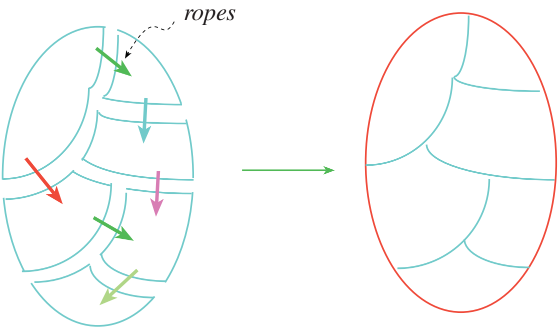
A survey on exotic smooth structures on 4-Manifolds

Selman Akbulut

April 12, 2017

Assembling 4-manifolds by “ropes”

Glue standard manifolds with boundary with ropes:



PIECES: Bundles, branched covers, sugeries, log transforms..etc

Tools to construct exotic 4-manifolds

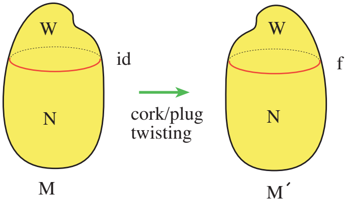
Large Alteration tools:

- Log-transforms (Kodaira, Donaldson)
- Knot-Surgery, Rational blow-downs (Fint-Stern, Meng-Taubes)
- Gluck twisting (Gluck)

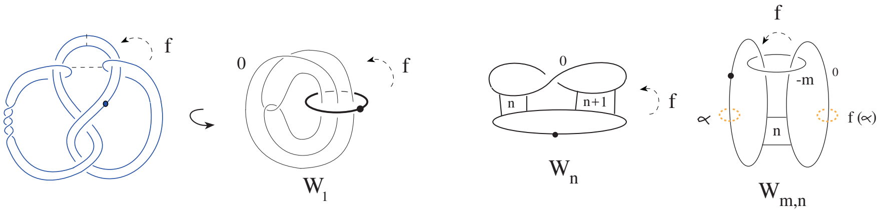
Gauge theory tools to distinguish smooth structures:

- Symplectic manifold (X,ω) with $b_+(X) > 1$ has $SW_X \neq 0$ (Taubes)
- Adjunction: $2g - 2 \geq \Sigma.\Sigma + |K_S.\Sigma|$ (Kron-Mrowka, Mor-Szabo-T)
- Stein manifolds W can be constructed by handles. (Eliashberg)
- Stein compactifies to symplectic $W \subset (M,\omega)$ (Lisc-Matt, Ozb-A)

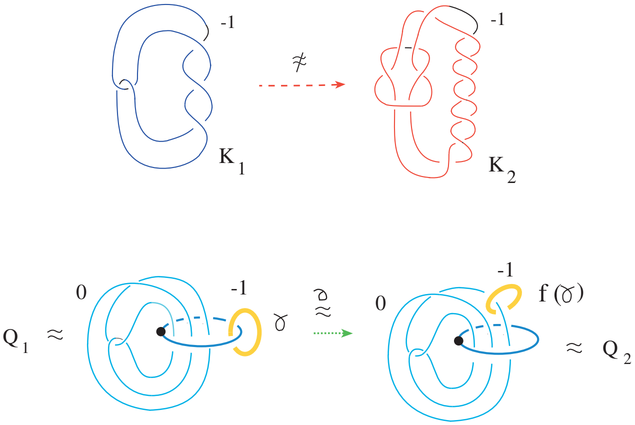
Atomic pieces: Corks, plugs, and anti-corks



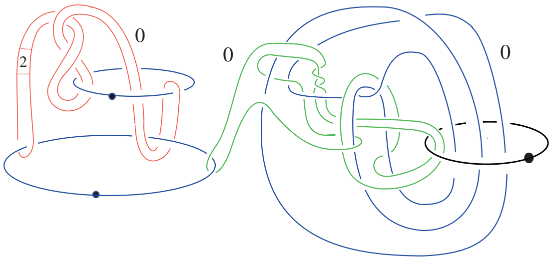
- Cork ≃ point (A, Mat. and Curt-Fr-Hsiang-Stong) (e.g. W_n below)
- Plug ≃ S^2 (A-Yasui). It is usually $B^4 \cup 2-$handle. (e.g. $W_{n.m}$ below)
- Anticork ≃ S^1



