

CALIBRATED MANIFOLDS AND GAUGE THEORY

SELMAN AKBULUT AND SEMA SALUR

ABSTRACT. By a theorem of Mclean, the deformation space of an associative submanifold Y of an integrable G_2 -manifold (M, φ) can be identified with the kernel of a Dirac operator $\mathcal{D} : \Omega^0(\nu) \rightarrow \Omega^0(\nu)$ on the normal bundle ν of Y . Here, we generalize this to the non-integrable case, and also show that the deformation space becomes smooth after perturbing it by natural parameters, which corresponds to moving Y through ‘pseudo-associative’ submanifolds. Infinitesimally, this corresponds to twisting the Dirac operator $\mathcal{D} \mapsto \mathcal{D}_A$ with connections A of ν . Furthermore, the normal bundles of the associative submanifolds with $Spin^c$ structure have natural complex structures, which helps us to relate their deformations to Seiberg-Witten type equations.

If we consider G_2 manifolds with 2-plane fields (M, φ, Λ) (they always exist) we can split the tangent space TM as a direct sum of an associative 3-plane bundle and a complex 4-plane bundle. This allows us to define (almost) Λ -associative submanifolds of M , whose deformation equations, when perturbed, reduce to Seiberg-Witten equations, hence we can assign local invariants to these submanifolds. Using this we can assign an invariant to (M, φ, Λ) . These Seiberg-Witten equations on the submanifolds are restrictions of global equations on M . We also discuss similar results for the Cayley submanifolds of a $Spin(7)$ manifold.

0. INTRODUCTION

We first study deformations of associative submanifolds Y^3 of a G_2 manifold (M^7, φ) , where $\varphi \in \Omega^3(M)$ is the G_2 structure. We prove a generalized version of the McLean’s theorem where integrability condition of the underlying G_2 structure is not necessary. This deformation space might be singular, but by perturbing it with some natural parameters it can be made smooth. This amounts to deforming Y through the associatives in (M, φ) with varying φ , or alternatively deforming Y through the pseudo-associative submanifolds (Y ’s whose tangent planes become associative after rotating by a generic element of the gauge group of TM). Infinitesimally, these perturbed deformations correspond to the kernel of the twisted Dirac operator $\mathcal{D}_A : \Omega^0(\nu) \rightarrow \Omega^0(\nu)$, twisted by some connection A in $\nu(Y)$.

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The associative submanifolds with $Spin^c$ structures in (M, φ) are useful objects to study, because their normal bundles have natural complex structures. Also we can view (M, φ) as an analog of a symplectic manifold, and view a non-vanishing 2-plane field Λ on M as an analog of a complex structure taming φ . Note that 2-plane fields are stronger versions of $Spin^c$ structures on M^7 , and they always exist by [T]. The data (M^7, φ, Λ) determines an interesting splitting of the tangent bundle $TM = \mathbf{E} \oplus \mathbf{V}$, where \mathbf{E} is the bundle of associative 3-planes, and \mathbf{V} is the complementary 4-plane bundle with a complex structure, which is a spinor bundle of \mathbf{E} . Then the integral submanifolds Y^3 of \mathbf{E} , which we call Λ -associative submanifolds, can be viewed as analogues of J-holomorphic curves; because their normal bundles come with an almost complex structure. Even if they may not always exist, their perturbed versions, i.e. *almost Λ -associative submanifolds*, always do. Almost Λ -associative submanifolds are the transverse sections of the bundle $\mathbf{V} \rightarrow M$. We can deform such Y by using the connections in the determinant line bundle of $\nu(Y)$ and get a smooth deformation space, which is described by the twisted Dirac equation. Then by constraining this new variable with another natural equation we arrive to Seiberg-Witten type equations for Y . So we can assign an integer to Y , which is invariant under small isotopies through almost Λ -associative submanifolds.

In fact it turns out that (M^7, φ, Λ) gives a finer splitting $TM = \bar{\mathbf{E}} \oplus \xi$, where $\bar{\mathbf{E}}$ is a 6-plane bundle with a complex structure, and ξ is a real line bundle. In a way this structure of (M, φ) mimics the structure of $(\text{Calabi-Yau}) \times S^1$ manifolds, and by ‘rotating’ ξ inside of TM we get a new insight for so-called “Mirror manifolds” which is investigated in [AS1].

There is a similar process for the deformations of Cayley submanifolds $X^4 \subset N^8$ of a $Spin(7)$ manifold (N^8, Ψ) , which we discuss at the end. So in a way Λ -associative (or Cayley) manifolds in a G_2 (or $Spin(7)$) manifold, behave much like higher dimensional analogue of holomorphic curves in a Calabi-Yau manifold.

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1. PRELIMINARIES

Here we first review basic properties of the manifolds with special holonomy (most material can be found in [B2], [B3], [H], [HL]), and then proceed to prove some new results. Recall that the set of octonions $\mathbb{O} = \mathbb{H} \oplus l\mathbb{H} = \mathbb{R}^8$ is an 8-dimensional division algebra generated by $\langle 1, i, j, k, l, li, lj, lk \rangle$. On the set of the imaginary octonions $im\mathbb{O} = \mathbb{R}^7$ we have the cross product operation $\times : \mathbb{R}^7 \times \mathbb{R}^7 \rightarrow \mathbb{R}^7$, defined by $u \times v = im(\bar{v}.u)$. The exceptional Lie group G_2 can be defined as the linear automorphisms of $im\mathbb{O}$ preserving this cross product operation, $G_2 = Aut(\mathbb{R}^7, \times)$. There is also another useful description in terms of the orthogonal 3-frames in \mathbb{R}^7 :

$$(1) \quad G_2 = \{(u_1, u_2, u_3) \in (im\mathbb{O})^3 \mid \langle u_i, u_j \rangle = \delta_{ij}, \langle u_1 \times u_2, u_3 \rangle = 0\}$$

Alternatively, G_2 can be defined as the subgroup of the linear group $GL(7, \mathbb{R})$ which fixes a particular 3-form $\varphi_0 \in \Omega^3(\mathbb{R}^7)$. Denote $e^{ijk} = dx^i \wedge dx^j \wedge dx^k \in \Omega^3(\mathbb{R}^7)$, then

$$G_2 = \{A \in GL(7, \mathbb{R}) \mid A^*\varphi_0 = \varphi_0\}$$

$$(2) \quad \varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

Definition 1. A smooth 7-manifold M^7 has a G_2 structure if its tangent frame bundle reduces to a G_2 bundle. Equivalently, M^7 has a G_2 structure if there is a 3-form $\varphi \in \Omega^3(M)$ such that at each $x \in M$ the pair $(T_x(M), \varphi(x))$ is isomorphic to $(T_0(\mathbb{R}^7), \varphi_0)$.

Here are some useful properties, discussed more fully in [B2]: Any G_2 structure φ on M^7 gives an orientation $\mu \in \Omega^7(M)$ on M , and this μ determines a metric $g = \langle \cdot, \cdot \rangle$ on M , and a cross product structure \times on its tangent bundle of M as follows: Let i_v denote the interior product with a vector v then

$$(3) \quad \langle u, v \rangle = [i_u(\varphi) \wedge i_v(\varphi) \wedge \varphi] / 6\mu$$

$$(4) \quad \varphi(u, v, w) = \langle u \times v, w \rangle$$

To emphasize the dependency on φ sometimes g is denoted by g_φ . In particular, the 14-dimensional Lie group G_2 imbeds into $SO(7)$ subgroup of $GL(7, \mathbb{R})$. Note that because of the way we defined $G_2 = G_2^{\varphi_0}$, this imbedding is determined by φ_0 .

