# AN ABSOLUTELY EXOTIC CONTRACTIBLE 4-MANIFOLD

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ABSTRACT. We give examples of absolutely exotic smooth structures on smooth contractible 4-manifolds. In this context, *absolute* means that the exotic structure is not relative to a particular parameterization of the boundary. Previous examples of exotic structures on contractible manifolds, known as corks, arise from diffeomorphisms of the boundary that do not extend to diffeomorphisms of the interior. Our examples are constructed by modifying a cork by adding an invertible homology cobordism along the boundary.

## 1. INTRODUCTION

One goal of 4-dimensional topology is to find exotic smooth structures on the simplest of closed 4-manifolds, such as $S^4$ and $\mathbb{CP}^2$. Amongst manifolds with boundary, there are very simple exotic structures coming from the phenomenon of corks, which are *relatively* exotic contractible manifolds discovered by the first-named author [2]. More specifically, a cork is a compact smooth contractible manifold $W$ together with a diffeomorphism $f : \partial W \to \partial W$ which does not extend to a self-diffeomorphism of $W$, although it does extend to a self-homeomorphism $F : W \to W$. This gives an exotic smooth structure on $W$ relative to its boundary, namely the pullback smooth structure by F. This smooth structure is not *absolute*, in the sense that it is diffeomorphic to $W$ if we don't fix the identification of the boundary. We will explain this distinction more precisely below.

In this paper we construct exotic smoothings of contractible manifolds that are absolutely exotic. The smallest previously known such manifolds

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were homotopy equivalent to $S^2$ and were constructed as 4-balls with a single 2-handle attached [1].

**Theorem 1.1.** *There are homeomorphic but not diffeomorphic smooth compact contractible* 4*-manifolds* $V$ *and* $V'$ *with diffeomorphic boundaries.*

The cork theorem [11, 24] implies that $V'$ is obtained from $V$ by a cork-twisting operation in the interior; that is how we will obtain our example. In a final section, we will extend the technique to show how the existence of infinitely many relatively exotic contractible 4-manifolds implies the existence of infinitely many absolutely exotic ones.

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1.1. **Smoothings and markings of the boundary.** The material discussed here is standard but we review it to fix our terminology.

**Definition 1.2.** Let $W^{n+1}$ be a compact topological manifold with boundary, and let $M^n$ be a closed smooth manifold. A marking of the boundary is a homeomorphism $j : M \to \partial W$. A smoothing of $W$ relative to the marking $j$ is a smooth structure on $W$, so that $j$ is a diffeomorphism. Two relative smoothings $(W, j)$ and $(W', j')$ are equivalent (relatively diffeomorphic) if there is a diffeomorphism $F : W \to W'$ with $F \circ j = j'$.

$$\begin{array}{ccccc} W & & \xrightarrow{\quad F \quad} & & W' \\ \uparrow \subseteq & & & & \uparrow \subseteq \\ \partial W & \xleftarrow{\;j\;} & M & \xrightarrow{\;j'\;} & \partial W' \end{array}$$

In the terminology of current 3-manifold topology [23], this notion is described under the name of bordered manifold. As an example, a smooth structure on $W^4$ induces, in a canonical way, a relative smoothing with $M^3 = \partial W^4$ and $j$ the identity. By composition with $j$, the set of diffeomorphisms of $M$, up to isotopy (more precisely, pseudo-isotopy) acts on the set of relative diffeomorphism classes of manifolds with boundary $M$; if $M = \partial W$, this amounts to replacing $j = \mathrm{id}$ by an arbitrary self-diffeomorphism. Corks are relative smoothings in this sense; the *Mazur cork* shown in Figure 1 was shown to be relatively exotic (in different terminology) in [2, 3].