Cork ⇒ An absolutely exotic 4-manifold



Cork ⇒ An absolutely exotic contractible 4-manifold (A-Ruberman)



Plug $\Rightarrow$ Exotic Dolgachev surface

- Plug $W_{m,n} = B^4 + K^r$, $K \subset S^3$, where $r = -2n - n^2m^2$, and K is some torus knot. So upside plug looks like a 2-handle attached to different place at top.

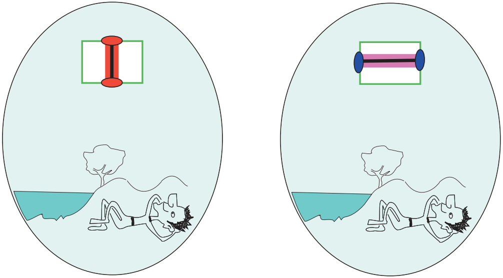
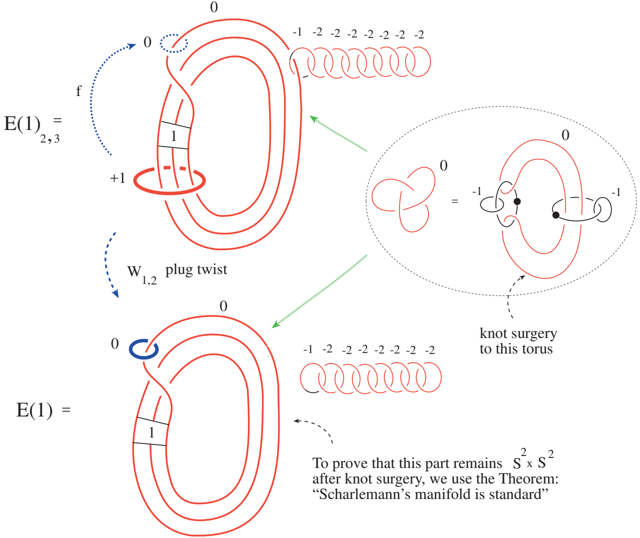


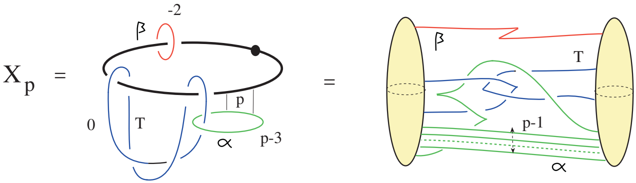
Figure:


E(1)_{2,3} =
f
0
0
+1
1
-1 -2 -2 -2 -2 -2 -2 -2
W_{1,2} plug twist
E(1) =
0
0
1
-1 -2 -2 -2 -2 -2 -2 -2
0
=
-1
0
-1
knot surgery
to this torus
To prove that this part remains S^2 x S^2
after knot surgery, we use the Theorem:
“Scharlemann’s manifold is standard”

Adjunction inequality ⇒ Absolutely exotic Stein mnflds

An example:

Theorem (A-Yasui) ∃ *infinitely many* 1*-connected Stein manifolds (of Betti number 2), which are exotic copies of each other. They are Stein fillings of the same contact* 3*-manifold*.

