

STEIN 4-MANIFOLDS AND CORKS

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ABSTRACT. It is known that every compact Stein 4-manifolds can be embedded into a simply connected, minimal, closed, symplectic 4-manifold. Using this property, we give simple constructions of various cork structures of 4-manifolds. We also give an example of infinitely many disjoint embeddings of a fixed cork into a non-compact 4-manifold which produce infinitely many exotic smooth structures (the authors previously gave examples of arbitrary many disjoint embeddings of different corks in a closed manifold inducing mutually different exotic structures). Furthermore, here we construct arbitrary many simply connected compact codimension zero submanifolds of S^4 which are mutually homeomorphic but not diffeomorphic.

1. INTRODUCTION

It is known that every smooth structure on a simply connected closed smooth 4-manifold is obtained from the given manifold by a cork twist (Matveyev [18], Curtis-Freedman-Hsiang-Stong [11], Akbulut-Matveyev [4]). It is thus important to investigate cork structures of 4-manifolds. The first author [1], [3], Bižaca-Gompf [10], and the authors [6] have found cork structures of (surgered) elliptic surfaces. However, finding cork structures of given exotic pairs of smooth 4-manifolds is usually a quite difficult task.

In [7] (and also [6]), rather than trying to locate corks in exotic manifold pairs we constructed exotic manifold pairs from given corks. Our strategy in [7] was: (1) Construct a “suitable” 4-manifold with boundary which contains candidates of corks; (2) Embed it into a closed 4-manifold with the non-vanishing Seiberg-Witten invariant, “appropriately”; (3) Do surgeries and compute Seiberg-Witten invariants; (4) Relate surgeries and cork twists. Nevertheless, the step (2) is generally not easy.

Eliashberg [12] proved that compact Stein 4-manifolds can be recognized by handlebody pictures, that is, just by checking Thurston-Bennequin framings of its 2-handles (e.g. Gompf-Stipsicz [16], Ozbagci-Stipsicz [19]). Furthermore, compact Stein 4-manifolds satisfy the following useful embedding theorem (though “simply connected” is not claimed in [17] and [5], this is obvious from the proof in [5]):

Theorem 1.1 (Lisca-Matić [17], Akbulut-Ozbagci [5]). *Every compact Stein 4-manifold with boundary can be embedded into a simply connected, minimal, closed, symplectic 4-manifold with $b_2^+ > 1$. Here minimal means that there are no smoothly embedded 2-sphere with the self-intersection number -1 .*

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In this paper, by using this embedding theorem, we give simple constructions of the various cork structures which we previously constructed in [7]. We also construct a new example in the non-compact case. Our new strategy is as follows (the non-closed case is different): (1) Construct a “suitable” compact Stein 4-manifold which contains candidates of corks (possibly after blow ups); (2) Embed it into a minimal closed symplectic 4-manifold; (3) Do surgeries (possibly after blow ups) and compute Seiberg-Witten invariants; (4) Relate surgeries and cork twists. Now, by this approach, the previously difficult step (2) is automatically achieved, though in this case computations of Seiberg-Witten invariants require more careful treatment.

It is a natural question whether every smooth structure on a 4-manifold can be induced from a fixed cork (C, τ) . The following theorem (see Section 3) shows that infinitely many different smooth structures on a closed 4-manifold can be obtained from a fixed cork:

Theorem 1.2 ([7]). *There exist a compact contractible Stein 4-manifold C , an involution τ on the boundary ∂C , and infinitely many simply connected closed smooth 4-manifolds X_n ($n \geq 0$) with the following properties:*

- (1) X_n ($n \geq 0$) are mutually homeomorphic but not diffeomorphic;
- (2) For each $n \geq 1$, X_n is obtained from X_0 by a cork twist along (C, τ) .
Consequently, the pair (C, τ) is a cork of X_0 .

In particular, from X_0 we can produce infinitely many different smooth structures by the cork twist along (C, τ) . Consequently, these embeddings of C into X_0 are mutually non-isotopic (knotted copies of each other).

The next theorem (see Section 6) says that we can put finitely many corks into mutually disjoint positions in closed 4-manifolds so that corresponding cork twists produce mutually different exotic smooth structures on the 4-manifolds:

Theorem 1.3 ([7]). *For each $n \geq 1$, there exist simply connected closed smooth 4-manifolds X_i ($0 \leq i \leq n$) and corks (C_i, τ_i) ($0 \leq i \leq n$) of X_0 with the following properties:*

- (1) The submanifolds C_i ($1 \leq i \leq n$) of X_0 are mutually disjoint;
- (2) X_i ($1 \leq i \leq n$) is obtained from X_0 by the cork twist along (C_i, τ_i) ;
- (3) X_i ($0 \leq i \leq n$) are mutually homeomorphic but not diffeomorphic.

This theorem easily gives the following corollary which says that, for an embedding of a cork into a closed 4-manifold, cork twists can produce finitely many different exotic smooth structures on the 4-manifold by only changing the involution of the cork without changing its embedding:

Corollary 1.4 ([7]). *For each $n \geq 1$, there exist simply connected closed smooth 4-manifolds X_i ($0 \leq i \leq n$), an embedding of a compact contractible Stein 4-manifold C into X_0 , and involutions τ_i ($1 \leq i \leq n$) on the boundary ∂C with the following properties:*

- (1) For each $1 \leq i \leq n$, X_i is obtained from X_0 by the cork twist along (C, τ_i) where the above embedding of C is fixed;
- (2) X_i ($0 \leq i \leq n$) are mutually homeomorphic but not diffeomorphic, hence the pairs (C, τ_i) ($1 \leq i \leq n$) are mutually different corks of X_0 .

In [6], we introduced new objects which we called *plugs*. We can also give simple constructions of various examples for plug structures.

In this paper we also give a new example of cork structures in the non-compact case by using the above embedding theorem. The theorem below (see Section 7) shows that we can embed a fixed cork into infinitely many mutually disjoint positions in a non-compact 4-manifold so that corresponding cork twists produce infinitely many mutually different exotic smooth structures on the 4-manifold:

Theorem 1.5. *There exist infinitely many simply connected non-compact smooth 4-manifolds X_n ($n \geq 0$), a cork (C, τ) and infinitely many embedded copies C_n ($n \geq 1$) of C into X_0 with the following properties:*

- (1) C_n ($n \geq 0$) are mutually disjoint in X_0 ;
- (2) X_n ($n \geq 1$) is obtained from X_0 by the cork twist along (C_n, τ) ;
- (3) X_n ($n \geq 0$) are mutually homeomorphic but not diffeomorphic.

Consequently, these infinitely many disjoint embeddings of C into X_0 are mutually non-isotopic (knotted copies of each other).