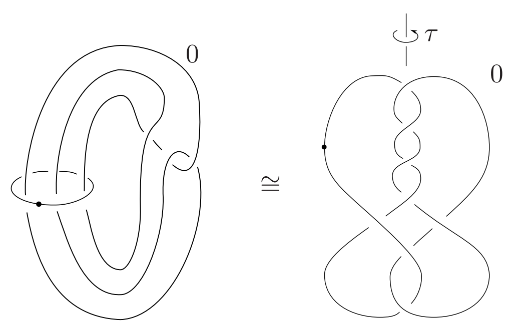


FIGURE 1. The Mazur cork $M$

In contrast, an absolute smoothing of $W$ is just a smooth structure without a marking of the boundary, considered up to diffeomorphism. If we are given a particular relative (resp. absolute) smooth structure on $W$, then a relatively (resp. absolutely) inequivalent smoothing will be referred to as *exotic*. Sometimes there is no distinction between relative and absolute, as in the following simple lemma.

**Lemma 1.3.** *Suppose that every self-diffeomorphism of $\partial W$ extends to a diffeomorphism of $W$. Then the natural forgetful map from relative to absolute smoothings of $W$ is a bijection.*

There are some well-known instances where this hypothesis is satisfied, for instance [7] if $W = B^4$ or more generally [22] if $W = \natural^n S^1 \times B^3$. We will give another example as part of our main theorem.

## 2. An absolutely exotic contractible 4-manifold

The construction of our example requires several ingredients from knot theory and 3-dimensional topology. We explain the basic idea first, and then show how to find those ingredients. We start with a standard definition.

**Definition 2.1.** An invertible cobordism $X^{n+1}$ from $M^n$ to $N^n$ is a smooth manifold with $\partial X = -M \cup N$, such that there is a manifold $Y$ with

$$\partial Y = -N \cup M \text{ and } X \cup_N Y \cong M \times I.$$

We will implicitly assume that there are markings of $\partial X$ and $\partial Y$ that are used in gluing $X$ to $Y$ along $N$, and that the diffeomorphism between $X \cup_N Y$ and $M \times I$ respects the markings of the $M$ boundary components. It will be the case in our examples that the inclusion of $M$ into $X$ induces an isomorphism on homology, so that $X$ is an invertible homology cobordism. It is easy to see that the inclusion of $N$ into $X$ induces a surjection on the fundamental group. We will be exclusively concerned with $n = 3$.

We can now explain the basic idea of the proof of Theorem 1.1. Start with a cork $(W, \tau)$, and form the union $V = W \cup_M X$, where $X$ is an invertible homology cobordism from $M = \partial W$ to some other 3-manifold $N$. It is clear that $V$ is a homology ball, but it is not automatic that $V$ is actually contractible. Doing a cork twist on the embedded copy of $W$ in $V$ results in a manifold $V'$, and the invertibility of $X$ will show that $V'$ is exotic relative to the identity marking on $\partial V' = N$. To show that $V'$ is absolutely exotic, we will choose $N$ carefully so that all of its self-diffeomorphisms extend over $V'$. Philosophically, we have used the invertible homology cobordism to 'kill' the $\tau$-symmetry of $M$.

### 2.1. Constructing $N$ and the invertible cobordism.

By [5] the boundary of the Mazur cork $(W, \tau)$ (Figure 1) is diffeomorphic to the 3-manifold $M^3$ given by $+1$ surgery on the pretzel knot $K = P(-3, 3, -3)$ of Figure 3. This diffeomorphism $f : \partial W \to \partial K^{+1}$ is explained by the steps in Figure 2.

