or } \beta = 0 \tag{25}$$

This argument eventually calculates $SW_K(X) = 1$ ([**W**]). We will not repeat this argument here, instead we will review a stronger result of C.Taubes for symplectic manifolds below, which implies this result.

We call an almost complex manifold with Hermitian metric $\{X, I, g\}$ syplectic if $d\omega = 0$. Clearly a nondegenerate closed form ω and a hermitian metric determines the almost complex structure I. Given ω then I is called an almost complex structure taming the symplectic form ω

By Section 2.1 there is a unique connection A_0 in $K \longrightarrow X$ such that the induced Dirac operator D_{A_0} on W^+ restricted to the trivial summand $\underline{\mathbb{C}} \to X$ is the exterior derivative d. Let u_0 be the section of W^+ with constant $\mathbb{C}$ component and $||u_0|| = 1$. Taubs's first fundamental observation is

$$D\!\!\!/_A(u_0) = 0 \quad \text{ if and only if } \quad d\omega = 0$$

This can be seen by applying the Dirac operator to both sides of $iu_0 = \rho(\omega).u_0$, and observing that by the choice of u_0 the term $\nabla_{A_0}(u_0)$ lies entirely in K^{-1} component:

$$\begin{aligned} iD\!\!\!/_{A_0}(u_0) &= \sum \rho(e^i)\nabla_i\ (\rho(\omega)\ u_0) \\ &= \sum \rho(e^i)\ [\ \nabla_i\ (\rho(\omega))\ u_0 + \rho(\omega)\ \nabla_i\ (u_0)\ ] \\ &= \sum \rho(e^i)\nabla_i\ (\rho(\omega))\ u_0 - i\ \sum \rho(e^i\ )\nabla_i\ (u_0) \\ 2i\ D\!\!\!/_{A_0}(u_0) &= \sum \rho(e^i)\nabla_i\ (\rho(\omega))\ u_0 = \rho((d+d^*)\omega)\ u_0 = \rho((d - *d)\omega)\ u_0 \end{aligned}$$

Last equality holds since $\omega \in \Lambda^+(X)_{\mathbb{C}} \oplus \mathbb{C}$, and by naturality, the Dirac operator on $\Lambda^*(X)_{\mathbb{C}}$ is $d + d^*$, and since $d = - * d*$ on 2 forms and ω is self dual

$$2i\ D\!\!\!/_{A_0}(u_0) = -\rho(*d\omega)u_0$$

Theorem (Taubes) : Let (X, ω) be a closed symplectic manifold such that $b_2(X)^+ \geq 2$, then $SW_K(X) = \pm 1$.

Proof: Write $\psi = \alpha u_0 + \beta \in \Gamma(W^+) = \Gamma(\mathbb{C} \oplus K^{-1})$ where $\alpha : X \to \mathbb{C}$, and u_0 is the section as above. Consider the perturbed Seiberg-Witten equations : For $(A, \psi) \in \mathcal{A}(L) \times \Gamma(W^+)$:

$$D\!\!\!/_A(\psi) = 0 \tag{26}$$
$$\rho(F_A^+) = \rho(F_{A_0}^+) + r\ [\ \sigma(\psi) + i\ \rho(\omega)\ ] \tag{27}$$

By (20) the second equation is equivalent to:

$$F_A^+ - F_{A_0}^+ \quad = \quad r \;\; [\; (\frac{|\beta|^2 - |\alpha|^2}{2} + 1) \; i\omega - \frac{1}{4}\alpha\bar{\beta} + \frac{1}{4}\bar{\alpha}\beta \;] \tag{28}$$

We will show that up to gauge equivalence there is a unique solution to these equations. Write $A = A_0 + a$, after a gauge transformation we can assume that a is coclosed, i.e. $d^*(a) = 0$. Clearly $(A, \psi) = (A_0, u_0)$, and $r = 0$ satisfy these equations. It suffices to show that for $r \longmapsto \infty$ these equations admit only (A_0, u_0) as a solution. From Weitzenbock formulas (7), (8) and abbreviating $\nabla_{A_0}(u_0) = b$ we get

$$\not{D}_A^2(\psi) = \not{D}_A^2(\beta) + (\nabla_a^*\nabla_a\alpha)u_0 - 2 < \nabla_a\alpha, b > + \frac{1}{2}\alpha \; \rho(F_A^+ - F_{A_0}^+) \; u_0 \tag{29}$$

$$\not{D}_A^2(\beta) = (\nabla_A^*\nabla_A \; \beta) + \frac{s}{4} \; \beta + \frac{1}{4}\rho(F_{A_0}^+) \; \beta + \frac{1}{4}\rho(F_A^+ - F_{A_0}^+) \; \beta \tag{30}$$