Since $GL(7, \mathbb{R})$ acts on $\Lambda^3(\mathbb{R}^7)$ with stabilizer G_2 , its orbit $\Lambda_+^3(\mathbb{R}^7)$ is open for dimension reasons, so the choice of φ_0 in the above definition is generic (in fact it has two orbits containing $\pm\varphi_0$). G_2 has many copies G_2^φ inside $GL(7, \mathbb{R})$, which are all conjugate to each other, since G_2 has only one 7 dimensional representation. Hence the space of G_2 structures on M^7 are identified with the sections of the bundle:

$$(5) \quad \mathbb{R}P^7 \simeq GL(7, \mathbb{R})/G_2 \rightarrow \Lambda_+^3(M) \rightarrow M$$

which are called *the positive 3-forms*, these are the set of 3-forms $\Omega_+^3(M)$ that can be identified pointwise by φ_0 . Each G_2^φ imbeds into a conjugate of one standard copy $SO(7) \subset GL(7, \mathbb{R})$. The space of G_2 structures φ on M , which induce the same metric on M , that is all φ 's for which the corresponding G_2^φ lies in the standard $SO(7)$, are the sections of the bundle (whose fiber is the orbit of φ_0 under $SO(7)$):

$$(6) \quad \mathbb{R}P^7 = SO(7)/G_2 \rightarrow \tilde{\Lambda}_+^3(M) \longrightarrow M$$

which we will denote by $\tilde{\Omega}_+^3(M)$. The set of smooth 7-manifolds with G_2 -structures coincides with the set of 7-manifolds with spin structure, though this correspondence is not 1 – 1. This is because $Spin(7)$ acts on S^7 with stabilizer G_2 inducing the fibrations

$$G_2 \rightarrow Spin(7) \rightarrow S^7 \rightarrow BG_2 \rightarrow BSpin(7)$$

and so there is no obstruction to lifting maps $M^7 \rightarrow BSpin(7)$ to BG_2 , and there are many liftings. Cotangent frame bundle $\mathcal{P}^*(M) \rightarrow M$ of a manifold with G_2 structure (M, φ) can be expressed as $\mathcal{P}^*(M) = \cup_{x \in M} \mathcal{P}_x^*(M)$, where each fiber is:

$$\mathcal{P}_x^*(M) = \{u \in Hom(T_x(M), \mathbb{R}^7) \mid u^*(\varphi_0) = \varphi(x)\}$$

Throughout this paper we will denote the cotangent frame bundle by $\mathcal{P}^*(M) \rightarrow M$ and its adapted frame bundle by $\mathcal{P}(M)$. They can be G_2 or $SO(7)$ frame bundles; to emphasize it sometimes we will specify them by the notations $\mathcal{P}_{SO(7)}(M)$ or $\mathcal{P}_{G_2}(M)$. Also we will denote the sections of a bundle $\xi \rightarrow Y$ by $\Omega^0(Y, \xi)$ or simply by $\Omega^0(\xi)$, and the bundle valued p -forms by $\Omega^p(\xi) = \Omega^0(\Lambda^p T^*Y \otimes \xi)$, and the *sphere bundle* of ξ by $S(\xi)$. There is a notion of a G_2 structure φ on M^7 being *integrable*, which corresponds to φ being an harmonic form:

Definition 2. *A manifold with G_2 structure (M, φ) is called a G_2 manifold if the holonomy group of the Levi-Civita connection (of the metric g_φ) lies inside of G_2 . Equivalently (M, φ) is a G_2 manifold if φ is parallel with respect to the metric g_φ i.e. $\nabla_{g_\varphi}(\varphi) = 0$; this condition is equivalent to $d\varphi = 0 = d(*_{g_\varphi}\varphi)$.*

In short one can define a G_2 manifold to be any Riemannian manifold (M^7, g) whose holonomy group is contained in G_2 , then φ and the cross product \times come as a consequence. It turns out that the condition φ being harmonic is equivalent to the condition that at each point $x_0 \in M$ there is a chart $(U, x_0) \rightarrow (\mathbb{R}^7, 0)$ on which φ equals to φ_0 up to second order term, i.e. on the image of U

$$(7) \quad \varphi(x) = \varphi_0 + O(|x|^2)$$

Remark 1. *For example if (X^6, ω, Ω) is a complex 3-dimensional Calabi-Yau manifold with Kähler form ω , and a nowhere vanishing holomorphic 3-form Ω , then $X \times S^1$ has holonomy group $SU(3) \subset G_2$, hence is a G_2 manifold. In this case*

$$(8) \quad \varphi = Re \Omega + \omega \wedge dt.$$

Definition 3. Let (M, φ) be a manifold with a G_2 structure. A 4-dimensional submanifold $X \subset M$ is called an *co-associative* if $\varphi|_X = 0$. A 3-dimensional submanifold $Y \subset M$ is called an *associative* if $\varphi|_Y \equiv \text{vol}(Y)$; this condition is equivalent to $\chi|_Y \equiv 0$, where $\chi \in \Omega^3(M, TM)$ is the tangent bundle valued 3-form defined by the identity:

$$(9) \quad \langle \chi(u, v, w), z \rangle = * \varphi(u, v, w, z)$$

The equivalence of these conditions follows from the ‘associator equality’ of [HL]

$$(10) \quad \varphi(u, v, w)^2 + |\chi(u, v, w)|^2/4 = |u \wedge v \wedge w|^2$$

In general, if $\{e^1, e^2, \dots, e^7\}$ is any orthonormal coframe on (M, φ) , then the expression (2) for φ hold on a chart. By calculation $*\varphi$, and using (9) we can calculate the expression of χ (note the error in the the second term of 6th line of the corresponding formula (5.4) of [M]):

$$(11) \quad \begin{aligned} * \varphi &= e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247} \\ \chi &= (e^{256} + e^{247} + e^{346} - e^{357}) e_1 \\ &\quad + (-e^{156} - e^{147} - e^{345} - e^{367}) e_2 \\ &\quad + (e^{245} + e^{267} - e^{146} + e^{157}) e_3 \\ &\quad + (-e^{567} + e^{127} + e^{136} - e^{235}) e_4 \\ &\quad + (e^{126} + e^{467} - e^{137} + e^{234}) e_5 \\ &\quad + (-e^{457} - e^{125} - e^{134} - e^{237}) e_6 \\ &\quad + (e^{135} - e^{124} + e^{456} + e^{236}) e_7 \end{aligned}$$

Also χ can be expressed in terms of cross product operation (c.f. [H], [HL], [Ka]):

$$(12) \quad \chi(u, v, w) = -u \times (v \times w) - \langle u, v \rangle w + \langle u, w \rangle v$$

When $d\varphi = 0$, the associative submanifolds are volume minimizing submanifolds of M (calibrated by φ). Even in the general case of a manifold with a G_2 structure (M, φ) , the form χ imposes an interesting structure near associative submanifolds:

Notice (9) implies that, χ maps every oriented 3-plane in $T_x(M)$ to the orthogonal subspace $T_x(M)^\perp$, so if we choose local coordinates (x_1, \dots, x_7) for M^7 we get

$$(13) \quad \chi = \sum a_J^\alpha dx^J \otimes \frac{\partial}{\partial x_\alpha}$$

where $dx^J = dx^i \wedge dx^j \wedge dx^k$, and the summation is taken over the multi-index $J = \{i, j, k\}$ and α such that $\alpha \notin J$. So if $Y \subset M$ is given by (x_1, x_2, x_3) coordinates, then locally the condition Y to be associative is given by the equations:

$$(14) \quad a_{123}^\alpha = 0$$

From (9) it is easy to calculate $a_{ijk}^\alpha = *\varphi_{ijks}g^{s\alpha}$, where $g^{-1} = (g^{ij})$ is the inverse of the metric $g = (g_{ij})$, and of course the metric g can be expressed in terms of φ . By evaluating χ on the orientation form of Y we get a normal vector field so:

Lemma 1. *To any 3-dimensional submanifold $Y^3 \subset (M, \varphi)$, χ associates a normal vector field, which vanishes when Y is associative.*

Hence χ defines an interesting flow on 3 dimensional submanifolds of (M, φ) , fixing associative submanifolds. On the associative submanifolds with a $Spin^c$ structure, χ rotates their normal bundles and imposes a complex structure on them:

Lemma 2. *To any associative manifold $Y^3 \subset (M, \varphi)$ with a non-vanishing oriented 2-plane field, χ defines an almost complex structure on its normal bundle $\nu(Y)$ (notice that in particular any coassociative submanifold $X \subset M$ has an almost complex structure if its normal bundle has a non-vanishing section).*