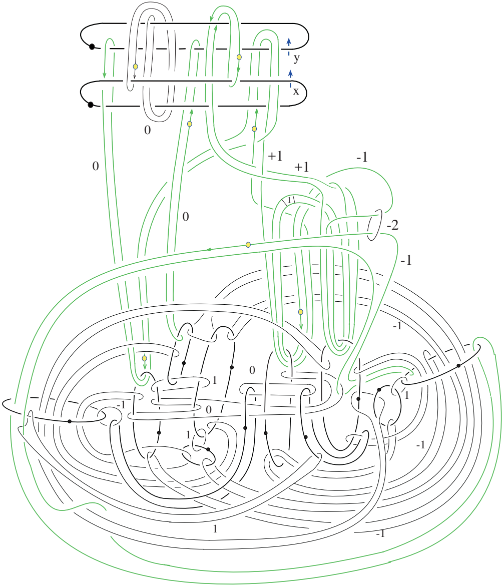


proof: Apply adjunction to $\Sigma = \alpha - p\beta + qT$, where $q = p^2 - (p-3)/2$

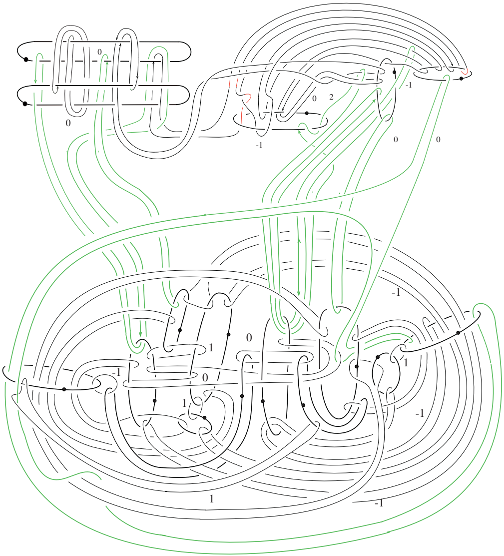
$$2g(\Sigma) - 2 \geq \Sigma.\Sigma + |K_S.\Sigma| = 0 + |p - 1|$$

$$\Rightarrow g(\Sigma) \to \infty \ \text{ as } \ p \to \infty$$

An exotic copy of $CP^2 \# -3\overline{CP^2}$ (A-P, and F-S, B-K)



An exotic copy of $CP^2\# -2CP^2$ (A-P)

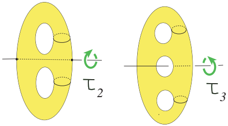


Example: Constructing a complex surface by ropes

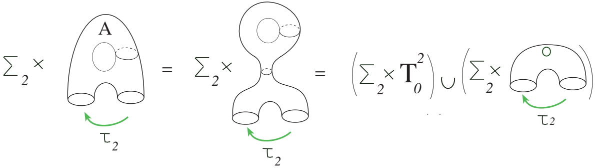
CaCiMe (Catanese-Ciliberto-Mendes-Lopes) is a complex surface (a particular Σ_2-bundle over Σ_2, where Σ_g is a genus g surface):

$$M = (\Sigma_2 \times \Sigma_3)/\tau_2 \times \tau_3$$

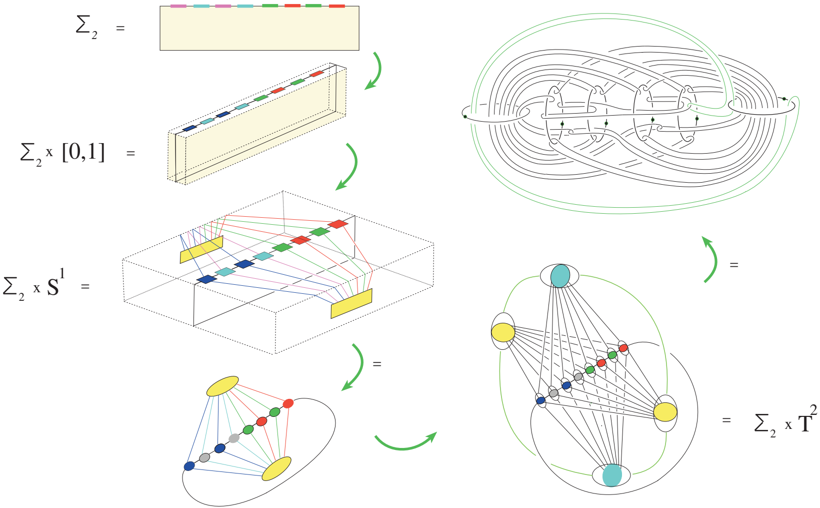
where $\tau_2 : \Sigma_2 \to \Sigma_2$ is the hyperelliptic involution, $\tau_3 : \Sigma_3 \to \Sigma_3$ is the free involution induced by 180^o rotation.