From (28) and (*) we see that

$$\frac{1}{2}\alpha \; \rho(F_A^+ - F_{A_0}^+)u_0 \quad = \quad \frac{r}{4}\alpha \; (\; |\alpha|^2 - |\beta|^2 - 2) \; u_0 + \frac{r}{2} \; |\alpha|^2\beta \tag{31}$$

$$\frac{1}{4} \; \rho(F_A^+ - F_{A_0}^+) \; \beta \quad = \quad -\frac{r}{8} \; (\; |\alpha|^2 - |\beta|^2 - 2) \; \beta + \frac{r}{4}\alpha|\beta|^2 u_0 \tag{32}$$

By substituting (31) in (29) we get

$$\begin{aligned} \not{D}_A^2(\psi - \beta) \quad = \quad & [\; \nabla_a^*\nabla_a\alpha + \frac{r}{4}\alpha \; (|\alpha|^2 - |\beta|^2 - 2) \;] \; u_0 \\ & -2 < \nabla_a\alpha, b > + \frac{r}{2} \; |\alpha \; |^2\beta \end{aligned} \tag{33}$$

By substituting (32) in (30), then substituting (30) in (33) we obtain:

$$\begin{aligned} 0 = \not{D}_A^2(\psi) \quad = \quad & [\; \nabla_a^*\nabla_a\alpha + \frac{r}{4}\alpha \; (|\alpha|^2 - 2) \;] \; u_0 - 2 < \nabla_a\alpha, b > \\ & + [\; \nabla_A^*\nabla_A + \frac{s}{4} + \frac{1}{4} \; \rho(F_{A_0}^+) + \frac{r}{8}(\; 3|\alpha|^2 + |\beta|^2 + 2) \;] \; \beta \end{aligned} \tag{34}$$

By recalling that β and u_0 are orthogonal sections of W^+, we take inner product of both sides of (8) with β and integrate over X and obtain:

$$\int_X (\; |\nabla_A\beta \; |^2 + \frac{r}{8} \; |\beta|^4 + \frac{r}{4}|\beta|^2 + \frac{3r}{8}|\alpha|^2|\beta|^2 \;) \quad =$$

$$2\int_X (<< \nabla_a\alpha \; , \; b > \; , \; \beta > - \frac{s}{4} \; |\beta|^2 - \; \frac{1}{4} < \rho(F_{A_0}^+)\beta \; , \; \beta >$$

Hence $\displaystyle\int_X \; |\nabla_A\beta \; |^2 + \frac{r}{8} \; |\beta|^4 + \frac{r}{4}|\beta|^2 + \frac{3r}{8}|\alpha|^2|\beta|^2 \; \leq \int_X c_1|\beta|^2 + c_2|\beta||\nabla_a\alpha|$

where c_1 and c_2 are positive constants depending on the Riemanian metric and the base connection A_0. Choose $r \gg 1$, by calling $c_2 = 2c_3$ we get :

$$\int_X (\ |\nabla_A \beta \ |^2 + \frac{r}{8} \ |\beta|^4 + \frac{r}{8}|\beta|^2 + \frac{3r}{8}|\alpha|^2|\beta|^2 \ ) \leq \int_X (c_1 - \frac{r}{8}) \ |\beta \ |^2 + 2c_3|\beta \ ||\nabla_a \alpha| =$$

$$- \int_X \Big[\ (r/8 - c_1)^{1/2} \ |\beta \ | - c_3(r/8 - c_1)^{-1/2} \ |\nabla_a \alpha| \ \Big]^2 + \frac{c_3^2}{(r/8 - c_1)}|\nabla_a \alpha|^2 \leq \int_X \frac{C}{r}|\nabla_a \alpha|^2$$

