Visualising 4-manifolds

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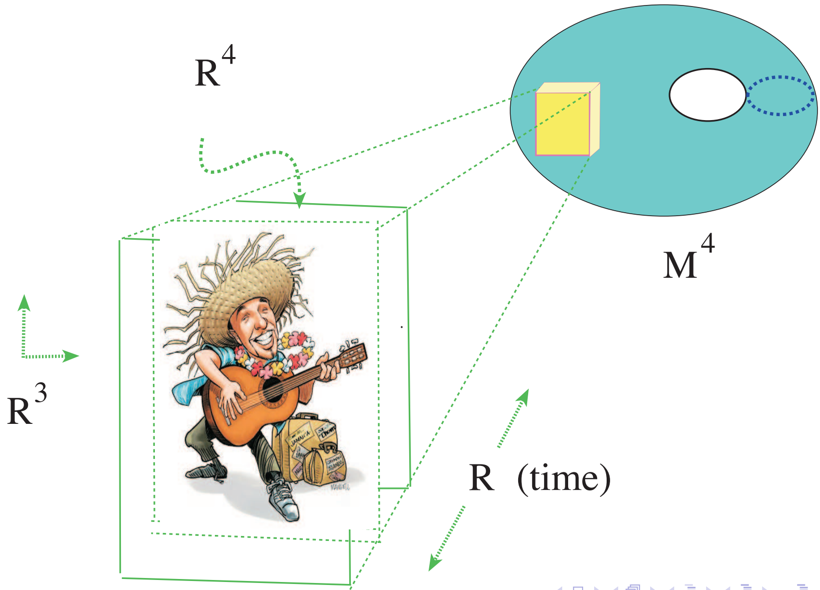
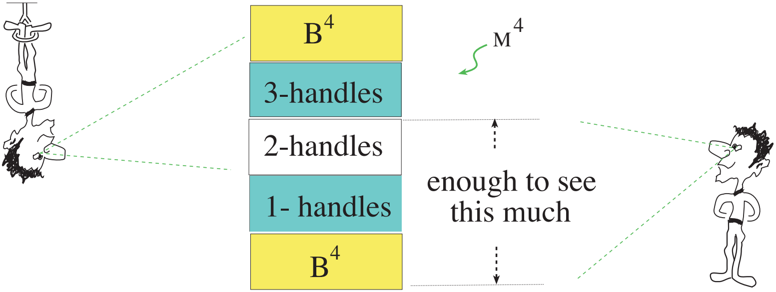
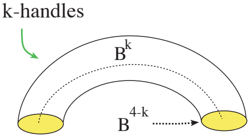
4-manifold

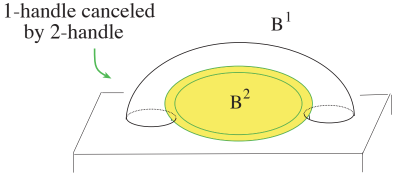


4-manifold as a handlebody (Morse function)

A k-handle is just a ball $B^4 = \mathbf{B}^k \times B^{4-k}$ $(k = 0, 1, 2, 3, 4)$ attached along $\partial \mathbf{B}^k \times B^{4-k} = \mathbf{S}^{k-1} \times B^{4-k}$.