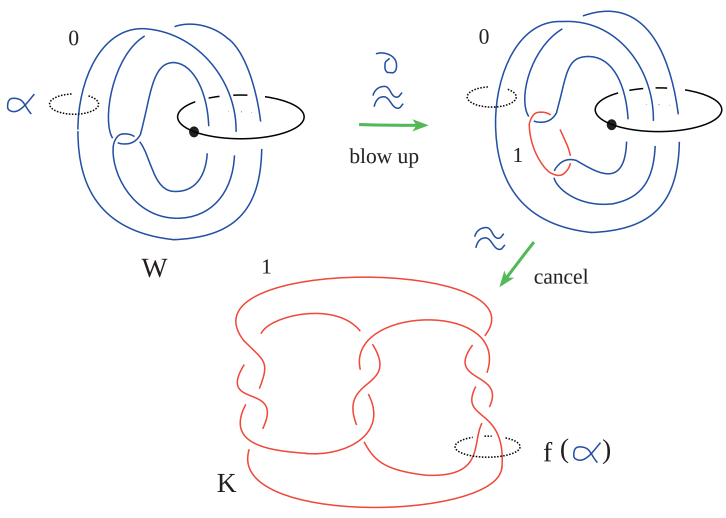


FIGURE 2. The diffeomorphism $f : \partial W \to M$

The symmetry group of $K$ (up to isotopy) is a $Z_2 \oplus Z_2$ with generators $\sigma$ and $\tau$ as indicated in Figure 3; both of these extend over the surgery to symmetries of $M$ which is, by construction (see also [6]) $\tau$-equivariantly diffeomorphic to $\partial W$. The choice of $\eta$ was made in order to disrupt the symmetries of $K$; we will explain below a computer calculation that rigorously confirms what seems clear visually.

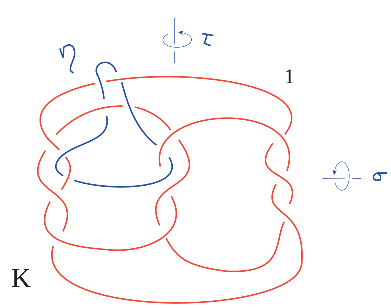


FIGURE 3. $K^{+1}$

We define the manifold $P$ to be the complement of $\eta$ in $M$, or equivalently the result of $+1$ surgery on the $K$ component of the link $(K, \eta)$.

The second ingredient is a knot $J$ with the following properties: it is to be doubly slice, hyperbolic, and with trivial symmetry group. We know of three such knots: the Kinoshita-Terasaka [21] knot $11n42$ (Figure 4) as well as $12n0313$, and $12n0430$. These were found, starting with a list of doubly slice knots supplied by Jeff Meier, by a search on Knotinfo [9] and some computations with SnapPy [10]. Such invariants are computed numerically, and in principle require a rigorous verification. Fortunately, the recent paper [12] shows how to certify the symmetry of certain 3-manifolds using interval arithmetic. The arxiv listing for that paper contains code (based in turn on [17], which verifies hyperbolicity) that can be run, starting with a triangulation found via SnapPy, and will rigorously compute the symmetry group. All properties of the manifolds used in our construction were verified in this way; files describing the triangulations are available upon request to the authors.

We summarize the output of these calculations.

**Proposition 2.2.** *The manifold $P$ is hyperbolic, and has trivial symmetry group. The knots $11n42$, $12n0313$, and $12n0430$ are doubly slice, and are hyperbolic with trivial symmetry group.*

*Proof.* The statements about hyperbolicity and symmetry were proved by computation, as described above. We will show that $J = 11n42$ is doubly slice; this seems to be a well-known fact. The others are left to the interested reader, as only $J$ is used in this paper.

The dotted line in Figure 4 indicates the slice move for $J = 11n42$ (specifying a disk $D$ which $J$ bounds in $B^4$).