For some C depending on the metric and A_0. In particular we have

$$\begin{aligned} \int_X r \ |\beta|^2 - \frac{8C}{r} \ |\nabla_a \alpha|^2 \quad &\leq \quad 0 \\ \int_X 8c_2 \ |\beta \ | \ |\nabla_a \alpha| - \frac{8C}{r}|\nabla_a \alpha|^2 \quad &\leq \quad \int_X (r - 8c_1) \ |\beta \ |^2 \end{aligned}$$

$$\text{Hence} \qquad \int_X c_2 \ |\beta \ | \ |\nabla_a \alpha| - \frac{2C}{r}|\nabla_a \alpha|^2 \quad \leq \quad 0 \tag{35}$$

Now by self adjointness of the Dirac operator, and by $\alpha u_0 = \psi - \beta$ we get:

$$\begin{aligned} < \not{D}_A^2(\psi) \ , \ \alpha u_0 > \quad &= \quad < \not{D}_A^2(\psi - \beta) \ , \ \alpha u_0 > + < \not{D}_A^2 \ (\beta) \ , \ \alpha u_0 > \\ &= \quad < \not{D}_A^2(\psi - \beta) \ , \ \alpha u_0 > + < \beta \ , \ \not{D}_A^2(\psi - \beta) > \end{aligned} \tag{36}$$

We can calculate (36) by using (33) and obtain the inequalities:

$$\begin{aligned} 0 =< \not{D}_A^2(\psi) \ , \ \alpha u_0 > \quad = \quad & |\nabla_a \alpha|^2 + \frac{r}{4}|\alpha|^4 - \frac{r}{4}|\alpha|^2|\beta|^2 - \frac{r}{2}|\alpha|^2 \\ & + \frac{r}{2} \ |\alpha|^2|\beta|^2 - 2 << \nabla_a \alpha, b >, \beta > \end{aligned}$$

$$\begin{aligned} \int_X |\nabla_a \alpha|^2 + \frac{r}{4}|\alpha|^4 - \frac{r}{2}|\alpha|^2 \quad &\leq \quad \int_X 2 << \nabla_a \alpha, b >, \beta > - \frac{r}{4}|\alpha|^2|\beta|^2 \\ &\leq \quad \int_X 2 << \nabla_a \alpha, b >, \beta > \leq \int_X c_2|\nabla_a \alpha| \ |\beta \ | \end{aligned}$$

By choosing $c_4 = 1 - 2C/r$ and by (35), we see

$$\int_X c_4 \ |\nabla_a \alpha|^2 + \frac{r}{4}|\alpha|^4 - \frac{r}{2}|\alpha|^2 \quad \leq \quad 0 \tag{37}$$

Since for a connection A in $K \longrightarrow X$ the class $(i/2\pi)F_A$ represents the Chern class $c_1(K)$, and since ω is a self dual two form we can write:

$$\int_X \omega \wedge F_A = -2\pi \ i \ \omega c_1(K) \qquad \int_X \omega \wedge F_A = \int_X \omega \wedge F_A^+$$

$$\int_X \omega \wedge (F_A^+ - F_{A_0}^+) = 0$$

By (28) this implies:

$$\frac{r}{2}\int_X (2 - |\alpha|^2 + |\beta|^2) = 0 \tag{38}$$

By adding (38) to (37) we get

$$\int_X c_4\ |\nabla_a \alpha|^2 + \frac{r}{2}|\beta|^2 + r(1 - \frac{1}{2}|\alpha|^2)^2 \leq 0 \tag{39}$$

Assume $r \gg 1$, then $c_4 \geq 0$ and hence $\nabla_a \alpha = 0$ and $\beta = 0$ and $|\alpha| = \sqrt{2}$, hence:

$$\beta = 0 \ \text{ and } \ \alpha = \sqrt{2}e^{i\ \theta} \quad \text{and} \ \ \nabla_a(\alpha) = d(e^{i\ \theta}) + i\ a\ e^{i\ \theta} = 0$$

Hence $a = d(-\theta)$, recall that we also have $d^*(a) = 0$ which gives

$$0 =< d^*d(\theta), \theta >=< d(\theta), d(\theta >) >= ||d(\theta)||^2$$

Hence $a = 0$ and $\alpha =$constant. So up to a gauge equivalence $(A, \psi) = (A_0, u_0)$ □

5 Applications

Let X be a simply connected closed smooth 4-manifold. By J.H.C.Whitehead the intersection form

$$q_X : H_2(X;\mathbb{Z}) \otimes H_2(X;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

determines the homotopy type of X. By C.T.C Wall in fact q_X determines the h-cobordism class of X. Donaldson (c.f. [**DK**]) showed that if q_X is definite then it is dioganalizable, i.e.