Proof. Let $L \subset \mathbb{R}^7$ be an associative 3-plane, that is $\varphi|_L = vol(L)$. Then to every pair of orthonormal vectors $\{u, v\} \subset L$, the form χ defines a complex structure on the orthogonal 4-plane L^\perp , as follows: Define $j : L^\perp \rightarrow L^\perp$ by

$$(15) \quad j(X) = \chi(u, v, X)$$

This is well defined i.e. $j(X) \in L^\perp$, because when $w \in L$ we have:

$$\langle \chi(u, v, X), w \rangle = *\varphi(u, v, X, w) = -*\varphi(u, v, w, X) = \langle \chi(u, v, w), X \rangle = 0$$

Also $j^2(X) = j(\chi(u, v, X)) = \chi(u, v, \chi(u, v, X)) = -X$. We can check the last equality by taking an orthonormal basis $\{X_j\} \subset L^\perp$ and calculating

$$\begin{aligned} \langle \chi(u, v, \chi(u, v, X_i)), X_j \rangle &= *\varphi(u, v, \chi(u, v, X_i), X_j) = \\ -*\varphi(u, v, X_j, \chi(u, v, X_i)) &= -\langle \chi(u, v, X_j), \chi(u, v, X_i) \rangle = -\delta_{ij} \end{aligned}$$

The last equality holds since the map j is orthogonal, and the orthogonality can be seen by polarizing the associator equality (10), and by noticing $\varphi(u, v, X_i) = 0$. Observe that the map j only depends on the oriented 2-plane $l = \langle u, v \rangle$ generated by $\{u, v\}$. So the result follows. \square

In fact, for any unit vector field ξ on an associative Y (i.e. a $Spin^c$ structure) defines a complex structure $J_\xi : \nu(Y) \rightarrow \nu(Y)$ by $J_\xi(z) = z \times \xi$, and the complex structure defined in Lemma 2 corresponds to $J_{u \times v}$, because from (12):

$$\chi(u, v, z) = \chi(z, u, v) = -z \times (u \times v) - \langle z, u \rangle v + \langle z, v \rangle u = J_{v \times u}(z).$$

Also recall that the complex structures on any $SO(4)$ bundle such as $\nu \rightarrow Y$ are given by the unit sections of the associated $SO(3)$ bundle $\lambda_+(\nu) \rightarrow Y$, which is induced by the left reductions $SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2 \rightarrow SU(2)/\mathbb{Z}_2 = SO(3)$.

Definition 4. A Riemannian 8-manifold (N^8, g) is called a $Spin(7)$ manifold if the holonomy group of its Levi-Civita connection lies in $Spin(7) \subset GL(8, \mathbb{R})$.

Equivalently a $Spin(7)$ manifold (N, Ψ) is a Riemannian 8-manifold with a triple cross product \times on its tangent bundle, and a harmonic 4-form $\Psi \in \Omega^4(N)$ with

$$\Psi(u, v, w, z) = g(u \times v \times w, z)$$

It is easily checked that if (M, φ) is a G_2 manifold, then $(M \times S^1, \Psi)$ is a $Spin(7)$ manifold where $\Psi = \varphi \wedge dt - *\varphi$.

Definition 5. A 4-dimensional submanifold X of a $Spin(7)$ manifold (N, Ψ) is called Cayley if $\Psi|_X \equiv \text{vol}(X)$. This is equivalent to $\tau|_X \equiv 0$ where $\tau \in \Omega^4(N, E)$ is a certain vector-bundle valued 4-form defined by the “four-fold cross product” of the imaginary octonions $\tau(v_1, v_2, v_3, v_4) = v_1 \times v_2 \times v_3 \times v_4$ (see [M], [HL]).

2. GRASSMANN BUNDLES

Let $G(3, 7)$ be the Grassmann manifold of oriented 3-planes in \mathbb{R}^7 . Let M^7 be an oriented smooth 7-manifold, and let $\tilde{M} \rightarrow M$ be the bundle oriented 3-planes in TM , which is defined by the identification $[p, L] = [pg, g^{-1}L] \in \tilde{M}$:

$$(16) \quad \tilde{M} = \mathcal{P}_{SO(7)}(M) \times_{SO(7)} G(3, 7) \rightarrow M.$$

This is just the bundle $\tilde{M} = \mathcal{P}_{SO(7)}(M)/SO(3) \times SO(4) \rightarrow \mathcal{P}_{SO(7)}(M)/SO(7) = M$. Let $\xi \rightarrow G(3, 7)$ be the universal \mathbb{R}^3 bundle, and $\nu = \xi^\perp \rightarrow G(3, 7)$ be the dual \mathbb{R}^4 bundle. Therefore, $Hom(\xi, \nu) = \xi^* \otimes \nu \rightarrow G(3, 7)$ is the tangent bundle $TG(3, 7)$. ξ, ν extend fiberwise to give bundles $\Xi \rightarrow \tilde{M}, \mathbb{V} \rightarrow \tilde{M}$ respectively, and let Ξ^* be the dual of Ξ . Notice that $Hom(\Xi, \mathbb{V}) = \Xi^* \otimes \mathbb{V} \rightarrow \tilde{M}$ is the bundle of vertical vectors $T^v(\tilde{M})$ of $T(\tilde{M}) \rightarrow M$, i.e. the tangents to the fibers of $\pi: \tilde{M} \rightarrow M$, hence

$$(17) \quad T\tilde{M} \cong T^v(\tilde{M}) \oplus \pi^*TM = (\Xi^* \otimes \mathbb{V}) \oplus \Xi \oplus \mathbb{V}.$$

That is, $T\tilde{M}$ is the vector bundle associated to principal $SO(3) \times SO(4)$ bundle $\mathcal{P}_{SO(7)} \rightarrow \tilde{M}$ by the obvious representation of $SO(3) \times SO(4)$ to $(\mathbb{R}^3)^* \otimes \mathbb{R}^4 + \mathbb{R}^3 + \mathbb{R}^4$. The identification (17) is defined up to gauge automorphisms of bundles Ξ and \mathbb{V} .

Note that the bundle $\mathbb{V} = \Xi^\perp$ depends on the metric, and hence it depends on φ when metric is induced from a G_2 structure (M, φ) . To emphasize this fact we can denote it by $\mathbb{V}_\varphi \rightarrow \tilde{M}$. But when we are considering G_2 structures coming from G_2 subgroups of a fixed copy of $SO(7) \subset GL(7, \mathbb{R})$, they induce the same metric and so this distinction is not necessary.

Let $\mathcal{P}(\mathbb{V}) \rightarrow \tilde{M}$ be the $SO(4)$ frame bundle of the vector bundle \mathbb{V} , identify \mathbb{R}^4 with the quaternions \mathbb{H} , and identify $SU(2)$ with the unit quaternions $Sp(1) = S^3$. Recall that $SO(4)$ is the equivalence classes of pairs $[q, \lambda]$ of unit quaternions

$$SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2$$

Hence $\mathbb{V} \rightarrow \tilde{M}$ is the associated vector bundle to $\mathcal{P}(\mathbb{V})$ via the $SO(4)$ representation

$$(18) \quad x \mapsto qx\lambda^{-1}$$

There is a pair of $\mathbb{R}^3 = im(\mathbb{H})$ bundles over \tilde{M} corresponding to the left and right $SO(3)$ reductions of $SO(4)$, which are given by the $SO(3)$ representations

$$(19) \quad \begin{aligned} \lambda_+(\mathbb{V}) &: x \mapsto qxq^{-1} \\ \lambda_-(\mathbb{V}) &: y \mapsto \lambda y \lambda^{-1} \end{aligned}$$

The map $x \otimes y \mapsto xy$ gives actions $\lambda_+(\mathbb{V}) \otimes \mathbb{V} \rightarrow \mathbb{V}$ and $\mathbb{V} \otimes \lambda_-(\mathbb{V}) \rightarrow \mathbb{V}$; by combining we can think of them as one conjugation action

$$(20) \quad (\lambda_+(\mathbb{V}) \otimes \lambda_-(\mathbb{V})) \otimes \mathbb{V} \rightarrow \mathbb{V}$$