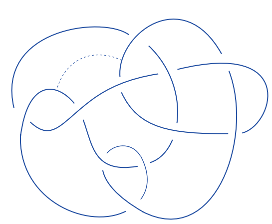

FIGURE 4. J (the dotted line indicates the slice move)

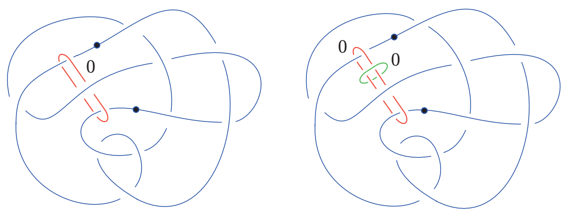


FIGURE 5. $B^4 - D$ and $S^4 - S^2$

The first picture of Figure 5 is the handlebody of the complement of the disk $D$ in $B^4$, the second picture is the complement of the $S^2$ (which is the double of $D$) in $S^4$ (the reader can verify this by Section 1.4 of [4]). After an isotopy, Figure 5 becomes Figure 6, where the doubly sliceness of $J$ is now evident, i.e. the complement of $S^2$ in $S^4$ is $S^1 \times B^3$, so $S$ is an unknotted 2-sphere [15]. $\square$

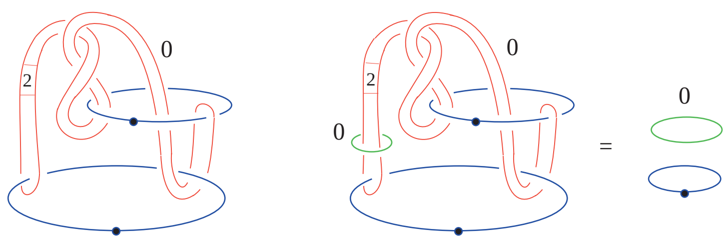


FIGURE 6. $B^4 - D$ and $S^4 - S^2$

Let $C \subset S^3 \times [0,1]$ be the invertible concordance, so that $C \cap S^3 \times 0 = U$ (the unknot) and $C \cap S^3 \times 1 = J$. By definition there is another concordance $C' \subset S^3 \times [1,2]$ with $C' \cap S^3 \times 1 = J$ and $C' \cap S^3 \times 2 = U$, and $C \cup C' \subset S^3 \times [0,2] = U \times [0,2]$. For the knots named above, $C'$ may be obtained by turning $C$ upside down to obtain the concordance $\bar{C}$.

Now let $\eta$ be the knot in $\partial K^{+1}$ as indicated in Figure 3. A patient reader can check that under the diffeomorphism $f : \partial W \to \partial K^{+1}$ of Figure 2, $\eta$ corresponds to the curve $\eta$ in $\partial W$ indicated in Figure 7

Glue the knot exteriors of $E(\eta)$ in Figure 3, and $E(J)$ by identifying the meridian of $\eta$ by the longitute of $J$ (and vice versa), denote the resulting

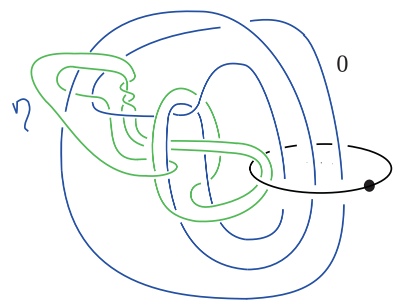


FIGURE 7. V

satellite knot $K_J$. It's useful to describe $K_J$ as the 'infection' of K by J using the indicated curve $\eta$ in the picture. Let $N$ be given by $+1$ surgery on $K_J$. We claim that there is an invertible cobordism $X$ from $M$ to $N$, with inverse $\bar{X}$, so that $X \cup_N \bar{X} = M \times [0,2]$. This is a standard argument [16]; glue $S^3 \times [0,1] - \nu(C)$ into $E(K) \times [0,1]$ and do $+1$ surgery on K at every level to make $X$, and do the same thing with $\bar{C}$ to obtain $\bar{X}$. From the identification of Figure 2, we see that this corresponds to the handlebody in Figure 8 (see 5.3 of [4]). Hence Figure 8 describes $V$ with $\partial V = N$. Notice that In this figure $W$ can easily be identified inside of $V$, and $V$ is built from $W$ by attaching two 1/2-handle pairs (which is $X$). Also from Figure 8, the reader can easily verify that $V$ is simply connected. This fact can alternatively be seen via van Kampen's theorem by noting that the fundamental group of $X$ is normally generated by the meridian of $J$, but this meridian dies in $\pi_1(W)$.