$$q_X =< 1 > \oplus < 1 > \oplus ... < 1 >$$

We call q_X is even if q(a,a) is even for all a, otherwise we call q_X odd. Since integral liftings c of the second Steifel Whitney calass w_2 of X are characterized by $c.a = a.a$ for all $a \in H_2(X;\mathbb{Z})$, the condition of q_X being even is equivalent to X being spin. From classification of unimodular even integral quadratic forms and the Rohlin theorem it follows that the intersection form of a closed smooth spin manifold is in the form:

$$q_X = 2kE_8 \oplus lH \tag{40}$$

where E_8 is the 8×8 intersection matrix given by the Dynikin diagram

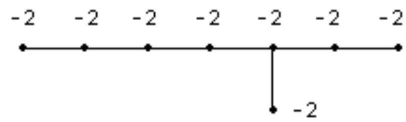

Figure 1:

and H is the form $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. The intersection form of the manifold $S^2 \times S^2$ realizes the form H, and the K3 surface (quadric in $\mathbb{CP}^3$) realizes $2E_8 \oplus 3H$. Donaldson had shown that if $k = 1$, then $l \geq 3$ ([**D**]). Clearly connected sums of K3 surface realizes $2kE_8 \oplus 3kH$. In general it is a conjecture that in (40) we must necessarily have $l \geq 3k$ (sometimes this is called 11/8 conjecture). Recently by using Seiberg-Witten theory M.Furuta has shown that

Theorem (Furuta) : Let X be a simply connected closed smooth spin 4-manifold with the intersection form $q_X = 2kE_8 \oplus lH$, then $l \geq 2k + 1$

Proof: We will only sketch the proof of $l \geq 2k$. We pick $L \longrightarrow X$ to be the trivial bundle (it is characteristic since X is spin). Notice that the spinor bundles

$$V^{\pm} = P \times_{\rho_{\pm}} \mathbb{C}^2 \longrightarrow X$$

$\rho_{\pm} : x \longmapsto q_{\pm}x$, are quoternionic vector bundles. That is, there is an action $j : V^{\pm} \rightarrow V^{\pm}$ defined by $[p, x] \rightarrow [p, xj]$, which is clearly well defined. This action commutes with

$$\partial\!\!\!/ : \Gamma(V^+) \longrightarrow \Gamma(V^-)$$

Let A_0 be the trivial connection, and write $\pm A = A_0 \pm i\ a \in \mathcal{A}(\mathcal{L})$

$$\begin{aligned} \partial\!\!\!/_A(\psi j) &= \sum \rho(e^k)\ [\ \nabla_k + i\ a\ ]\ (\psi j) = \sum \rho(e^k)\ [\ \nabla_k(\psi)j + \psi jia\ ] \\ &= \sum \rho(e^k)\ [\ \nabla_k(\psi)j - ia\psi j\ ] = \partial\!\!\!/_{-A}(\psi)j \end{aligned}$$

$\mathbb{Z}_4$ action $(A, \psi) \longmapsto (-A, \psi j)$ on $\Omega^1(X) \times \Gamma(V^+)$ preserves the compact set

$$\mathcal{M}_0 = \tilde{\mathcal{M}} \cap\ ker(d^*) \oplus \Gamma(V^+)$$

where $\tilde{\mathcal{M}} = \{(a, \psi) \in \Omega^1(X) \oplus \Gamma(V^+) \mid \partial\!\!\!/_A(\psi) = 0\ , \quad \rho(F_A^+) = \sigma(\psi)\ \}$
For example from the local description of σ in (2) we can check

$$\sigma(\psi j) = \sigma(z + jw)j = \sigma(-\bar{w} + j\bar{z}) = -\sigma(\psi) = -\ F_A^+ = F_{-A}^+$$