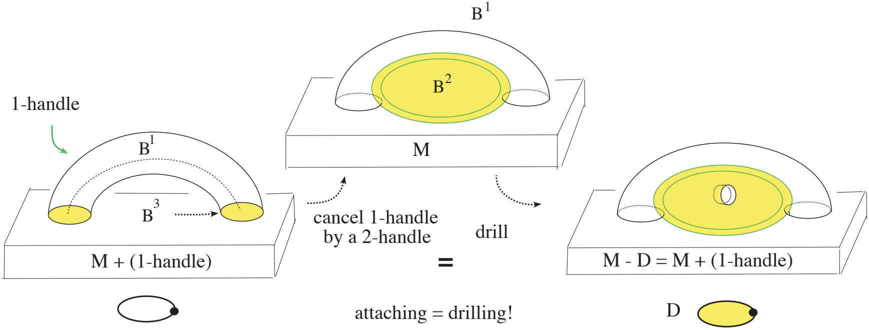




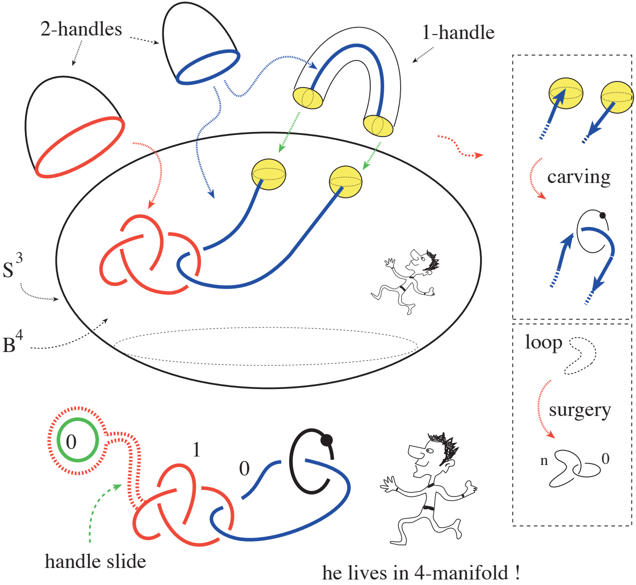


3 basic principles for 4-manifold handlebodies

(1) 0, 1, and 2-handles are sufficient (Laudenbach-Poenaru/1972)
(2) "**Carving**" : Adding 1-handle = Subtracting 2-handle (A/1977)
(3) When stuck don't despair, turn handlebody upside down!



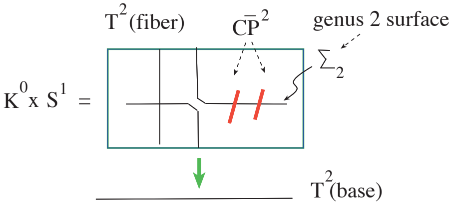
Visualizing 4-manifolds



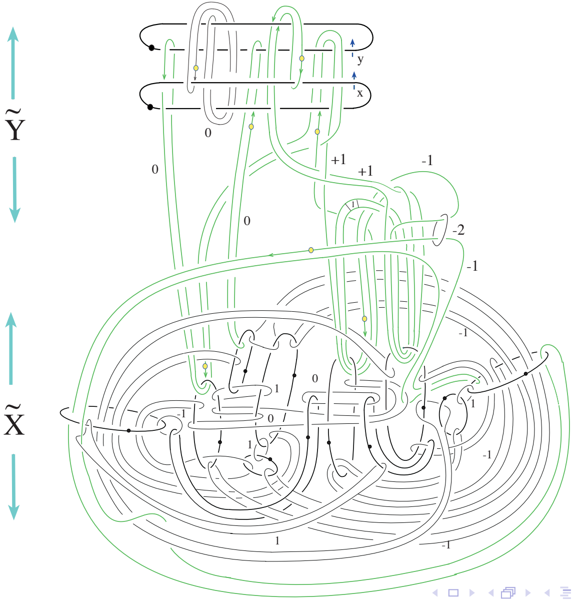
Akhmedov-Park exotic $CP^2\#3\bar{CP}^2$ (description)

$$M = \tilde{X} \smile_\partial \tilde{Y}$$

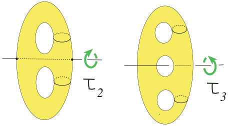
- $\Sigma_2 =$ Genus 2 surface $< a_1, b_1, a_2, b_2 >$
- $T_0^2 =$ punctured torus $< C, D >$
- $X = \Sigma_2 \times T_0^2$
- $\tilde{X} =$ Log transform $X < a_1 \times C, b_1 \times C, a_2 \times C, a_2 \times D >$
- $Y = (K^0 \times S^1)\#2\bar{CP}^2$ (where $K \subset S^3$ trefoil)
- $\tilde{Y} = Y - \Sigma_2 \times D^2$



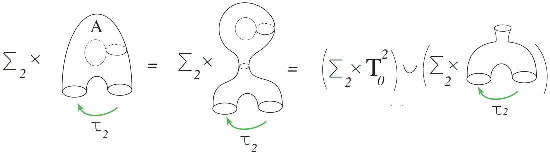
Akhmedov-Park exotic $CP^2 \# 3\bar{CP}^2$ (picture)



Cacime surface (description)