If the $SO(4)$ bundle $\mathcal{P}(\mathbb{V}) \rightarrow \tilde{M}$ lifts to a $Spin(4) = SU(2) \times SU(2)$ bundle (locally it does), we get two additional bundles over \tilde{M}

$$(21) \quad \begin{aligned} \mathcal{S} &: y \mapsto qy \\ \mathbb{E} &: y \mapsto y\lambda^{-1} \end{aligned}$$

They identify \mathbb{V} as a tensor product of two quaternionic line bundles $\mathbb{V} = \mathcal{S} \otimes_{\mathbb{H}} \mathbb{E}$. In particular, $\lambda_+(\mathbb{V}) = ad(\mathcal{S})$ and $\lambda_-(\mathbb{V}) = ad(\mathbb{E})$, i.e. they are the $SO(3)$ reductions of the $SU(2)$ bundles \mathcal{S} and \mathbb{E} . Also there is a multiplication map $\mathcal{S} \otimes \mathbb{E} \rightarrow \mathbb{V}$. Recall the identifications: $\Lambda^2(\mathbb{V}) = \Lambda_+^2(\mathbb{V}) \oplus \Lambda_-^2(\mathbb{V}) = \lambda_-(\mathbb{V}) \oplus \lambda_+(\mathbb{V}) = \lambda(\mathbb{V}) = gl(\mathbb{V}) = ad(\mathbb{V})$.

2.1. Associative Grassmann Bundles.

Now consider the *Grassmannian of associative 3-planes* $G^\varphi(3,7)$ in \mathbb{R}^7 , consisting of elements $L \in G(3,7)$ with the property $\varphi_0|_L = vol(L)$ (or equivalently $\chi_0|_L = 0$). G_2 acts on $G^\varphi(3,7)$ transitively with the stabilizer $SO(4)$, so it gives the identification $G^\varphi(3,7) = G_2/SO(4)$. If we identify the imaginary octonions by $\mathbb{R}^7 = Im(\mathbb{O}) \cong im(\mathbb{H}) \oplus \mathbb{H}$, then the action of the subgroup $SO(4) \subset G_2$ on \mathbb{R}^7 is

$$(22) \quad \begin{pmatrix} \rho(A) & 0 \\ 0 & A \end{pmatrix}$$

where $\rho : SO(4) = (SU(2) \times SU(2))/\mathbb{Z}_2 \rightarrow SO(3)$ is the projection of the first factor ([HL]), that is for $[q, \lambda] \in SO(4)$ the action is given by $(x, y) \mapsto (qxq^{-1}, qy\lambda^{-1})$. So the action of $SO(4)$ on the 3-plane $L = im(\mathbb{H})$ is determined by its action on L^\perp . Now let M^7 be a G_2 manifold. Similar to the construction before, we can construct the bundle of associative Grassmannians over M (which is a submanifold of \tilde{M}):

$$(23) \quad \tilde{M}_\varphi = \mathcal{P}_{G_2}(M) \times_{G_2} G^\varphi(3,7) \rightarrow M$$

which is just the quotient bundle $\tilde{M}_\varphi = \mathcal{P}_{G_2}(M)/SO(4) \longrightarrow \mathcal{P}_{G_2}(M)/G_2 = M$. As in the previous section, the restriction of the universal bundles ξ , $\nu = \xi^\perp \rightarrow G^\nu(3, 7)$ induce 3 and 4 plane bundles $\Xi \rightarrow \tilde{M}_\varphi$ and $\mathbb{V} \rightarrow \tilde{M}_\varphi$ (by restricting from \tilde{M}). Also

$$(24) \quad T\tilde{M}_\varphi \cong T^v(\tilde{M}_\varphi) \oplus \Xi \oplus \mathbb{V}$$

From (22) we see that in the associative case, we have an important identification: $\Xi = \lambda_+(\mathbb{V})$ (as bundles over \tilde{M}_φ), and the dual of the action $\lambda_+(\mathbb{V}) \otimes \mathbb{V} \rightarrow \mathbb{V}$ gives a Clifford multiplication:

$$(25) \quad \Xi^* \otimes \mathbb{V} \rightarrow \mathbb{V}$$

In fact this is just the map induced from the cross product operation [AS2]. Recall that $T^v(\tilde{M}) = \Xi^* \otimes \mathbb{V} \rightarrow \tilde{M}$ is the subbundle of vertical vectors of $T(\tilde{M}) \rightarrow M$. The total space $E(\nu_\varphi)$ of the normal bundle of the imbedding $\tilde{M}_\varphi \subset \tilde{M}$ should be thought of an open tubular neighborhood of \tilde{M}_φ in \tilde{M} , and it has a nice description:

Lemma 3. ([M]) *Normal bundle ν_φ of $\tilde{M}_\varphi \subset \tilde{M}$ is isomorphic to \mathbb{V} , and the bundle of vertical vectors $T^v(\tilde{M}_\varphi)$ is the kernel of the Clifford multiplication $c : \Xi^* \otimes \mathbb{V} \rightarrow \mathbb{V}$. We have $T^v(\tilde{M})|_{\tilde{M}_\varphi} = T^v(\tilde{M}_\varphi) \oplus \nu_\varphi$, and the following exact sequence over \tilde{M}_φ*

$$T^v(\tilde{M}_\varphi) \rightarrow \Xi^* \otimes \mathbb{V}|_{\tilde{M}_\varphi} \xrightarrow{c} \mathbb{V}|_{\tilde{M}_\varphi} \rightarrow 0$$

Hence the quotient bundle, $T^v(\tilde{M})/T^v(\tilde{M}_\varphi)$ is isomorphic to \mathbb{V} .

Proof. This is because the Lie algebra inclusion $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ is given by

$$\begin{pmatrix} a & \beta \\ -\beta^t & \rho(a) \end{pmatrix}$$

where $a \in \mathfrak{so}(4)$ is $y \mapsto qy - yq$, and $\rho(a) \in \mathfrak{so}(3)$ is $x \mapsto qx - xq$. So the tangent space inclusion of $G_2/SO(4) \subset SO(7)/SO(4) \times SO(3)$ is given by the matrix $\beta \in (\text{im}\mathbb{H})^* \otimes \mathbb{H}$. Therefore, if we write β as column vectors of three quaternions $\beta = (\beta_1, \beta_2, \beta_3) = i^* \otimes \beta_1 + j^* \otimes \beta_2 + k^* \otimes \beta_3$, then $\beta_1 i + \beta_2 j + \beta_3 k = 0$ ([M]). \square

The reader can consult Lemma 5 of [AS2] for a more self contained proof of this fact, where the Clifford multiplication is identified with the cross product operation.

3. ASSOCIATIVE SUBMANIFOLDS

Any imbedding of a 3-manifold $f : Y^3 \hookrightarrow M^7$ induces an imbedding $\tilde{f} : Y \hookrightarrow \tilde{M}$:

$$(26) \quad \begin{array}{ccc} & & \tilde{M} \supset \tilde{M}_\varphi \\ \tilde{f} & \nearrow & \downarrow \\ Y & \xrightarrow{f} & M \end{array}$$

and the pull-backs $\tilde{f}^*\Xi = T(Y)$ and $\tilde{f}^*\mathbb{V} = \nu(Y)$ give the tangent and normal bundles of Y . Furthermore, if f is an imbedding of an associative submanifold into a G_2 manifold (M, φ) , then the image of \tilde{f} lands in \tilde{M}_φ . We will denote this canonical lifting of any 3-manifold $Y \subset M$ by $\tilde{Y} \subset \tilde{M}$. Also since we have the dependency $\mathbb{V} = \mathbb{V}_\varphi$, we can denote $\nu(Y) = \nu(Y)_\varphi = \nu_\varphi$ when needed.