This $\mathbb{Z}_4$ in fact extends to an action of the subgroup G of $SU(2)$ which is generated by $< S^1\ ,\ j >$, where S^1 acts trivially on Ω^* and by complex multiplication on $\Gamma(V^+)$, and j acts by -1 on Ω^* and by quaternionic multiplication on $\Gamma(V^+)$ In particular we get a G-equivariant map $\varphi = L + \theta : \mathcal{V} \rightarrow \mathcal{W}$ where:

$$\varphi : \mathcal{V} = ker(d^*) \oplus \Gamma(V^+) \longrightarrow \mathcal{W} = \Omega^2_+ \oplus \Gamma(V^-)$$

$$L = \begin{pmatrix} d^+ & 0 \\ 0 & \partial\!\!\!/ \end{pmatrix} \quad \text{and} \quad \theta(a,\psi) = (\ \sigma(\psi)\ ,\ a\psi\ )$$

with $\varphi^{-1}(0) = \mathcal{M}_0$ and $\varphi(v) = L(v) + \theta(v)$ with L linear Fredholm and θ quadratic. We apply the "usual" Kuranishi technique (cf [**L**]) to obtain a finite dimensional local model $V \longmapsto W$ for φ.

We let $\mathcal{V} = \oplus V_\lambda$ and $\mathcal{W} = \oplus W_\lambda$, where V_λ and W_λ be λ eigenspaces of $L^*L : V \to V$ and $LL^* : W \to W$ repectively. Since LL^* is a multiplication by λ on V_λ, for $\lambda > 0$ we have isomorphisms $L : V_\lambda \xrightarrow{\cong} W_\lambda$. Now pick $\Lambda > 0$ and consider projections:

$$\oplus_{\lambda \leq \Lambda} W_\lambda \ \xleftarrow{p_\Lambda}\ W \ \xrightarrow{1-p_\Lambda}\ \oplus_{\Lambda > \Lambda} W_\lambda$$

Consider the local diffeomorphism $f_\Lambda : V \longrightarrow V$ given by:

$$u = f_\Lambda(v) = v + L^{-1}(1-p_\Lambda)\theta(v) \iff L(u) = L(v) + (1-p_\Lambda)\theta(v)$$

The condition $\varphi(v) = 0$ is equivalent to $p_\Lambda\ \varphi(v) = 0$ and $(1-p_\Lambda)\ \varphi(v) = 0$, but

$$(1-p_\Lambda)\ \varphi(v) = 0 \iff (1-p_\Lambda)\ L(v) + (1-p_\Lambda)\ \theta(v) = 0 \iff$$
$$(1-p_\Lambda)\ L(v) + L(u) - L(v) = 0 \iff L(u) = p_\Lambda\ L(v) \iff u \in \oplus_{\lambda \leq \Lambda} V_\lambda$$

Hence $\varphi(v) = 0 \iff p_\Lambda\ \varphi(v) = 0$ and $u \in \oplus_{\lambda \leq \Lambda} V_\lambda$, let

$$\varphi_\Lambda : V = \oplus_{\lambda \leq \Lambda} V_\lambda \longrightarrow W = \oplus_{\lambda \leq \Lambda} W_\lambda \quad \text{where} \quad \varphi_\Lambda(u) = p_\Lambda\ \varphi\ f_\Lambda^{-1}(u)$$

Hence in the local diffeomorphism $f_\Lambda : \mathcal{O} \xrightarrow{\approx} \mathcal{O} \subset \mathcal{V}$ takes the piece of the compact set $f_\lambda(\mathcal{O} \cap \mathcal{M}_0)$ into the finite dimensional subspace $V \subset \mathcal{V}$, where $\mathcal{O}$ is a neighborhood of $(0,0)$. As a side fact note that near $(0,0)$ we have

$$\mathcal{M}(L) \approx \mathcal{M}_0(L)/S^1$$

We claim that for $\lambda \gg 1$, the local diffeomorphism $f_\Lambda : \mathcal{O} \xrightarrow{\approx} \mathcal{O} \subset V$ extends to a ball B_R of large radius R containig the compact set $\mathcal{M}_0(L)$, i.e. we can make the zero set $\varphi_\Lambda^{-1}(0)$ a compact set.