The proof of this theorem immediately give the following corollary which says that the smallest 4-manifold S^4 has finitely many compact submanifolds which are mutually homeomorphic but not diffeomorphic. Furthermore they are obtained by a disjointly embedded fixed cork.

Corollary 1.6. *For each $n \geq 1$, there exist simply connected compact smooth 4-manifolds $X_i^{(n)}$ ($0 \leq i \leq n$), a cork (C, τ) and embedded copies C_i ($1 \leq i \leq n$) of C into X_0 with the following properties:*

- (1) C_i ($1 \leq i \leq n$) are mutually disjoint in X_0 ;
- (2) $X_i^{(n)}$ ($1 \leq i \leq n$) is obtained from X_0 by the cork twist along (C_i, τ) ;
- (3) $X_i^{(n)}$ ($0 \leq i \leq n$) can be embedded into S^4 ;
- (4) $X_i^{(n)}$ ($0 \leq i \leq n$) are mutually homeomorphic but not diffeomorphic.

More examples of exotic pairs of (Stein) 4-manifolds will be given in our forthcoming paper.

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2. CORKS

In this section, we recall corks. For details, see [6].

Definition 2.1. Let C be a compact contractible Stein 4-manifold with boundary and $\tau : \partial C \rightarrow \partial C$ an involution on the boundary. We call (C, τ) a *Cork* if τ extends to a self-homeomorphism of C , but cannot extend to any self-diffeomorphism of C . For a cork (C, τ) and a smooth 4-manifold X which contains C , a *cork twist of X along (C, τ)* is defined to be the smooth 4-manifold obtained from X by removing the submanifold C and regluing it via the involution τ . Note that, a cork twist does not change the homeomorphism type of X (see the remark below). A cork (C, τ) is called a cork of X if the cork twist of X along (C, τ) is not diffeomorphic to X .

Remark 2.2. In this paper, we always assume that corks are contractible. (We did not assume this in the more general definition of [6].) Note that Freedman's theorem tells us that every self-diffeomorphism of the boundary of C extends to a self-homeomorphism of C when C is a compact contractible smooth 4-manifold.

Definition 2.3. Let W_n be the contractible smooth 4-manifold shown in Figure 1. Let $f_n : \partial W_n \rightarrow \partial W_n$ be the obvious involution obtained by first surgering $S^1 \times B^3$ to $B^2 \times S^2$ in the interior of W_n , then surgering the other imbedded $B^2 \times S^2$ back to $S^1 \times B^3$ (i.e. replacing the dot and "0" in Figure 1). Note that the diagram of W_n comes from a symmetric link.

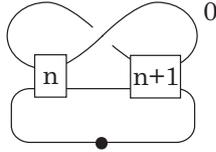


FIGURE 1. W_n

In [6] a quick proof of the following theorem was given, by using the embedding theorem of Stein 4-manifolds.

Theorem 2.4 ([6, Theorem 2.5]). *For $n \geq 1$, the pair (W_n, f_n) is a cork.*

3. INFINITELY MANY KNOTTED CORKS

Here we prove Theorem 1.2 by using the embedding theorem of Stein 4-manifolds.

Definition 3.1. Let S be the compact 4-manifold with boundary in Figure 2. Note that S disjointly contains a cusp neighborhood and W_1 .

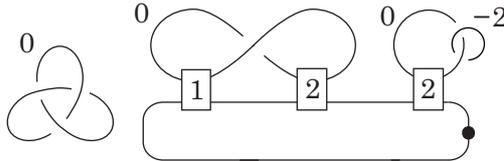


FIGURE 2. S

The embedding theorem of Stein 4-manifold gives the minimal symplectic closed 4-manifold below:

Proposition 3.2. *There exists a simply connected, minimal, closed, symplectic 4-manifold \tilde{S} with the following properties:*

- (1) $b_2^+(\tilde{S}) > 1$;
- (2) \tilde{S} contains the 4-manifold S ;
- (3) A naturally embedded torus in the cusp neighborhood of S represents a non-zero second homology class of \tilde{S} .

Proof. Change the diagram of S into the Legendrian diagram in Figure 3 (in particular change the notation of the 1-handle from the dotted unknot to pair of balls). Here the coefficients of the 2-handles denote the contact framings. Then attach a 2-handle to S along the dotted meridian in Figure 3 with contact -1 framing. Since each framing is contact -1 framing, this new handlebody is also a compact Stein 4-manifold. Hence Theorem 1.1 gives us a simply connected, minimal, closed, symplectic 4-manifold \tilde{S} with $b_2^+(\tilde{S}) > 1$ which contains this handlebody. The 4-manifold \tilde{S} has the property (3), because the torus in the cusp neighborhood algebraically intersects a sphere with the self-intersection number -2 . \square

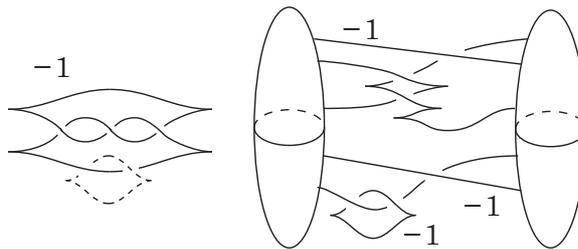


FIGURE 3. Legendrian diagram of S with contact framings

Though the rest of argument in this section is almost the same as [7], we proceed the proof for the completeness.

Definition 3.3.

- (1) Let X be the cork twist of \tilde{S} along (W_1, f_1) , where this copy of W_1 is the one contained in S . Note that X contains a cusp neighborhood because the copy of W_1 in S is disjoint from the cusp neighborhood of S .
- (2) Let K be a knot in S^3 , and X_K denote the manifold obtained by the (Fintushel-Stern's) knot surgery operation with K in the cusp neighborhood of X ([14]).
- (3) Let \tilde{S}_K be the knot surgered \tilde{S} with K in the cusp neighborhood of \tilde{S} .

The corollary below clearly follows from Proposition 3.2, Definition 3.3 and the diagram of S in Figure 2. See also Figure 4.

Corollary 3.4.

- (1) The copy of W_1 in X (given in Definition 3.3.) is disjoint from the cusp neighborhood of X .
- (2) X splits off $S^2 \times S^2$ as a connected summand. Consequently, the Seiberg-Witten invariant of X vanishes. Furthermore, the cusp neighborhood of X is disjoint from $S^2 \times S^2$, in this connected sum decomposition of X .
- (3) \tilde{S}_K is obtained from X_K by a cork twist along (W_1, f_1) . In particular, X_K is homeomorphic to \tilde{S}_K .

Corollary 3.5.