\tilde{M}_φ can be thought of as a universal space parameterizing associative submanifolds of M . In particular, if $\tilde{f} : Y \hookrightarrow \tilde{M}_\varphi$ is the lifting of an associative submanifold, by pulling back we see that the principal $SO(4)$ bundle $\mathcal{P}(\mathbb{V}) \rightarrow \tilde{M}_\varphi$ induces an $SO(4)$ -bundle $\mathcal{P}(Y) \rightarrow Y$, and gives the following vector bundles via the representations:

$$(27) \quad \begin{aligned} \nu(Y) & : y \mapsto qy\lambda^{-1} \\ T(Y) & : x \mapsto qxq^{-1} \end{aligned}$$

where $[q, \lambda] \in SO(4)$, $\nu = \nu(Y)$ and $T(Y) = \lambda_+(\nu)$. Also we can identify T^*Y with TY by the induced metric. From above we have the action $T^*Y \otimes \nu \rightarrow \nu$ inducing actions $\Lambda^*(T^*Y) \otimes \nu \rightarrow \nu$.

Let $\mathbb{L} = \Lambda^3(\Xi) \rightarrow \tilde{M}$ be the determinant (real) line bundle. Recall that the definition (9) implies that χ maps every oriented 3-plane in $T_x(M)$ to its complementary subspace, so χ gives a bundle map $\mathbb{L} \rightarrow \mathbb{V}$ over \tilde{M} , which is a section of $\mathbb{L}^* \otimes \mathbb{V} \rightarrow \tilde{M}$. Since Ξ is oriented \mathbb{L} is trivial, so χ actually gives a section

$$(28) \quad \chi = \chi_\varphi \in \Omega^0(\tilde{M}, \mathbb{V})$$

Clearly $\tilde{M}_\varphi \subset \tilde{M}$ is the codimension 4 submanifold which is the zeros of this section. Associative submanifolds $Y \subset M$ are characterized by the condition $\chi|_{\tilde{Y}} = 0$, where $\tilde{Y} \subset \tilde{M}$ is the canonical lifting of Y . Similarly φ defines a map $\varphi : \tilde{M} \rightarrow \mathbb{R}$.

3.1. Pseudo-associative submanifolds.

Here we generalize associative submanifolds to a more flexible class of submanifolds. To do this we first generalize the notion of imbedded submanifolds.

Definition 6. *A Grassmann-framed 3-manifold in (M, φ) is a triple (Y^3, f, F) , where $f : Y \hookrightarrow M$ is an imbedding, $F : Y \rightarrow \tilde{M}$, such that the following commute*

$$(29) \quad \begin{array}{ccc} & & \tilde{M} \\ & F \nearrow & \downarrow \\ Y & \xrightarrow{f} & M \end{array}$$

We call (Y, f, F) a pseudo-associative submanifold if in addition $\text{Image}(F) \subset \tilde{M}_\varphi$. So a pseudo-associative submanifold (Y, f, F) with $F = \tilde{f}$ is associative.

Remark 2. The bundle $\tilde{M} \rightarrow M$ always admits a section, in fact the subbundle $\tilde{M}_\varphi \rightarrow M$ has a section. This is because by [T] every orientable 7-manifold admits a non-vanishing linearly independent 2-frame field $\Lambda = \{v_1, v_2\}$ ¹. By Gram-Schmidt process with metric g_φ , we can assume that Λ is orthonormal. The cross product assigns Λ to an orthonormal 3-frame field $\{v_1, v_2, v_1 \times_\varphi v_2\}$ on M , then 3-plane generated by $\{v_1, v_2, v_1 \times_\varphi v_2\} := \langle v_1, v_2, v_1 \times_\varphi v_2 \rangle$ gives a section of $\lambda_\varphi : M \rightarrow \tilde{M}_\varphi$. Let

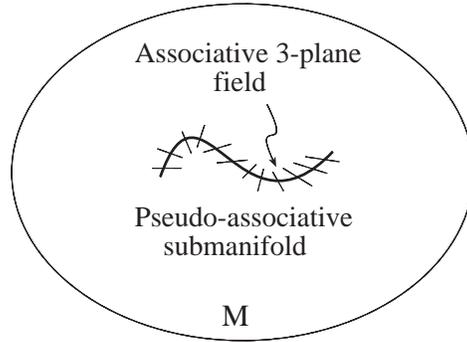


FIGURE 1

$\mathcal{Z}(M)$ and $\mathcal{Z}_\varphi(M)$ denote the set of Grassmann-framed and the pseudo-associative submanifolds, respectively, and let $\mathcal{A}_\varphi(M)$ be the set of associative submanifolds. We have inclusions $\mathcal{A}_\varphi(M) \hookrightarrow \mathcal{Z}_\varphi(M) \hookrightarrow \mathcal{Z}(M)$, where the first map is given by $(Y, f) \mapsto (Y, f, \tilde{f})$. So there is an inclusion $Im(Y, M) \hookrightarrow \mathcal{Z}(M)$, where $Im(Y, M)$ is the space of imbeddings. This inclusion can be thought of the canonical sections of a bundle

$$(30) \quad \mathcal{Z}(Y) \xrightarrow{\pi} Im(Y, M)$$

with fibers $\pi^{-1}(f) = \Omega^0(Y, f^* \tilde{M})$. We also have the subbundle $\mathcal{Z}_\varphi(Y) \xrightarrow{\pi} Im(Y, M)$ with fibers $\pi^{-1}(f) = \Omega^0(Y, f^* \tilde{M}_\varphi)$. So $\mathcal{Z}(Y)$ is the set of triples (Y, f, F) (in short just set of F 's), where $F : Y \rightarrow \tilde{M}$ is a lifting of the imbedding $f : Y \hookrightarrow M$. Also $\mathcal{Z}_\varphi(Y) \subset \mathcal{Z}(Y)$ is a smooth submanifold, since $\tilde{M}_\varphi \subset \tilde{M}$ is smooth. There is the canonical section $\Phi : Im(Y, M) \rightarrow \mathcal{Z}(Y)$ given by $\Phi(f) = \tilde{f}$. Therefore, $\Phi^{-1} \mathcal{Z}_\varphi(Y) := Im_\varphi(Y, M)$ is the set of associative imbeddings $Y \subset M$. Also, any 2-frame field Λ as above gives to a section $\Phi_\Lambda(f) = \lambda_\varphi \circ f$. To make these definitions parameter free we also have to divide $Im(Y, M)$ by the diffeomorphism group of Y .

¹We thank T.Onder for pointing out [T]

There are also the vertical tangent bundles of $\mathcal{Z}(Y)$ and $\mathcal{Z}_\varphi(Y)$

$$\begin{array}{ccc} T^v \mathcal{Z}(Y) & \xrightarrow{\pi} & \mathcal{Z}(Y) \\ \cup & & \cup \\ T^v \mathcal{Z}_\varphi(Y) & \xrightarrow{\pi|} & \mathcal{Z}_\varphi(Y) \end{array}$$

with fibers $\pi^{-1}(F) = \Omega^0(Y, F^*(\Xi^* \otimes \mathbb{V}))$. By Lemma 3 the fibers of $T^v(\mathcal{Z}_\varphi)$ can be identified with the kernel of the map induced by the Clifford multiplication

$$(31) \quad c : \Omega^0(Y, F^*(\Xi^* \otimes \mathbb{V})) \rightarrow \Omega^0(Y, F^*(\mathbb{V}))$$

One of the nice properties of a pseudo-associative submanifold (Y, f, F) is that there is a Clifford multiplication action (by pull back)

$$(32) \quad F^*(\Xi^*) \otimes F^*(\mathbb{V}) \rightarrow F^*(\mathbb{V})$$

If F is close to \tilde{f} , by parallel translating the fibers over $F(x)$ and $\tilde{f}(x)$ along geodesics in \tilde{M} we get canonical identifications:

$$(33) \quad F^*(\Xi) \cong TY \quad F^*(\mathbb{V}) \cong \nu_f$$

inducing Clifford multiplication between the tangent and the normal bundles. So if $\forall x \in Y$ the distance between $F(x)$ and $\tilde{f}(x)$ is less than the injectivity radius $j(\tilde{M})$, there is a Clifford multiplication between the tangent and normal bundles of Y .