We see this by applying the Banach contraction principle. For example for a given $u \in B_R$, showing that there is $v \in V$ such that $f_\Lambda(v) = u$ is equivalent of showing that the map $T_u(v) = u - L^{-1}(1-p_\Lambda)\theta(v)$ has a fixed point. Since $L^{-1}(1-p_\Lambda)$ has eigenvalues $1/\lambda$ on each W_λ in appropriate Sobolev norm we can write

$$||T_u(v_1) - T_u(v_2)|| \leq \frac{C}{\Lambda}||\theta(v_1) - \theta(v_2)|| \leq \frac{C}{\Lambda}||v_1 - v_2||$$

Vector subspaces V_λ and W_λ are either quaternionic or real depending on whether they are subspaces of $\Gamma(V^\pm)$ or $\Omega^*(X)$. For a generic metric we can make the cokernel of $\partial\!\!\!/$ zero hence the dimension of the kernel (as a complex vector space) is $ind(\partial\!\!\!/) = -\sigma/8 = 2k$, and since $H^1(X) = 0$ the dimension of the cokernel of d^+ (as a real vector space) is $b^+ = l$. Hence φ_Λ gives a G-equivariant map

$$\varphi : \mathbb{H}^{k+y} \oplus \mathbb{R}^x \longrightarrow \mathbb{H}^y \oplus \mathbb{R}^{l+x}$$

with compact zero set. From this Furuta shows that $l \geq 2l + 1$. Here we give an easier argument of D.Freed which gives a slightly weaker result of $l \geq 2k$. Let E_0 and E_1 be the complexifications of the domain an the range of φ; consider E_0 and E_1 as bundles over a point x_0 with projections $\pi_i : E_i \to x_0$, and with 0-sections $s_i : x_0 \to E_i$, $i = 0, 1$. Recall $K_G(x_0) = R(G)$, and we have Bott isomorphisms $\beta(\rho) = \pi_i^*(\rho)\ \lambda_{E_i}$, for $i = 0, 1$ where λ_{E_i} are the Bott classes. By compactness we get an induced map φ^*:

$$\begin{array}{ccc} K_G(B(E_1), S(E_1)) & \stackrel{\varphi^*}{\longrightarrow} & K_G(B(E_0), S(E_0)) \\ \approx \uparrow \beta & & \approx \uparrow \beta \\ R(G) & & R(G) \end{array}$$

Consider $s_i^*(\lambda_{E_i}) = \sum(-1)^k \Lambda^k(E_i) = \Lambda_{-1}(E_i) \in R(G)$, then by some ρ we have

$$\Lambda_{-1}(E_1) = s_1^*(\lambda_{E_1}) = s_0^* \varphi^*(\lambda_{E_1}) = s_0^*(\pi_0^*(\rho)\ \lambda_{E_0}) = \rho\ \Lambda_{-1}(E_0)$$

So in particular $tr_j(\Lambda_{-1}(E_0))$ divides $tr_j(\Lambda_{-1}(E_1))$. By recalling $j : E_i \to E_i$

$$tr_j(\Lambda_{-1}(E_i)) = det(I - j) \quad \text{for } i = 0, 1$$

Since $(z, w)j = (z + jw)j = -\bar{w} + j\bar{z} = (-\bar{w}, \bar{z})$ j acts on $\mathbb{H} \otimes \mathbb{C}$ by matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

so $det(I - A) = 4$, and j acts on $\mathbb{R} \otimes \mathbb{C}$ by $j(x) = -x$ so $det(I - (-I)) = 2$. Hence $4^{k+y}\ 2^x$ divides $4^y\ 2^{l+x}$ which implies $l \geq 2k$ □

There is another nice application of Seiberg-Witten invariants: It is an old problem whether the quotient of a simply connected smooth complex surface by an antiholomorphic involution $\sigma : \tilde{X} \to \tilde{X}$ (an involution which anticommutes with the almost complex homomorphism $\sigma_* J = -J\sigma_*$) is a "standard" manifold (i.e. connected sums of $S^2 \times S^2$ and $\pm\mathbb{CP}^2$). A common example of a antiholomorphic involution is the complex conjugation on a complex projective algebraic surface with real coefficients. It is known that the quotient of $\mathbb{CP}^2$ by complex conjugation is S^4 (Arnold, Massey, Kuiper); and for every d there is a curve of degree d in $\mathbb{CP}^2$ whose two fold branched cover has a standard quotient ([**A**]). This problem makes sense only if the antiholomorphic involution has a fixed point, otherwise the quotient space has fundamental group $\mathbb{Z}_2$ and hence it can not be standard. By "connected sum" theorem, Seiberg-Witten invariants of "standard" manifolds vanish, so it is natural question to ask whether Seiberg-Witten invariants of the quotients vanish. Shugang Wang has shown that this is the case for free antiholomorphic involutions.