- (1) X_K is diffeomorphic to X . In particular, the Seiberg-Witten invariant of X_K vanishes.
- (2) For each knot K in S^3 , \tilde{S}_K is obtained from X by a cork twist along (W_1, f_1) .
- (3) If K in S^3 has non-trivial Alexander polynomial, then \tilde{S}_K is homeomorphic but not diffeomorphic to X , in particular (W_1, f_1) is a cork of X .

Proof. The claim (1) follows from Corollary 3.4.(2), the definition of X_K and the stabilization theorem of knot surgery by the first author [2] and Auckly [9]. Corollary 3.4.(3) thus shows the claim (2). Since the Seiberg-Witten invariants of \tilde{S}_K does not vanish (Fintushel-Stern [14]), the claim (3) follows from the claim (1). \square

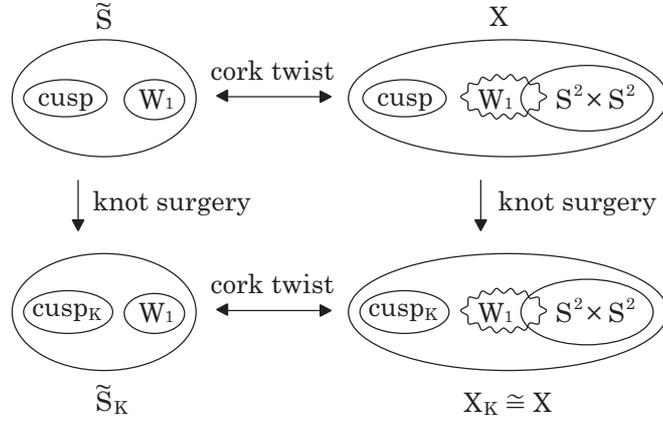


FIGURE 4. relation between \tilde{S} , X , \tilde{S}_K and X_K

Now we can easily prove Theorem 1.2.

Proof of Theorem 1.2. Let $X_0 := X$, and K_n ($n \geq 1$) be knots in S^3 with mutually different non-trivial Alexander polynomials. Define $X_n = \tilde{S}_{K_n}$. Then the claim easily follows from Corollary 3.5 and the Fintushel-Stern's formula ([14]) of the Seiberg-Witten invariant of knot surgered manifolds. \square

Remark 3.6. We proved Theorem 1.2 for the cork (W_1, f_1) . One can similarly prove Theorem 1.2 for many other corks, including (W_n, f_n) .

4. RATIONAL BLOWDOWN

In this section we review the rational blowdown introduced by Fintushel-Stern [13], and recall relations between rational blowdowns and corks.

Let C_p and B_p be the smooth 4-manifolds defined by handlebody diagrams in Figure 5, and u_1, \dots, u_{p-1} elements of $H_2(C_p; \mathbf{Z})$ given by corresponding 2-handles in the figure such that $u_i \cdot u_{i+1} = +1$ ($1 \leq i \leq p-2$). The boundary ∂C_p of C_p is diffeomorphic to the lens space $L(p^2, p-1)$, and also diffeomorphic to the boundary ∂B_p of B_p .

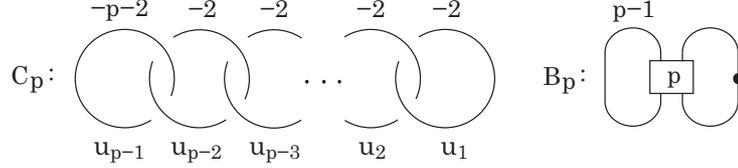


FIGURE 5

Suppose that C_p embeds in a smooth 4-manifold Z . Let $Z_{(p)}$ be the smooth 4-manifold obtained from Z by removing C_p and gluing B_p along the boundary. The smooth 4-manifold $Z_{(p)}$ is called the rational blowdown of Z along C_p . Note that $Z_{(p)}$ is uniquely determined up to diffeomorphism by a fixed pair (Z, C_p) (see Fintushel-Stern [13]). This operation preserves b_2^+ , decreases b_2^- , may create torsion in the first homology group.

Rational blowdowns have the following relations with corks.

Theorem 4.1 ([7], see also [6]). *Let D_p be the smooth 4-manifold in Figure 6 (notice that D_p is C_p with two 2-handles attached). Suppose that a smooth 4-manifold Z contains D_p . Let $Z_{(p)}$ be the rational blowdown of Z along the copy of C_p contained in D_p . Then the submanifold D_p of Z contains W_{p-1} such that $Z_{(p)} \# (p-1)\overline{\mathbf{CP}}^2$ is obtained from Z by the cork twist along (W_{p-1}, f_{p-1}) .*

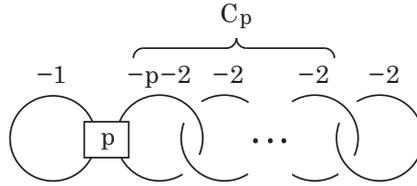


FIGURE 6. D_p

5. SEIBERG-WITTEN INVARIANTS

In this section, we briefly review basic facts about the Seiberg-Witten invariants. For more details, see, for example, Fintushel-Stern [15], Gompf-Stipsicz [16], Ozbagci-Stipsicz [19].

Suppose that Z is a simply connected closed smooth 4-manifold with $b_2^+(Z) > 1$ and odd. Let $\mathcal{C}(Z)$ be the set of characteristic elements of $H^2(Z; \mathbf{Z})$. Then the Seiberg-Witten invariant $SW_Z : \mathcal{C}(Z) \rightarrow \mathbf{Z}$ is defined. Let $e(Z)$ and $\sigma(Z)$ be the Euler characteristic and the signature of Z , respectively, and $d_Z(K)$ the even integer defined by $d_Z(K) = \frac{1}{4}(K^2 - 2e(Z) - 3\sigma(Z))$ for $K \in \mathcal{C}(Z)$. If $SW_Z(K) \neq 0$, then K is called a Seiberg-Witten basic class of Z . It is known that if K is a Seiberg-Witten basic class of Z , then $-K$ is also a Seiberg-Witten basic class of Z . We denote $\beta(Z)$ as the set of the Seiberg-Witten basic classes of Z . The blow up formula is as follows:

Theorem 5.1 (Witten [22], cf. Gompf-Stipsicz [16]). *Suppose that Z is a simply connected closed smooth 4-manifold with $b_2^+(Z) > 1$. If $\beta(Z)$ is not empty, then*

$$\beta(Z\#\overline{n\mathbf{CP}^2}) = \{K \pm E_1 \pm E_2 \pm \cdots \pm E_n \mid K \in \beta(Z)\}.$$

Here E_1, E_2, \dots, E_n denotes the standard orthogonal basis of $H^2(n\overline{\mathbf{CP}^2}; \mathbf{Z})$ such that $E_i^2 = -1$ ($1 \leq i \leq n$).