3.2. Dirac operator.

The normal bundle $\nu = \nu(Y)$ of any orientable 3-manifold Y in a G_2 manifold (M, φ) has a $Spin(4)$ structure (e.g. [B2]). Hence we have $SU(2)$ bundles S and E over Y such that $\nu = S \otimes_{\mathbb{H}} E$ (18), with $SO(3)$ reductions $adS = \lambda_+(\nu)$, and $adE = \lambda_-(\nu)$ which is also the bundle of endomorphisms $End(E)$. If Y is associative, then the bundle $ad(S)$ becomes isomorphic to TY , i.e. S becomes the spinor bundle of Y , so $\nu(Y)$ becomes a twisted spinor bundle.

The Levi-Civita connection of the G_2 metric of (M, φ) induces connections on the associated bundles \mathbb{V} and Ξ on \tilde{M} . In particular, it induces connections on the tangent and normal bundles of any submanifold $Y^3 \subset M$. We will call these connections the *background connections*. Let \mathbb{A}_0 be the induced connection on the normal bundle $\nu = S \otimes E$. From the Lie algebra decomposition $so(4) = so(3) \oplus so(3)$, we can write $\mathbb{A}_0 = B_0 \oplus A_0$, where B_0 and A_0 are connections on S and E , respectively.

Let $\mathcal{A}(E)$ and $\mathcal{A}(S)$ be the set of connections on the bundles E and S . Hence $A \in \mathcal{A}(E)$, $B \in \mathcal{A}(S)$ are in the form $A = A_0 + a$, $B = B_0 + b$, where $a \in \Omega^1(Y, ad E)$ and $b \in \Omega^1(Y, ad S)$. So $\Omega^1(Y, \lambda_{\pm}(\nu))$ parametrizes connections on S and E , and the connections on ν are in the form $\mathbb{A} = B \oplus A$. To emphasize the dependency on b and a we sometimes denote $\mathbb{A} = \mathbb{A}(b, a)$, and $\mathbb{A}_0 = \mathbb{A}(0, 0) = A_0$.

Now, let $Y^3 \subset M$ be any smooth manifold. We can express the covariant derivative $\nabla_{\mathbb{A}} : \Omega^0(Y, \nu) \rightarrow \Omega^1(Y, \nu)$ on ν by $\nabla_A = \sum e^i \otimes \nabla_{e_i}$, where $\{e_i\}$ and $\{e^i\}$ are orthonormal tangent and cotangent frame fields of Y , respectively. Furthermore, if Y is an associative submanifold, we can use the Clifford multiplication of (25) (i.e. the cross product) to form the twisted Dirac operator $\mathcal{D}_{\mathbb{A}} : \Omega^0(Y, \nu) \rightarrow \Omega^0(Y, \nu)$

$$(34) \quad \mathcal{D}_{\mathbb{A}} = \sum e^i \cdot \nabla_{e_i}$$

The sections lying in the kernel of this operator are usually called harmonic spinors twisted by (E, \mathbb{A}) . Elements of the kernel of \mathcal{D}_{A_0} are called the harmonic spinors twisted by E , or just the twisted harmonic spinors.

4. DEFORMATIONS

In [M], McLean showed that the space of associative submanifolds of a G_2 manifold (M, φ) , in a neighborhood of a fixed associative submanifold Y , can be identified with the harmonic spinors on Y twisted by E . Since the cokernel of the Dirac operator can vary, the dimension of its kernel is not determined (it has zero index since Y is odd dimensional). We will remedy this problem by deforming Y in a larger class of submanifolds. To motivate our approach we will first sketch a proof of McLean's theorem (adapting the explanation in [B3]). Let $Y \subset M$ be an associative submanifold, Y will determine a lifting $\tilde{Y} \subset \tilde{M}_{\varphi}$. Let us recall that the G_2 structure φ gives a metric connection on M , hence it gives a connection A_0 and a covariant differentiation in the normal bundle $\nu(Y) = \nu$

$$\nabla_{A_0} : \Omega^0(Y, \nu) \rightarrow \Omega^1(Y, \nu) = \Omega^0(Y, T^*Y \otimes \nu)$$

Recall that we identified $T_y^*(Y) \otimes \nu_y(Y)$ by the tangent space of the Grassmannian of 3-planes $TG(3, 7)$ in $T_y(M)$. So the covariant derivative lifts normal vector fields v of $Y \subset M$ to vertical vector fields \tilde{v} in $T(\tilde{M})|_{\tilde{Y}}$. We want the normal vector fields v of Y to move Y in the class of associative submanifolds of M , i.e. we want the liftings \tilde{Y}_v of the nearby copies Y_v of Y (pushed off by the vector field v) to lie in $\tilde{M}_{\varphi} \subset \tilde{M}$ upstairs, i.e. we want the component of \tilde{v} in the direction of the normal bundle $\tilde{M}_{\varphi} \subset \tilde{M}$ to vanish. By Lemma 3, this means $\nabla_{A_0}(v)$ should be in the kernel of the Clifford multiplication $c = c_{\varphi} : \Omega^0(T^*(Y) \otimes \nu) \rightarrow \Omega^0(\nu)$, i.e. $\mathcal{D}_{A_0}(v) = c(\nabla_{A_0}(v)) = 0$, where \mathcal{D}_{A_0} is the Dirac operator induced by the background connection A_0 , i.e. the composition

$$(35) \quad \Omega^0(Y, \nu) \xrightarrow{\nabla_{A_0}} \Omega^0(Y, T^*Y \otimes \nu) \xrightarrow{c} \Omega^0(Y, \nu)$$

The condition $\mathcal{D}_{A_0}(v) = 0$ implies φ must be integrable at Y , i.e. the $so(7)$ -metric connection ∇_{A_0} on Y coincides with G_2 -connection (c.f. [B2]).

Now we give a general version of the McLean's theorem, without integrability assumption on φ : Recall from (Section 3.1) that $\Phi^{-1}\mathcal{Z}_{\varphi}(Y)$ is the set of associative

submanifolds $Y \subset M$, where $\Phi : Im(Y, M) \rightarrow \mathcal{Z}(Y)$ is the canonical section (Gauss map) given by $\Phi(f) = \tilde{f}$. Therefore, if $f : Y \hookrightarrow M$ is the above inclusion, then $\Phi(f) \in \mathcal{Z}_\varphi$. So this moduli space is smooth if Φ was transversal to $\mathcal{Z}_\varphi(Y)$.

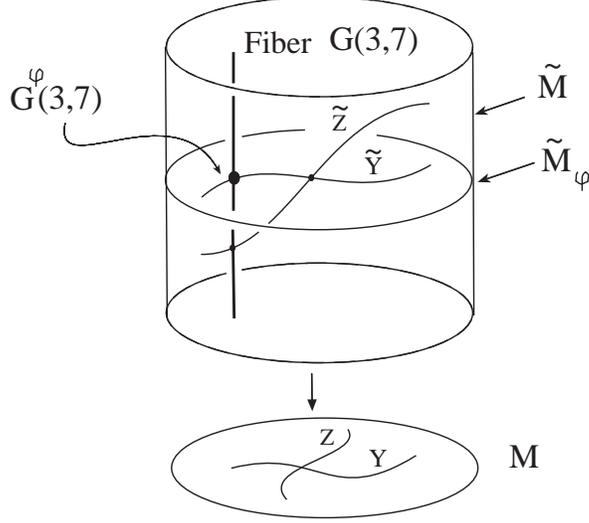


FIGURE 2

Theorem 4. *Let (M^7, φ) be a manifold with a G_2 structure, and $Y^3 \subset M$ be an associative submanifold. Then the tangent space of associative submanifolds of M at Y can be identified with the kernel of a Dirac operator $\mathcal{D}_A : \Omega^0(Y, \nu) \rightarrow \Omega^0(Y, \nu)$, where $A = A_0 + a$, and A_0 is the connection on ν induced by the metric g_φ , and $a \in \Omega^1(Y, ad(\nu))$. In the case φ is integrable $a = 0$. In particular, the space of associative submanifolds of M is smooth at Y if the cokernel of \mathcal{D}_A is zero.*

Proof. Let $f : Y \hookrightarrow M$ denote the imbedding. We consider unparameterized deformations of Y in $Im(Y, M)$ along its normal directions. Fix a trivialization $TY \cong im(\mathbb{H})$, by (17) we have an identification $\tilde{f}^*(T^v \tilde{M}) \cong TY^* \otimes \nu + TY + \nu$. We first claim $\Pi \circ d\Phi(v) = \nabla_A(v)$, where $d\Phi$ is the induced map on the tangent space and Π is the vertical projection.

$$\begin{array}{ccc}
 \Omega^0(Y, \nu) = T_f Im(Y, M) & \xrightarrow{d\Phi} & T_{\tilde{f}} \mathcal{Z}(Y) = \Omega^0(Y, \tilde{f}^*(T^v \tilde{M})) \xrightarrow{\Pi} \Omega^0(Y, T^*Y \otimes \nu) \\
 \downarrow exp & & \downarrow exp \\
 Im(Y, M) & \xrightarrow{\Phi} & \mathcal{Z}(Y)
 \end{array}$$

The two vertical maps $v \rightarrow f_v$, and $w \rightarrow (\tilde{f})_w$ are exponential projections of tangent vectors, i.e. $f_v(y) = \exp_{f(y)}(v)$ and $(\tilde{f})_w(y) = \exp_{\tilde{f}(y)}(w)$. It suffices to check this claim pointwise. Here for convenience view f as an inclusion $Y \subset M$.