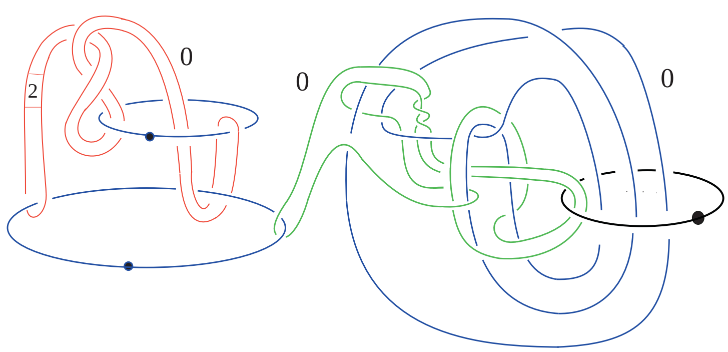


FIGURE 8. $W \subset V = W \cup_M X$

*Proof of Theorem 1.1.* Consider the manifold $P$ given by the complement of a tubular neighborhood of $\eta$ in $M$. $P$ is obtained by doing $+1$ framed surgery on the first component of the link $L = K \cup \eta \subset S^3$ in the complement of the tubular neighborhood of $\eta$. $P$ has boundary $T$, which is the boundary torus of $\eta$. Then $P$ is hyperbolic, with trivial symmetry group. We will use these facts to verify the following.

**Claim:** The group of diffeomorphisms of $N$ mod isotopy is isomorphic to $Z \oplus Z$, and all of these extend over the cobordism $X$ in such a way that they are isotopic to the identity on $M$.

**Proof of claim:** Write

$$N = P \cup_{T\times 0} T \times [0,1] \cup_{T\times 1} S^3 - \nu(J).$$

Since both $P$ and $S^3 - \nu(J)$ are hyperbolic, the JSJ decomposition [19, 20] of $N$ has just the one torus $T$. It follows that any self-diffeomorphism of $f : N \to N$ is isotopic to one that preserves $T \times [0,1]$. Since the symmetry groups of $J$ and $P$ are trivial, it follows that we can assume that in fact $f$ is the identity on $T \times 1$. Since $f$ on $T \times 0$ is homotopic to $f$ on $T \times 1$, we can assume that $f$ is the identity on $T \times 0$ as well. According to Waldhausen [26], the group of isotopy classes of diffeomorphisms of $T \times [0,1]$ (relative to the boundary) is isomorphic to the group of self-homotopy equivalences (again, relative to the boundary). The latter is readily seen to be $\mathbb{Z} \oplus \mathbb{Z}$, where the elements can be described as follows. The element $(a, b) \in \mathbb{Z} \oplus \mathbb{Z}$ corresponds to the Dehn twist of $S^1 \times S^1 \times [0,1]$ given by

$$(z, w, t) \to (e^{2\pi i a t} z, e^{2\pi i b t} w, t).$$

Any such diffeomorphism extends in a natural way over $S^1 \times D^2$. It is easy to see that the extension, as a diffeomorphism of $S^1 \times D^2$, is isotopic to the identity, via an isotopy that is the identity on the boundary. For example, take the disk $D^2$ to have radius 2, and write the diffeomorphism on $S^1 \times \{w \mid 1 \leq |w| \leq 2\}$. So the extension over $S^1 \times D^2$ is given by

$$F(z,w) = \begin{cases} (z,w) & \text{for } |w| \leq 1 \\ (e^{2\pi i a|w|}z, e^{2\pi i b|w|}w) & \text{for } 1 \leq |w| \leq 2. \end{cases}$$