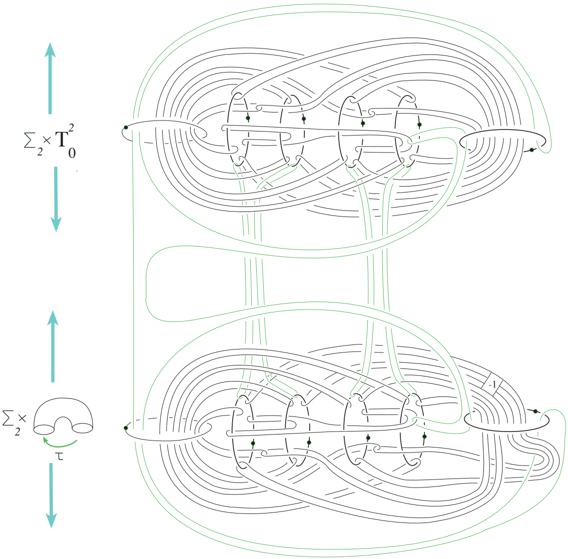
$$X = (\Sigma_2 \times \Sigma_3)/\tau_2 \times \tau_3 \longrightarrow \Sigma_2 \quad (\Sigma_2 \text{ bundle over } \Sigma_2)$$



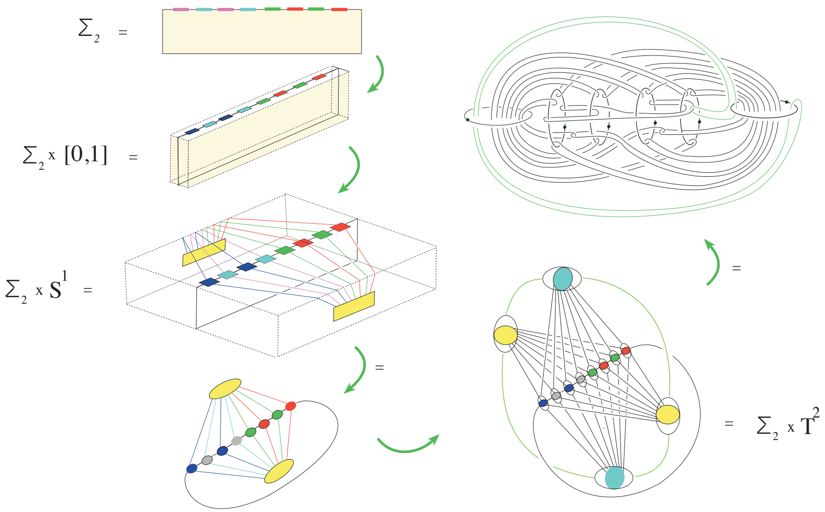
Theorem

(Hacon-Pardini/Pirola): If X is a complex surface $p_g(X) = q(X) = 3$.
Then X is one of the following
(a) $K^2 = 6 \Rightarrow X = Sym^2(\Sigma_3)$
(b) $K^2 = 8 \Rightarrow X =$ Cacime surface.

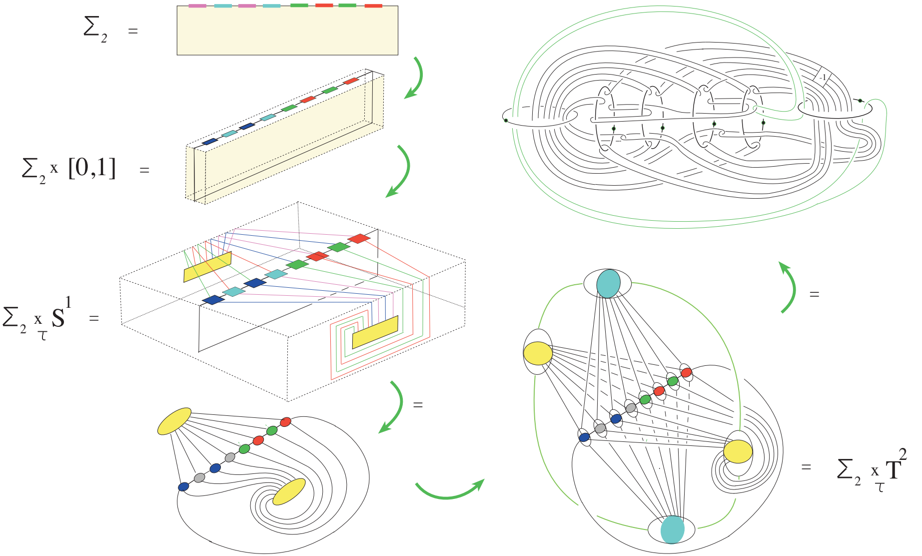
Cacime surface (picture)



Handlebody of $\Sigma_2 \times T^2 \to T^2$



Handlebody of $\Sigma_2 \times_\tau T^2 \to T^2$



Glue boundaries with $f_2 \circ f_1^{-1}$. Trick: Follow the arcs!