If every Seiberg-Witten basic class K of Z satisfies $d_Z(K) = 0$, then the 4-manifold Z is called of simple type. For example, it is known that every closed symplectic 4-manifold with $b_2^+ > 1$ has non-vanishing Seiberg-Witten invariant and is of simple type. For such a 4-manifold, the following adjunction inequality holds:

Theorem 5.2 (Ozsváth-Szabó [20], cf. Ozbagci-Stipsicz [19]). *Suppose that Z is a simply connected closed smooth 4-manifold of simple type with $b_2^+(Z) > 1$, and that $\Sigma \subset Z$ is a smoothly embedded, oriented, connected closed surface of genus $g > 0$. Let $[\Sigma]$ be the second homology class of Z represented by the embedded surface Σ . Then, for every Seiberg-Witten basic class K of X , the following adjunction inequality holds:*

$$[\Sigma]^2 + |\langle K, [\Sigma] \rangle| \leq 2g - 2.$$

5.1. Seiberg-Witten invariants of rational blowdowns. We here recall the change of the Seiberg-Witten invariants by rationally blowing down. Let Z be a simply connected closed smooth 4-manifold with $b_2^+(Z) > 1$ and odd. Suppose that Z contains a copy of C_p . Let $Z_{(p)}$ be the rational blowdown of Z along the copy of C_p . Assume that $Z_{(p)}$ is simply connected. The following theorems are obtained by Fintushel-Stern [13].

Theorem 5.3 (Fintushel-Stern [13]). *For every element K of $\mathcal{C}(Z_{(p)})$, there exists an element \tilde{K} of $\mathcal{C}(Z)$ such that $K|_{Z_{(p)}-B_p} = \tilde{K}|_{Z-C_p}$ and $d_{Z_{(p)}}(K) = d_Z(\tilde{K})$. Such an element \tilde{K} of $\mathcal{C}(Z)$ is called a lift of K .*

Theorem 5.4 (Fintushel-Stern [13]). *If an element \tilde{K} of $\mathcal{C}(Z)$ is a lift of some element K of $\mathcal{C}(Z_{(p)})$, then $SW_{Z_{(p)}}(K) = SW_Z(\tilde{K})$.*

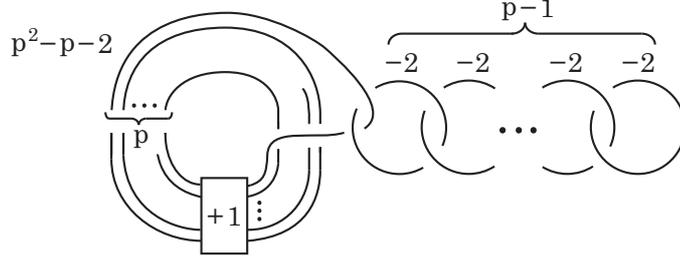
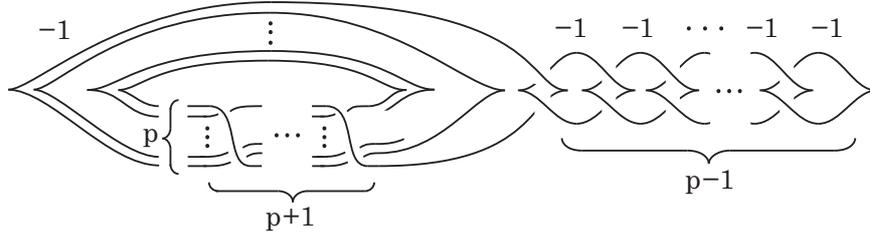
Theorem 5.5 (Fintushel-Stern [13], cf. Park [21]). *If an element \tilde{K} of $\mathcal{C}(Z)$ satisfies that $(\tilde{K}|_{C_p})^2 = 1 - p$ and $\tilde{K}|_{\partial C_p} = mp \in \mathbf{Z}_{p^2} \cong H^2(\partial C_p; \mathbf{Z})$ with $m \equiv p - 1 \pmod{2}$, then there exists an element K of $\mathcal{C}(Z_{(p)})$ such that \tilde{K} is a lift of K .*

Corollary 5.6. *If an element \tilde{K} of $\mathcal{C}(Z)$ satisfies $\tilde{K}(u_1) = \tilde{K}(u_2) = \cdots = \tilde{K}(u_{p-2}) = 0$ and $\tilde{K}(u_{p-1}) = \pm p$, then \tilde{K} is a lift of some element K of $\mathcal{C}(Z_{(p)})$.*

6. DISJOINTLY EMBEDDED CORKS

In this section we prove Theorem 1.3 and Corollary 1.4 by using the embedding theorem of Stein 4-manifolds.

Definition 6.1. Let \tilde{D}_p ($p \geq 2$) be the compact 4-manifold with boundary in Figure 7 (i.e. the blow down of D_p). Note that the left most knot in the figure is a $(p+1, p)$ torus knot. Define $\tilde{D}(p_1, p_2, \dots, p_n)$ as the boundary sum of $\tilde{D}_{p_1}, \tilde{D}_{p_2}, \dots, \tilde{D}_{p_n}$. We denote by $D(p_1, p_2, \dots, p_n)$ the boundary sum of $D_{p_1}, D_{p_2}, \dots, D_{p_n}$.

FIGURE 7. \tilde{D}_p FIGURE 8. Legendrian diagram of \tilde{D}_p with contact framings

Proposition 6.2. *For each $n \geq 1$ and each $p_1, p_2, \dots, p_n \geq 2$, there exists a simply connected, minimal, closed, symplectic 4-manifold $S(p_1, p_2, \dots, p_n)$ with the following properties:*

- (1) $b_2^+(S(p_1, p_2, \dots, p_n)) > 1$;
- (2) *The 4-manifold $S(p_1, p_2, \dots, p_n)$ contains $\tilde{D}(p_1, p_2, \dots, p_n)$;*
- (3) *Every Seiberg-Witten basic class K of $S(p_1, p_2, \dots, p_n)$ satisfies $\langle K, \alpha \rangle = 0$ for all $\alpha \in \iota_* H_2(\tilde{D}(p_1, p_2, \dots, p_n); \mathbf{Z})$. Here ι_* denotes the homomorphism induced by the inclusion $\iota : \tilde{D}(p_1, p_2, \dots, p_n) \hookrightarrow S(p_1, p_2, \dots, p_n)$.*

Proof. (1) and (2). Change the diagram of $\tilde{D}(p_1, p_2, \dots, p_n)$ into the Legendrian diagram as in Figure 8. Then, for each unknot, attach a 2-handle along a contact -1 -framed trefoil knot, as in Figure 9. Since every 2-handle has contact -1 -framings, this new handlebody is a compact Stein 4-manifold. The embedding theorem of Stein 4-manifolds thus gives a simply connected, minimal, closed, symplectic 4-manifold $S(p_1, p_2, \dots, p_n)$ with $b_2^+(S(p_1, p_2, \dots, p_n)) > 1$ which contains this handlebody. The claims (1) and (2) hence follows.