Then the isotopy is given by

$$F_s(z,w) = \begin{cases} (e^{2\pi i a s}z, e^{2\pi i b s}w) & \text{for } |w| \leq 1 \\ (e^{2\pi i a s(2-|w|)}z, e^{2\pi i b s(2-|w|)}w) & \text{for } 1 \leq |w| \leq 2. \end{cases}$$

Write $V'$ for $X \cup_\tau W$, the result of the cork twist along the embedded copy of $W$ in $V$. It has an obvious marking of the boundary coming from the identification of $N$ with a boundary component of $X$. If $V'$ were diffeomorphic to $V$, preserving this marking, then we could glue this diffeomorphism to the identity of $\bar{X}$ to get a diffeomorphism

$$\bar{X} \cup_N X \cup_\tau W \cong W \tag{1}$$

But $\bar{X} \cup_N X \cong N \times I$ (relative to the identity on the boundary) and hence $\tau$ extends to $\bar{X} \cup_N X$. It follows that (1) would result in a diffeomorphism of $(W, \tau)$ with $(W, \mathrm{id})$, contradicting [2]. Since $V$ and $V'$ are simply connected homology balls, they are contractible, hence homeomorphic. By the claim above and Lemma 1.3, there is no diffeomorphism between $V$ and $V'$. □

**Remark 2.3.** We originally tried this construction making use of a somewhat simpler curve $\eta_1$ in $M$ drawn below. We computed using SnapPy that the corresponding manifold $P_1$ (resulting from $+1$ surgery on $K$) is hyperbolic with symmetry group $\mathbb{Z}_2$, generated by $\sigma$. However, the procedure of [12] for verifying this numerical calculation of the symmetry group breaks down for $P_1$. The reason, as explained to us by Dunfield, is that not all of the cells in the Epstein-Penner canonical cellulation [8] of $P_1$ are tetrahedra. In this case we would also get a simpler $V_1$ as shown in Figure 10.

If we assume that the symmetry group of $P_1$ is as stated, then a slightly more elaborate argument with the JSJ decomposition then implies that the corresponding manifold $N_1$ has trivial symmetry group. This would imply $V_1'$

is an absolutely exotic copy of $V_1$, but proving this would require a rigorous verification of the symmetry group.

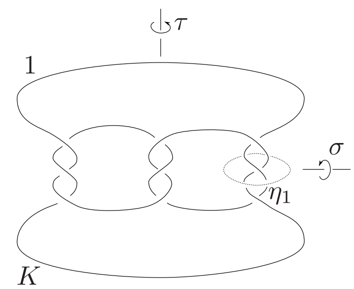


FIGURE 9. $P_1$

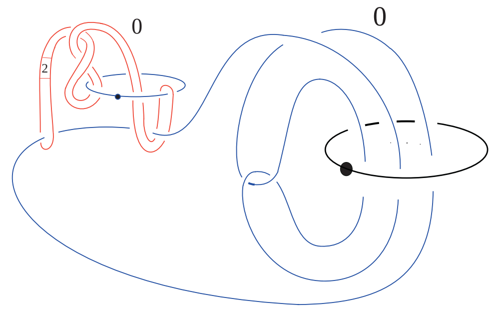


FIGURE 10. $W \subset V_1 = W \cup_M X_1$

## 3. From infinitely many relative exotic structures to absolutely exotic structures

In this section we will show how to modify a contractible manifold $W$ which admits infinitely many smooth structures relative to its boundary to infinitely many absolutely exotic smooth structures on a different contractible manifold $V$. The modification will not leave us with a full understanding of the symmetry group of the boundary, so we replace Lemma 1.3 by a weaker result.