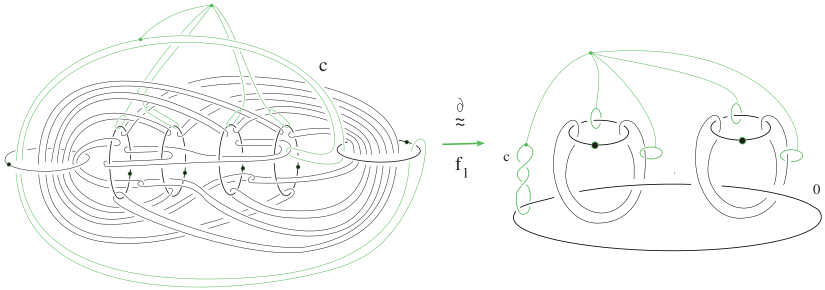
Cacime surface can be deformed into a union of two fibrations:



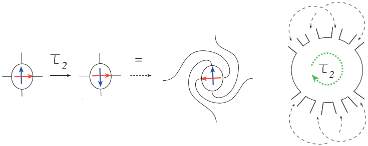
Drawing $\Sigma_2 \times T_0^2$

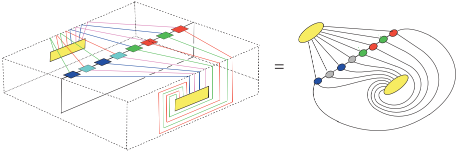


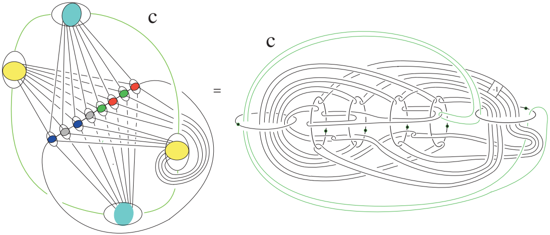
Reference diffeomorphism $f : \partial(\Sigma_2 \times T_0^2) \to \partial(\Sigma_2 \times D^2)$



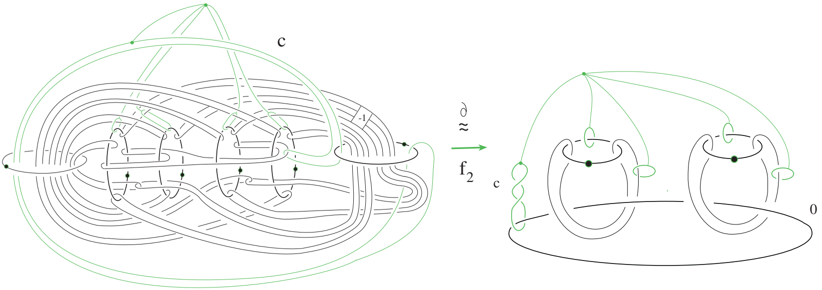
Drawing $\Sigma_2 \times_\tau T_0^2$



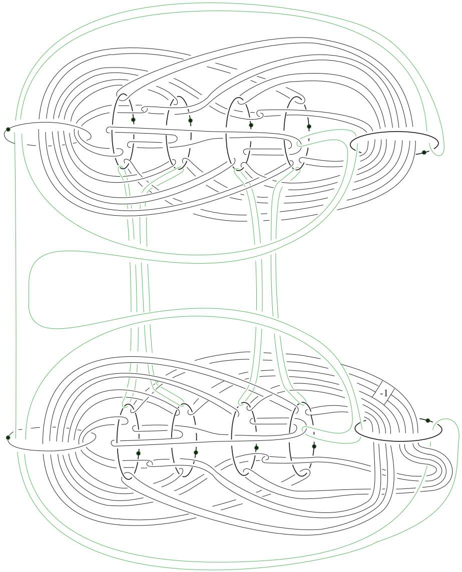




Reference diffeomorphism $f : \partial(\Sigma_2 \times_\tau T_0^2) \to \partial(\Sigma_2 \times D^2)$



Putting the two pieces together by ropes to get the Cacime surface



Basic fibration (a T^2- bundle over T^2):

$$T^2 \to S^3_0(K) \times S^1 \to T^2$$

$Z = F_2 \times T^2_0$ with four T^2-surgeries in its interior.