(3). Let u (resp. v) be the element of $H_2(S(p_1, p_2, \dots, p_n); \mathbf{Z})$ given by a contact -1 -framed unknot (resp. a contact -1 -framed trefoil knot), as in Figure 9. We can easily check, by a handle slide, that $u + v$ is represented by a torus with the self-intersection number 0. Let K be a Seiberg-Witten basic class of $S(p_1, p_2, \dots, p_n)$. Since v and $u + v$ are represented by tori with self-intersection numbers 0, the adjunction inequality gives $\langle K, v \rangle = 0$ and $\langle K, u + v \rangle = 0$. We thus have $\langle K, u \rangle = 0$. Let w_i be the element of $H_2(S(p_1, p_2, \dots, p_n); \mathbf{Z})$ given by the contact -1 -framed $(p_i + 1, p_i)$ torus knot of \tilde{D}_{p_i} . Note that w_i is represented by a genus $\frac{p_i(p_i-1)}{2}$ surface with the self-intersection number $p_i^2 - p_i - 2$. We can also easily check, by

the adjunction inequality, that $\langle K, w_i \rangle = 0$ ($1 \leq i \leq n$). Now (3) follows from the fact that $\iota_* H_2(\tilde{D}(p_1, p_2, \dots, p_n); \mathbf{Z})$ is generated by these classes u and w_i . \square

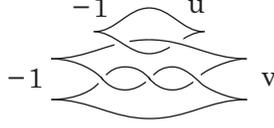


FIGURE 9

Let e_1, e_2, \dots, e_n be the standard basis of $H_2(n\overline{\mathbf{CP}}^2; \mathbf{Z})$ such that $e_i^2 = -1$ ($1 \leq i \leq n$) and $e_i \cdot e_j = 0$ ($i \neq j$). Then the proposition above together with the blow up formula immediately gives the following corollary.

Corollary 6.3. (1) For each $p_1, p_2, \dots, p_n \geq 2$, the 4-manifold $S(p_1, p_2, \dots, p_n) \# n\overline{\mathbf{CP}}^2$ contains $D(p_1, p_2, \dots, p_n)$ as in Figure 10. Here, $u_j^{(i)}$ ($1 \leq i \leq n$, $0 \leq j \leq p_i - 1$) in the figure denotes the second homology class given by corresponding 2-handle, and e_i ($1 \leq i \leq n$) represents the homology class given by the corresponding 2-handle. We orient e_i and $u_j^{(i)}$ so that $e_i \cdot u_{p_i-1}^{(i)} = p_i$ and $u_j^{(i)} \cdot u_{j+1}^{(i)} = 1$ ($1 \leq i \leq n$, $0 \leq j \leq p_i - 2$). (2) Every Seiberg-Witten basic class K of $S(p_1, p_2, \dots, p_n) \# n\overline{\mathbf{CP}}^2$ satisfies $\langle K, u_{p_i-1}^{(i)} \rangle = \langle K, -p_i e_i \rangle = \pm p_i$ and $\langle K, u_j^{(i)} \rangle = 0$ ($1 \leq i \leq n$, $0 \leq j \leq p_i - 2$).

Definition 6.4. (1) Define $X_0 := S(p_1, p_2, \dots, p_n) \# n\overline{\mathbf{CP}}^2$. Let X'_i ($1 \leq i \leq n$) be the rational blowdown of X_0 along the copy of C_{p_i} in Figure 10. Put $X_i := X'_i \# (p_i - 1)\overline{\mathbf{CP}}^2$.

(2) For $k_1, k_2, \dots, k_n \geq 1$, let $W(k_1, k_2, \dots, k_n)$ be the boundary sum $W_{k_1} \natural W_{k_2} \natural \dots \natural W_{k_n}$. Figure 11 is a diagram of $W(k_1, k_2, \dots, k_n)$. Let $f^i(k_1, k_2, \dots, k_n)$ be the involution on the boundary $\partial W(k_1, k_2, \dots, k_n)$ obtained by replacing the dot and zero of the component of W_{k_i} .

One can easily prove the lemma below by checking Thurston-Bennequin numbers of 2-handles. For a proof, see [7].

Lemma 6.5 ([7]). For each $k_1, k_2, \dots, k_n \geq 1$, the manifold $W(k_1, k_2, \dots, k_n)$ is a compact contractible Stein 4-manifold.

Proposition 6.6. (1) The 4-manifold X_0 contains mutually disjoint copies of $W_{p_1-1}, W_{p_2-1}, \dots, W_{p_n-1}$ such that, for each i , the 4-manifold X_i is obtained from X_0 by the cork twist along (W_{p_i-1}, f_{p_i-1}) .

(2) The 4-manifold X_0 contains a fixed copy of $W(p_1 - 1, p_2 - 1, \dots, p_n - 1)$ such that, for each i , the 4-manifold X_i is obtained from X_0 by the cork twist along $(W(p_1 - 1, p_2 - 1, \dots, p_n - 1), f^i(p_1 - 1, p_2 - 1, \dots, p_n - 1))$.

Proof. Corollary 6.3 and Theorem 4.1 clearly show the claims (1) and (2). \square

Remark 6.7. We here correct a misprint in [7]. Proposition 5.4.(1) of [7] should be changed as in Proposition 6.6.(1) of this paper. However, the claim itself of Proposition 5.4.(1) of [7] is correct, because we can easily replace (W_{p-1}, f_{p-1}) with (W_p, f_p) in Theorem 4.1 of this paper. See the proof of the theorem given in [7].