Let $y = (y_1, y_2, y_3)$ be the normal coordinates of Y centered around y_0 , and $\{e_j\}_{j=1}^3$ be an orthonormal frame field of M defined on Y , with $e_j(y_0) = \partial/\partial y_j$ for $j = 1, 2, 3$. To this data we can associate *Fermi coordinates* (y, t) around $f(y_0) \in M$ (they are a version of normal coordinates along a submanifold, see for example [G]):

$$(36) \quad (y, t) \longleftrightarrow f_{\sum t_\alpha e_\alpha}(y)$$

where $t = (t_1, \dots, t_3)$. Then we can write $\tilde{f}(y_0) = e_1 \wedge e_2 \wedge e_3$. Hence by definition we can express $d\Phi(v) = \widetilde{(f_v)_*} = (f_v)_*(e_1) \wedge (f_v)_*(e_2) \wedge (f_v)_*(e_3) := e_1(v) \wedge e_2(v) \wedge e_3(v)$.

$$(37) \quad d\Phi(v)(y_0) = \mathcal{L}_v(e_1 \wedge e_2 \wedge e_3) = \sum_{i=1}^3 (*e_i) \wedge \mathcal{L}_v(e_i)|_Y$$

where \mathcal{L}_v denotes Lie derivative along v , and $*$ is the star of Y . The metric connection is torsion free hence $\mathcal{L}_v(e_j) = \bar{\nabla}_{e_j}(v) - \bar{\nabla}_v(e_j)$, where $\bar{\nabla}$ is the metric connection of M . In case (M, φ) is a G_2 manifold (i.e. when φ integrable), by (2) and (7), up to quadratic term φ is φ_0 , therefore we can write:

$$0 = \bar{\nabla}_v(\varphi)|_Y = \bar{\nabla}_v(e^1 \wedge e^2 \wedge e^3)|_Y = \sum_{j=1}^3 (*e^j) \wedge \bar{\nabla}_v(e^j)|_Y, \quad \text{which implies}$$

$$(38) \quad \Pi \circ d\Phi(v)(y_0) = \sum_j (*e_j) \wedge \nabla_{e_j}(v)$$

where where $\{e^j\}$ is the dual coframe, and $\nabla_{e_j}(v)$ is the normal component of $\bar{\nabla}_{e_j}(v)$, i.e. it is the induced connection on $\nu(Y)$. The expression (38) can be viewed as an infinitesimal deformation of the 3-plane $\tilde{f}(y_0)$. By the identification $*e_j \leftrightarrow e^j$ we can view it as an element of the tangent space $T^*Y \otimes \nu$ of the Grassmannian of 3-planes in $T_{y_0}(M)$

$$(39) \quad \Pi \circ d\Phi(v)(y_0) = \sum e^j \otimes \nabla_{e_j}(v)(y_0) = \nabla_{A_0}(v)(y_0)$$

When φ is not integrable, there is an extra term which we can write

$$\sum (*e_j) \wedge \bar{\nabla}_v(e_j) = \sum (*e_j) \wedge \nabla_v(e_j)$$

where $\nabla_v(e_j)$ is the normal component of $\bar{\nabla}_v(e_j)$. Notice $\langle \bar{\nabla}_v(e_k), e_k \rangle = 0$, which is implied by $v \langle e_k, e_k \rangle = 0$. So in this case (39) becomes

$$(40) \quad \Pi \circ d\Phi(v)(y_0) = \nabla_{A_0}(v) + a(v) = \nabla_A(v)$$

where $a(v) = \sum e^j \otimes \nabla_v(e_j) \in \Omega^1(Y, ad(\nu))$ and $A = A_0 + a$. It easy to check that the expression $a(v)$ is independent of the choice the orthonormal frame $\{e_j\}$.

By (31) the vertical tangent space of $\mathcal{Z}_\varphi(Y)$ is given by the kernel of the Clifford multiplication $c_\varphi : \Omega^0(T^*Y \otimes \nu) \rightarrow \Omega^0(\nu)$. So, locally the moduli space of associative submanifolds of (M, φ) is given by the kernel of \mathcal{D}_A , i.e. the condition that $d\Phi(v)$ lies in $T_f^v \mathcal{Z}_\varphi(Y)$ is given by $\mathcal{D}_A(v) = 0$. The moduli space is smooth if Φ is transversal to $\mathcal{Z}_\varphi(Y)$, i.e. if the cokernel of \mathcal{D}_A is zero. Since $T_f^v \mathcal{Z}(Y) = \Omega^0(T^*Y \otimes \nu)$ and

$$T\mathcal{Z}(Y)/T\mathcal{Z}_\varphi(Y) = T^v\mathcal{Z}(Y)/T^v\mathcal{Z}_\varphi(Y)$$

to check transversality we look at the induced maps, and use $\Pi \circ d\Phi_f(v) = \nabla_A(v)$

$$\Omega^0(\nu) = T_f \text{Im}(Y, M) \xrightarrow{d\Phi} T_f \mathcal{Z}(Y) \xrightarrow{\Pi} T_f^v \mathcal{Z}(Y) \supset T_f^v \mathcal{Z}_\varphi(Y) \quad \square$$

Remark 3. *This theorem can also be proved by generalizing McLean's proof: The condition that an associative $Y \subset M$ remains associative, when moved via the exponential map along a normal vector field $v \in \Omega^0(Y, \nu)$, is $\mathcal{L}_v(\chi)|_Y = 0$. We can choose local coordinates (x_1, \dots, x_7) on M , such that (x_1, x_2, x_3) gives the coordinates of Y . By (13) and (14) $\chi = \sum a_j^\alpha dx^j \otimes \partial/\partial x_\alpha$, with $\alpha \notin J$ and $a_{123}^\alpha|_Y = 0$*

$$(41) \quad \mathcal{L}_v(\chi)|_Y = \sum v(a_{123}^\alpha) \frac{\partial}{\partial x_\alpha} + \sum a_j^\alpha \mathcal{L}_v(dx^j)|_Y \otimes \frac{\partial}{\partial x_\alpha} = 0$$

McLean treated integrable φ case, i.e when (M, φ) is a G_2 manifold. In this case the first term vanishes, and the second becomes $\mathcal{D}_{A_0}(v) \otimes dx^{123}$. But notice that the first term $a(v)$ is linear in v and takes values in $\Omega^0(Y, \nu)$, hence $a \in \Omega^0(Y, \text{ad}(\nu))$. So in the non-integrable case we get a twisted Dirac equation $\mathcal{D}_A(v) = 0$, where $A = A_0 + a$.