Theorem (**S.Wang**) Let $\tilde{X}$ be a minimal Kahler surface of general type, and $\sigma : \tilde{X} \to \tilde{X}$ be a free antiholomorphic involution, then the quotient $X = \tilde{X}/\sigma$ has all Seiberg-Witten invariants zero

Proof: Let h be the Kahler metric on $\tilde{X}$, i.e. $\omega(X,Y) = h(X, JY)$ is the Kahler form. Then $\tilde{g} = h + \sigma^* h$ is an invariant metric on $\tilde{X}$ with the Kahler form $\tilde{\omega} = \omega - \sigma^* \omega$. Let g be the "push-down" metric on X. Now we claim that all $SW_L(X) = 0$ for all $L \to X$, in fact we show that there are no solutions to Seiberg-Witten equations for X: Otherwise if $L \longrightarrow (X, g)$ is the characteristic line bundle supporting a solution (A, ψ), then the pull-back pair $(\tilde{A}, \tilde{\psi})$ is a solution for the pull-back line bundle $\tilde{L} \longrightarrow \tilde{X}$ with the pull-back $Spin_c$ structure, hence

$$0 \leq dim \mathcal{M}_{\tilde{L}}(\tilde{X}) = \frac{1}{4} c_1^2(\tilde{L}) - \frac{1}{4}(3\sigma(\tilde{X}) + 2\chi(\tilde{X}))$$

But $\tilde{X}$ being a minimal Kahler suface of general type $3\sigma(\tilde{X}) + 2\chi(\tilde{X}) = K_{\tilde{X}}^2 > 0$, hence $c_1^2(\tilde{L}) > 0$. This implies that $(\tilde{A}, \tilde{\psi})$ must be an irreducible solution (i.e. $\psi \neq 0$), otherwise $F_{\tilde{A}}^+ = 0$ would imply $c_1^2(\tilde{L}) < 0$. Now by (25) the nonzero solution $\psi = \alpha u_0 + \beta$ must have either one of α or β is zero (so the other one is nonzero), and since $\tilde{\omega} \wedge \tilde{\omega}$ is the volume element:

$$\tilde{\omega}.c_1(\tilde{L}) = \frac{i}{2\pi} \int \tilde{\omega} \wedge F_{\tilde{A}}^+ = \frac{i}{2\pi} \int \tilde{\omega} \wedge (\frac{|\beta|^2 - |\alpha|^2}{2}) \; i \; \tilde{\omega} \neq 0$$

But since $\sigma^*(\tilde{\omega}) = -\tilde{\omega}$, $\sigma^* c_1(\tilde{L}) = c_1(\tilde{L})$, and σ is an orientation preserving map we get a contradiction

$$\tilde{\omega}.c_1(\tilde{L}) = \sigma^*(\tilde{\omega}.c_1(\tilde{L})) = -\tilde{\omega}.c_1(\tilde{L}) \qquad \square$$

References

[A] S. Akbulut,Quotients of Complex Surfaces under complex conjugation, Journal reine angew. Math. 447 (1994), 83-90

[D] S. Donaldson Connections, cohomology and the intersection forms of 4-Manifolds, Jour.Diff.Geom. 24 (1986), 275-341

[DK] S. Donaldson and P. Kronheimer, The Geometry of 4-manifolds, Oxford Math. Monographs (1990)

[F] R. Fintushel, New Directions in 4-Manifold theory

[FS] E. Fintushel ans R. Stern, Immersed spheres in 4-manifolds and The immersed Thom Conjecture, Proceedings of Gokova Geometry-Topology Conference, 1994, and Turkish Journal of Math., vol.19, no.2 (1995), 145-157

[Fu] M. Furuta, (hand written lecture notes from (ICTP)

[KM] P. Kronheimer and T. Mrowka, The genus of the imbedded surfaces in the projective plane, Math. Research Letters,1 (1994), 797-808

[L] B. Lawson, The theory of gauge fields in four dimensions, Conference board of math. sciences regional series in math. no.58 (1983)

[LM] B. Lawson and M.Michelson, Spin geometry, Princeton University Press (1989)

[T] C. Taubes, The Seiberg-Witten invariants and symplectic forms, Math. Research Letters,1 (1994), 809-822

[W] E. Witten, Monopoles and four manifolds, Math. Research Letters,1 (1994), 769-796

[Wa] S. Wang, A Vanishing theorem for Seiberg-Witten invariants, Math. Research Letters, 2 (1995)