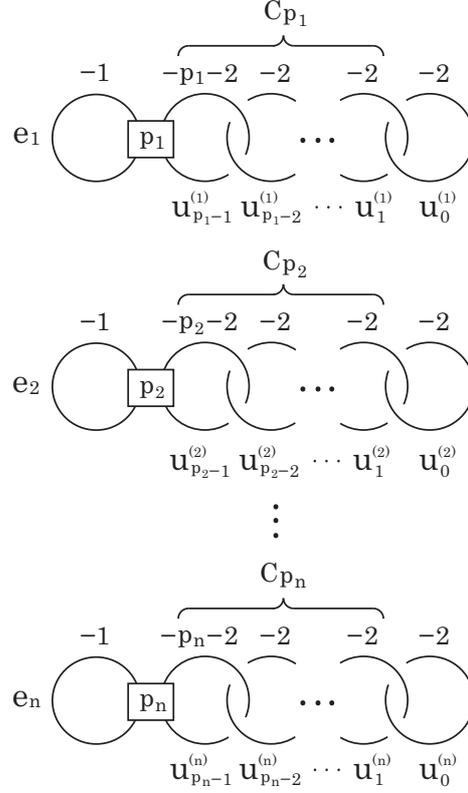


FIGURE 10. the submanifold $D(p_1, p_2, \dots, p_n)$ of $S(p_1, p_2, \dots, p_n) \# n\overline{\mathbf{CP}^2}$

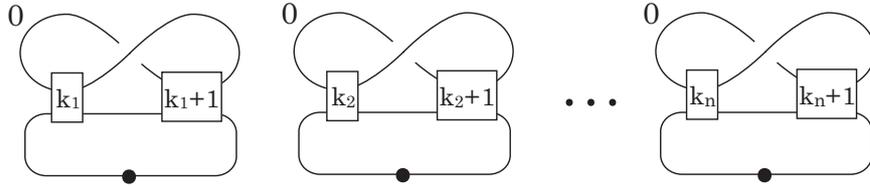


FIGURE 11. $W(k_1, k_2, \dots, k_n)$

6.1. Computation of SW invariants. In this subsection, we complete the proofs of Theorem 1.3 and Corollary 1.4 by computing the Seiberg-Witten invariants of the 4-manifolds X_i ($0 \leq i \leq n$) in Definition 6.4.

Lemma 6.8. *Fix an integer $i \in \{1, 2, \dots, n\}$. If Seiberg-Witten basic classes K and K' of X_0 satisfy $K \neq K'$, then restrictions $K|_{X_0 - C_{p_i}}$ and $K'|_{X_0 - C_{p_i}}$ are not equal to each other.*

Proof. Define an element α of $H_2(X_0; \mathbf{Z})$ by

$$\alpha = e_i + u_{p_i-1}^{(i)} + 2u_{p_i-2}^{(i)} + 3u_{p_i-3}^{(i)} + \dots + (p_i - 1)u_1^{(i)} + p_i u_0^{(i)}.$$

We get $\langle K, \alpha \rangle = \langle K, (1 - p_i)e_i \rangle$ and $\langle K', \alpha \rangle = \langle K', (1 - p_i)e_i \rangle$, because Corollary 6.3 gives $\langle K, u_{p_i-1}^{(i)} \rangle = \langle K, -p_i e_i \rangle$, $\langle K', u_{p_i-1}^{(i)} \rangle = \langle K', -p_i e_i \rangle$ and $\langle K, u_j^{(i)} \rangle = \langle K', u_j^{(i)} \rangle = 0$ ($0 \leq j \leq p_i - 2$). Corollary 6.3 implies

$$\alpha \cdot u_1^{(i)} = \alpha \cdot u_2^{(i)} = \cdots = \alpha \cdot u_{p_i-1}^{(i)} = 0.$$

Since $u_0^{(i)}$ satisfies

$$u_0^{(i)} \cdot u_1^{(i)} = 1 \text{ and } u_0^{(i)} \cdot u_2^{(i)} = u_0^{(i)} \cdot u_3^{(i)} = \cdots = u_0^{(i)} \cdot u_{p_i-1}^{(i)} = 0,$$

Lemma 5.1 of [23] shows that α is an element of $\iota_* H_2(X_0 - C_{p_i}; \mathbf{Z})$, where ι_* is the homomorphism induced by the inclusion $\iota : X_0 - C_{p_i} \hookrightarrow X_0$. Lemma 5.1 of [23] also gives $H_1(X_0 - C_{p_i}; \mathbf{Z}) = 0$. Thus Mayer-Vietoris exact sequence for $C_{p_i} \cup (X_0 - C_{p_i}) = X_0$ is as follows:

$$0 \rightarrow H_2(C_{p_i}; \mathbf{Z}) \oplus H_2(X_0 - C_{p_i}; \mathbf{Z}) \rightarrow H_2(X_0; \mathbf{Z}) \rightarrow \mathbf{Z}_{p_i^2} \rightarrow 0.$$

The case where $K|_{C_{p_i}} = K'|_{C_{p_i}}$: In this case, the exact sequence above together with the universal coefficient theorem for X_0 implies $K|_{X_0 - C_{p_i}} \neq K'|_{X_0 - C_{p_i}}$.

The case where $K|_{C_{p_i}} \neq K'|_{C_{p_i}}$: In this case, we have $\langle K, u_{p_i-1}^{(i)} \rangle \neq \langle K', u_{p_i-1}^{(i)} \rangle$, because $\langle K, u_j^{(i)} \rangle = \langle K', u_j^{(i)} \rangle = 0$ for $0 \leq j \leq p_i - 2$. We thus get $\langle K, -p_i e_i \rangle \neq \langle K', -p_i e_i \rangle$. This fact immediately gives $\langle K, \alpha \rangle \neq \langle K', \alpha \rangle$ and hence $K|_{X_0 - C_{p_i}} \neq K'|_{X_0 - C_{p_i}}$. \square

Though the rest of the proof is the same as that of Theorem 1.3 and 1.4 in [7], we proceed for the completeness. For a smooth 4-manifold Z we denote $N(Z)$ as the number of elements of $\beta(Z)$.

Lemma 6.9. $N(X_i) = 2^{p_i-1} N(X_0)$ ($1 \leq i \leq n$)

Proof. Corollary 6.3, Theorem 5.1 and Corollary 5.6 guarantees that every Seiberg-Witten basic class of X_0 is a lift of some element of $\mathcal{C}(X'_i)$. Lemma 6.8 shows that these basic classes of X_0 have mutually different restrictions to $X_0 - C_{p_i} (= X'_i - B_{p_i})$. Note that every element of $H^2(X'_i; \mathbf{Z})$ is uniquely determined by its restriction to $X'_i - B_{p_i}$. (We can easily check this by using the cohomology exact sequence for the pair $(X'_i, X'_i - B_{p_i})$.) Hence Theorems 5.3 and 5.4 give $N(X'_i) = N(X_0)$. Now the required claim follows from the blow-up formula. \square

Corollary 6.10. *If $p_1, p_2, \dots, p_n \geq 2$ are mutually different, then X_i ($0 \leq i \leq n$) are mutually homeomorphic but not diffeomorphic.*

Proof of Theorem 1.3 and Corollary 1.4. These clearly follow from the corollary above and Proposition 6.6. \square

Remark 6.11. If we appropriately choose p_1, p_2, \dots, p_n , then we can show that the natural combinations of cork twists of X_0 produce $2^n - 1$ distinct smooth structures. In fact, similarly to Lemma 6.9, we can show that the number of Seiberg-Witten basic classes of the combinations of cork twists of X_0 along $(W_{p_{i_1}-1}, f_{p_{i_1}-1})$, $(W_{p_{i_2}-1}, f_{p_{i_2}-1})$, \dots , $(W_{p_{i_k}-1}, f_{p_{i_k}-1})$ is $2^{p_{i_1}+p_{i_2}+\cdots+p_{i_k}-k} N(X_0)$.