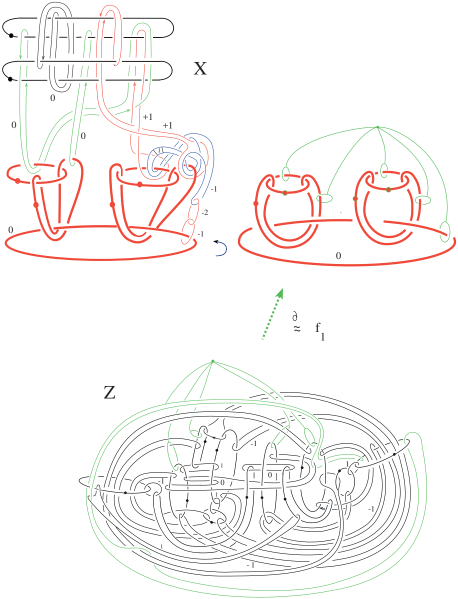
(1) Smooth $T^2_{vert} \cup T^2_{hor} \subset S^3_0(K) \times S^1$, then blow up $\Rightarrow$ get $F_2 \times D^2 \subset S^3_0(K) \times S^1 \# 2(-\mathbb{CP}^2) \Rightarrow$ Call its complement X

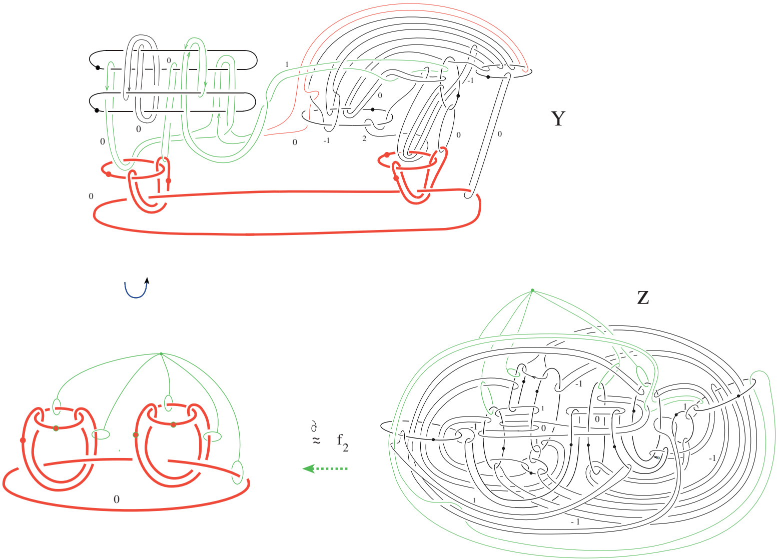
$$\textit{Exotic}\ \mathbb{CP}^2 \# 3(-\mathbb{CP}^2) = Z \smile_\partial X$$

(2) Blowup $S^3_0(K) \times S^1 \# (-\mathbb{CP}^2)$, then smooth $2T^2_{vert} \cup T^2_{hor} \Rightarrow$ $F_2 \times D^2 \subset S^3_0(K) \times S^1 \# (-\mathbb{CP}^2) \Rightarrow$ Call its omplement Y

$$\textit{Exotic}\ \mathbb{CP}^2 \# 2(-\mathbb{CP}^2) = Z \smile_\partial Y$$

Ropes used to put together exotic $\mathbb{CP}^2\#3(-\mathbb{CP}^2)$




Y
Z
∂
≈
f_2