**Lemma 3.1.** *Suppose that $M^n$ is a manifold such that $\pi_0(\mathrm{Diff}(M))$ is finite. If the manifold $V$ has infinitely many smoothings relative to some fixed identification $j : M \to \partial V$, then $V$ has infinitely many absolute smoothings.*

*Proof.* Suppose that $V$ has only finitely many different smooth structures that are absolutely distinct, and let $V_k$, $k \in \mathbb{N}$ be smoothings relative to $j$ that are not diffeomorphic relative to $j$. Then, replacing the $V_k$ by an appropriate subsequence, we may assume that all $V_k$ are diffeomorphic. Letting $F_k : V_1 \to V_k$ be a diffeomorphism, we must have that $j^{-1} \circ F_k|_{\partial V_k} \circ j$ are all distinct up to isotopy, contradicting our assumption that $\pi_0(\mathrm{Diff}(M))$ is finite. □

Now we make use of an old result of the second author [25], with a slight amplification.

**Theorem 3.2.** *Let $M$ be a closed 3 manifold. Then there is an invertible cobordism $X$ from $M$ to a hyperbolic manifold $N$, such that $\pi_1(M)$ normally generates $\pi_1(X)$.*

*Proof.* All but the last clause is Theorem 2.6 of [25]; the reader should beware that the labeling of boundary components $M$ and $N$ in that paper is reversed relative to this one. To see the last clause, we review the construction, introducing some new notation to lessen the confusion. The main ingredient is an invertible tangle concordance from the complement of a trivial $g$-string tangle in the 3-ball to a certain $g$-string tangle $T_g$. The complement of the trivial tangle is a genus-$g$ handlebody $H_g$, and so the complement $X_g$ of this concordance is an invertible homology cobordism (relative to the boundary) from $H_g$ to $A_g$, the complement of the tangle $T_g$. The main new observation is that the fundamental group of $X_g$ is normally generated by the meridians of the concordance, which are the same as the meridians of the trivial tangle. In other words, the fundamental group of $X_g$ is the normal closure of $\pi_1(H_g)$.

For $g \geq 3$, the manifold $A_g$ has the property that when it is glued to itself by any diffeomorphism of the boundary surface, the result is a hyperbolic manifold. Now, given a 3-manifold $M$, we choose a Heegaard splitting of genus at least 3, so that $M = H_g \cup_\varphi H_g$. Then

$$X = X_g \cup \varphi \times \mathrm{id}_I$$

is the required invertible homology cobordism $X$. It is straightforward to see that $\pi_1(M)$ normally generates $\pi_1(X)$. □

Now we have the main result of this section.

**Theorem 3.3.** *Suppose that $W$ is a contractible manifold, and let $M = \partial W$. Suppose that $f_j : M \to M$ are diffeomorphisms that extend to homeomorphisms $F_j : W \to W$, giving infinitely many smoothings of $W$ relative to the identity. Then there is a contractible manifold $V$ with infinitely many smoothings.*

*Proof.* We follow the proof of Theorem 1.1. Let $X$ be the invertible homology cobordism from $M$ to a hyperbolic manifold $N$ as in Theorem 3.2, and set $V = W \cup_M X$. Then we can form manifolds

$$V_j = W \cup_{f_j} X.$$

Since $W$ is contractible, and $\pi_1(M)$ normally generates $\pi_1(X)$, it follows that the $V_j$ are all simply-connected and hence contractible. Since they have the same boundary, they are all homeomorphic, but we claim that infinitely many of them are absolutely distinct smooth manifolds.

As in the proof of Theorem 1.1, the invertibility of $X$ implies that the $V_j$ are distinct smooth manifolds, relative to a fixed identification of $\partial V_j$ with $N$. But since $N$ is a hyperbolic manifold, $\pi_0(\mathrm{Diff}(N))$ is finite [13, 14] (a thorough discussion of such issues may be found in [18]). By Lemma 3.1, infinitely many of the $V_j$ are absolutely distinct. □

Although their proofs are similar, Theorems 3.3 and 1.1 are logically independent. That is because the manifold $N$ used in the proof of Theorem 3.3 may well have a non-trivial symmetry group. For instance, there may be some symmetries derived from the $g$-fold symmetry of the hyperbolic manifold $A_g$.

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