7. INFINITELY MANY DISJOINTLY EMBEDDED KNOTTED CORK

Here we prove Theorem 1.5 and Corollary 1.6. In the previous sections, we used the embedding theorem of Stein 4-manifolds to construct examples. However, in this section, we use the embedding theorem to detect smooth structures (i.e. to evaluate the minimal genera of surfaces which represent second homology classes).

Definition 7.1. (1) Let M_n and N_n ($n \geq 2$) be the compact smooth 4-manifolds with boundary given in Figure 12. Note that N_n is the cork twist of M_n along (W_1, f_1) .

(2) Let X_0 be the simply connected non-compact smooth 4-manifold given by attaching infinitely many handles to the boundary of $D^3 \times [0, \infty)$ as shown in Figure 13, where the $[0, \infty)$ component is horizontal line, and $\partial D^3 = \mathbf{R}^2 \cup \{\text{one point}\}$. In other words, X_0 is the infinite boundary sum of $\{M_i \mid i \geq 2\}$. Let X_n be the cork twist of X_0 along (W_1, f_1) , where W_1 is the one contained in the M_{n+1} component.

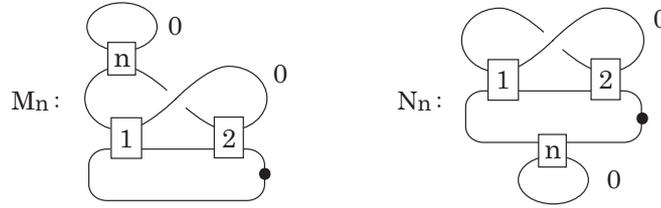


FIGURE 12

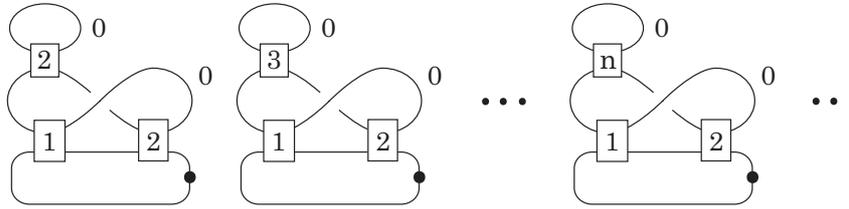


FIGURE 13. X_0

Lemma 7.2. (1) The generator α of $H_2(N_n; \mathbf{Z}) \cong \mathbf{Z}$ is represented by a smoothly embedded genus n surface.

(2) For $n \geq 1$, the 4-manifolds M_n and N_n can be embedded into the 4-ball D^4 .

Proof. (1) The lower 2-handle (call K) of N_n gives the generator α after sliding over the upper 2-handle n times. Thus all we have to check is to see that the knot K bounds a genus n surface in the interior of W_1 after sliding n times. See Figure 14. Introduce a 1-handle/2-handle pair and slide the upper 0-framed unknot twice, then we get the second picture. An isotopy gives the third picture. We slide the knot K over the 0-framed unknot n -times so that K does not link with the lower dotted circle. We get the fourth picture by ignoring two 2-handles and isotopy. We can now easily see that K bounds a genus n surface by the standard argument (cf.

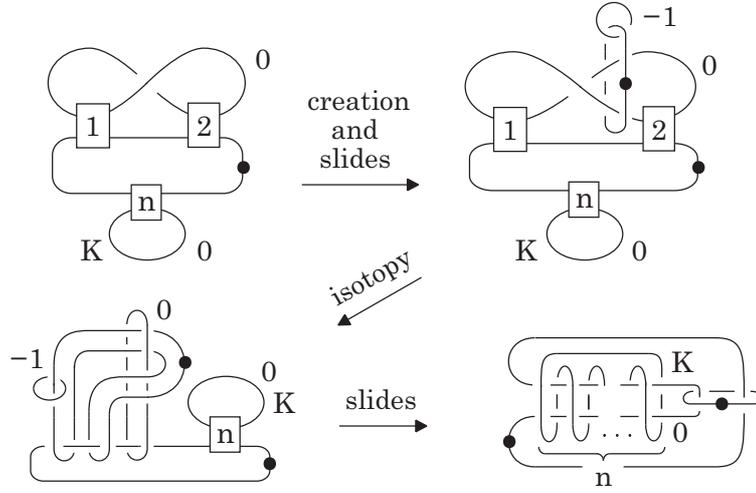


FIGURE 14

Gompf-Stipsicz [16, Exercise.4.5.12.(b)]). (Check that K is the boundary of D^2 with $2n$ bands attached.)

(2) Attach a 2-handle to the boundary of M_n as in the first picture of Figure 15. The second picture is given by sliding the upper left 2-handle over this new 2-handle n -times. Slide the middle 2-handle over its meridian as in the third diagram. Note that the middle 2-handle now links with the 1-handle geometrically once. Cancelling the 1-handle gives the last diagram. Attach two 3-handles cancelling these two 2-handles. So get an embedding of M_n into D^4 . The N_n case is similar, Figure 16. \square

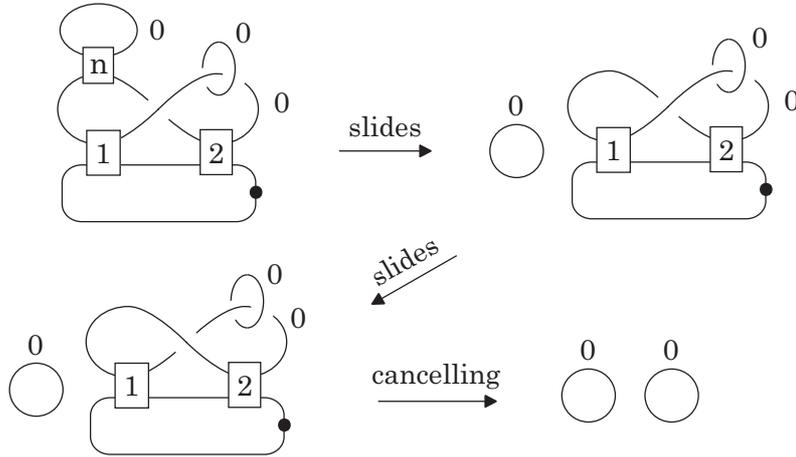


FIGURE 15

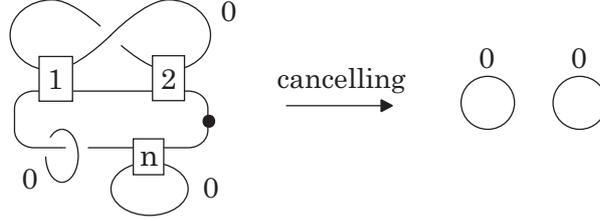


FIGURE 16

By using the embedding theorem of Stein 4-manifolds, we show the following.

Lemma 7.3. *Let α be the generator of $H_2(N_n; \mathbf{Z}) \cong \mathbf{Z}$. If $k\alpha$ is represented by a smoothly embedded surface with genus less than n , then $k = 0$.*

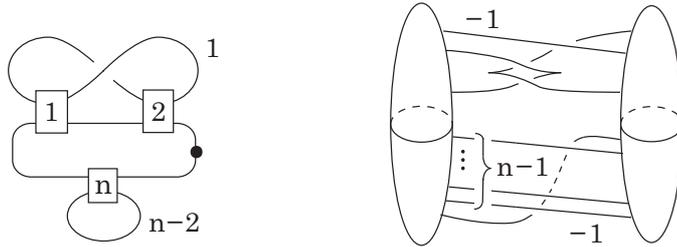
Proof. Let \tilde{N}_n be the compact smooth 4-manifold given in the left side of Figure 17. The right side is a Legendrian diagram of \tilde{N}_n with contact -1 -framings. Thus \tilde{N}_n is a compact Stein 4-manifold. Therefore there exists a simply connected minimal closed symplectic 4-manifold S_n which contains \tilde{N}_n .

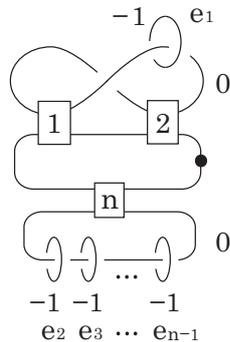
Let e_1, e_2, \dots, e_{n-1} be the standard orthogonal basis of $H_2((n-1)\overline{\mathbf{CP}^2}; \mathbf{Z})$ such that $e_i^2 = -1$ ($1 \leq i \leq n-1$). Blow up S_n as in Figure 18, where e_1, e_2, \dots, e_{n-1} denote the second homology classes given by corresponding 2-handles. Note that this picture contains N_n . The lower 0-framed unknot in this picture gives the generator α of $H_2(N_n; \mathbf{Z})$ after sliding over the upper 0-framed unknot n times. We thus get $\alpha \cdot e_1 = n$ and $\alpha \cdot e_i = 1$ ($2 \leq i \leq n-1$), where we view α as the element of $H_2(S_n \# (n-1)\overline{\mathbf{CP}^2})$ through the natural inclusion.

Let K be a Seiberg-Witten basic class of S_n . Then the blow up formula shows that $\pm K + e_1 + e_2 + \dots + e_{n-1}$ is a Seiberg-Witten basic class of $S_n \# (n-1)\overline{\mathbf{CP}^2}$. Therefore, there exists a Seiberg-Witten basic class L of $S_n \# (n-1)\overline{\mathbf{CP}^2}$ which satisfies the inequality below:

$$|\langle L, k\alpha \rangle| \geq |k(n + (n-2))| = |k|(2n-2).$$

Since $(k\alpha)^2 = 0$ and $n \geq 2$, we can now easily check the required claim by applying the adjunction inequality to $S_n \# (n-1)\overline{\mathbf{CP}^2}$. \square


 FIGURE 17. \tilde{N}_n and its legendrian diagram with contact framings

FIGURE 18. The subhandlebody of $S_n \# (n-1) \overline{\mathbf{CP}^2}$

Proposition 7.4. (1) For each $n \geq 1$, there exists a basis $\{\alpha, \beta_1, \beta_2, \dots\}$ of $H_2(X_n; \mathbf{Z})$ such that α is represented by a smoothly embedded surface with genus $n+1$, and that β_i ($i \geq 1$) is represented by a smoothly embedded sphere.

(2) For any $n \geq 1$, there exists no basis $\{\alpha, \beta_1, \beta_2, \dots\}$ of $H_2(X_n; \mathbf{Z})$ such that α is represented by a surface with genus less than $n+1$, and that β_i ($i \geq 1$) is represented by a smoothly embedded sphere.

Proof. (1) is obvious from Lemma 7.2.

(2). Let u_i ($i \geq 2$, $i \neq n+1$) and u_{n+1} be the basis of $H_2(M_i; \mathbf{Z})$ and $H_2(N_{n+1}; \mathbf{Z})$, respectively. Suppose that an element $v = \sum_{i=2}^{\infty} a_i u_i$ of $H_2(X_n; \mathbf{Z})$ is represented by a surface with genus less than $n+1$. Here $a_i = 0$ except finite number of i . Lemma 7.2 shows that X_n is embedded into N_{n+1} and that this embedding sends v to $a_{n+1} u_{n+1}$. Since $a_{n+1} u_{n+1} \in H_2(N_{n+1}; \mathbf{Z})$ is represented by a surface with genus less than $n+1$, Lemma 7.3 gives $a_{n+1} = 0$. This fact implies the required claim. \square

Proof of Theorem 1.5. The definition of X_n obviously shows (1), (2). The claim (3) follows from Proposition 7.4. \square

Remark 7.5. Lemma 7.2.(2) shows that X_n ($n \geq 0$) can be embedded into S^4 .

Definition 7.6. Let $X_0^{(n)}$ ($n \geq 1$) be the boundary sum of M_2, M_3, \dots, M_{n+1} . Define $X_i^{(n)}$ ($1 \leq i \leq n$) as the cork twist of $X_0^{(n)}$ along (W_1, f_1) where this W_1 is the one contained in the M_{i+1} component.

Proof of Corollary 1.6. This is almost the same as the proof of Theorem 1.5. \square

8. FURTHER REMARKS

In this section, we conclude this paper by making some remarks.

In [6], we introduced new objects which we call plugs. By using the embedding theorem of Stein 4-manifolds, we can easily construct examples of plug structures corresponding to Theorem 1.2, 1.3 and Corollary 1.4. The proofs are almost the same as that of cork structures. See also [7].

In Section 7, we used the embedding theorem to detect smooth structures. This technique is useful for constructions of exotic smooth structures on compact 4-manifolds with boundary. For details, see our forthcoming paper [8].

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