If in the proof of Theorem 4 we replace the G_2 structure φ with another G_2 structure ψ inducing the same metric, the identification of the bundle $TY^\perp = \nu$ doesn't change but the Clifford action c_φ changes to another one c_ψ , corresponding to another 4-dimensional Clifford representations of T^*Y . These two representations are conjugate by a gauge automorphism γ of ν .

$$\begin{array}{ccc} \Omega^0(T^*Y \otimes \nu) & \xrightarrow{c_\psi} & \Omega^0(\nu) \\ 1 \otimes \gamma \downarrow & & \gamma \downarrow \\ \Omega^0(T^*Y \otimes \nu) & \xrightarrow{c_\varphi} & \Omega^0(\nu) \end{array}$$

Therefore, if we call the Dirac operator induced by ψ by \mathcal{D}_{A_1} , we can write

$$\gamma(\mathcal{D}_{A_1}(w)) = \sum dy^j \cdot \gamma(\nabla_j(w)) = \sum dy^j \cdot (\nabla_j \gamma(w)) - dy^j \cdot (\nabla_j \gamma)(w)$$

where the dot "." denotes the Clifford product c_φ . So, $\mathcal{D}_{A_1}(w) = 0$ gives a twisted version of the Dirac equation $\mathcal{D}_{A_0}(v) = 0$ where $v = \gamma(w)$, this is because $\gamma(\mathcal{D}_{A_1}(w)) = \mathcal{D}_{A_0+a}(\gamma(w))$, where $a = -\sum dy^j \cdot (\nabla_j \gamma) \gamma^{-1}$. In Theorem 6 we will use the twisting of the Dirac operator, under deformations of φ , to obtain its surjectivity.

5. TRANSVERSALITY

We can make the cokernel of Dirac operator \mathcal{D}_{A_0} zero either by deforming the Gauss map $\Phi : \text{Im}(Y, M) \rightarrow \mathcal{Z}(Y)$, or by deforming the G_2 structure φ . Changing φ can be realized by deforming φ by a gauge transformation: Recall that the G_2 structures φ on M are the sections $\Omega_+^3(M)$ of the bundle (5). Also $GL(7, \mathbb{R})$ conjugates $G_2 = G_2^{\varphi_0}$ to any other G_2 subgroup G_2^φ of $GL(7, \mathbb{R})$ where

$$G_2^\varphi = \{A \in GL(7, \mathbb{R}) \mid A^*\varphi = \varphi\} \xrightarrow{\varphi} SO(7).$$

If we are interested in the G_2 structures inducing the same metric, we replace $GL(7, \mathbb{R})$ with $SO(7)$. $SO(7)$ acts on $G(3, 7)$ permuting submanifolds $G^\varphi(3, 7)$, where $\varphi \in \Omega_+^3(M)$. More generally the gauge group $\mathcal{G}(P)$ of $P = P_{SO(7)} \rightarrow M$ acts on \tilde{M} permuting \tilde{M}_φ 's. Recall that $\mathcal{G}(P) = \{P \xrightarrow{s} P \mid s(pg) = s(p)g\}$, which can be identified with sections $\Omega^0(M; Ad(P))$ of the bundle $Ad(P) \rightarrow M$ (c.f. [AMR]):

$$Ad(P) = P \times_{Ad} SO(7) = \{[p, h] \mid (p, h) \sim (pg, g^{-1}hg)\}.$$

$$\text{Hence : } \mathcal{G}(P) = \{s : P \rightarrow SO(7) \mid s(pg) = g^{-1}s(p)g\}.$$

The tangent space of $\mathcal{G}(P)$ at the identity I are the sections $\mathfrak{g}(P) = \Omega^0(M, ad(P))$ of the associated bundle of Lie algebras $ad(P) = P \times_{Ad} so(7) \rightarrow M$.

We can identify $T_s(\mathcal{G}_P(M)) \xrightarrow{\cong} \mathfrak{g}(P)$, by $s \mapsto s^{-1}ds$. There is also an action $\mathcal{G}(P) \times \tilde{M} \rightarrow \tilde{M}$ given by $(s, [p, L]) \rightarrow s[p, L] := [ps(p), s(p)L]$, which we will simply denote it by $(s, L) \mapsto s.L$. There is the pull-back action $\mathcal{G}(P) \times \Omega_+^3(M) \rightarrow \Omega_+^3(M)$ given by $(s, \varphi) \rightarrow s^*(\varphi)$. In particular, $s\tilde{M}_\varphi = \tilde{M}_{s^*\varphi}$. Put another way, if $\chi = \chi_\varphi$ is the 3-form of Definition 3 and $L \in \tilde{M}$, then $\chi|_L = 0 \iff s^*\chi|_{s^{-1}L} = 0$. Hence the 3-plane sL is φ -associative $\iff L$ is $s^*\varphi$ -associative (similar to the process in the Kleiman transversality, c.f. [AK])

From the above action, we see that the space of G_2 structures $\tilde{\Omega}_+^3(M)$ which induces the same metric on M has the following identification:

Lemma 5. *Let $\mathcal{G}(P_{G_2})$ be the stabilizer of the action of $\mathcal{G}(P)$ on $\tilde{\Omega}_+^3(M)$ (i.e. the gauge transformations fixing φ) then:*

$$\tilde{\Omega}_+^3(M) = \mathcal{G}(P)/\mathcal{G}(P_{G_2}) = \Omega^0(M, P \times_{SO(7)} \mathbb{R}\mathbb{P}^7).$$

Proof. Clearly $\mathcal{G}(P)$ acts transitively on $\tilde{\Omega}_+^3(M)$ with stabilizer $\mathcal{G}(P_{G_2})$. To see the second equality, we identify the fibers of the coset space with the fibers of $\tilde{\Lambda}_+^3(M) \rightarrow M$ by the map:

$$\mathbb{R}\mathbb{P}^7 = SO(7)/G_2^\varphi \rightarrow \tilde{\Lambda}_+^3(\mathbb{R}^7).$$

$G_2^\varphi s \mapsto s^* \varphi$. The adjoint action of $SO(7)$ on $SO(7)$ moves cosets

$$G_2^\varphi s \mapsto (g^{-1} G_2^\varphi g) g^{-1} s g = G^{g^* \varphi} g^{-1} s g$$

Hence by the above identification, on $\mathbb{R}\mathbb{P}^7$ it induces $\varphi \mapsto g^* \varphi$. \square

Now we can deform the canonical section $\Phi : Im(Y, M) \rightarrow \mathcal{Z}(Y)$ by the map

$$(42) \quad \tilde{\Phi} : \mathcal{G}(P) \times Im(Y, M) \rightarrow \mathcal{Z}(Y)$$

$\tilde{\Phi}(s, f) = \Phi_s(f) = s\Phi(f) = s(\tilde{f})$, that is $\tilde{\Phi}(s, f)(y) = s(f(y))\tilde{f}(y)$. Notice $\mathcal{G}(P)$ acts on the sections of the bundle $\mathcal{Z}(Y) \rightarrow Im(Y, M)$.

Theorem 6. *$\tilde{\Phi}$ is transversal to $\mathcal{Z}_\varphi(Y)$. Also Φ_s is transversal to $\mathcal{Z}_\varphi(Y)$ for a generic choice of s , equivalently Φ is transversal to $\mathcal{Z}_{s^* \varphi}(Y)$ for a generic s .*

Proof. : Let $\tilde{\Phi}(s, f) \in \mathcal{Z}_\varphi(Y)$. We can check transversality of $\tilde{\Phi}$ at (s, f) by computing its derivative. By the Leibnitz rule and Theorem 4 we can compute

$$s^{-1} \circ \Pi \circ d\tilde{\Phi}(h, v) : \mathfrak{g}_P(M) \oplus \Omega^0(\nu) \rightarrow T_{s(\tilde{f})}^v \mathcal{Z}(Y) \rightarrow T_{\tilde{f}}^v \mathcal{Z}(Y) = \Omega^0(T^*Y \otimes \nu)$$

where $d\tilde{\Phi}(h, v) = s(f) [\nabla_{A_0}(v) + s^{-1} ds(v)\tilde{f}]$, and $v = f_v$ is the perturbation of the inclusion f . Observe that $ad(P) = End(TM)$, and the map $y \mapsto s^{-1} ds(v)\tilde{f}(y)$ is a vertical deformation of the 3-plane $y \mapsto \tilde{f}(y) = T_y Y$, hence it is a section of the pull back of the vertical tangent bundle of $\tilde{M} \rightarrow M$ over Y , i.e. an element $a(v) \in T_{\tilde{f}}^v \mathcal{Z}(Y) = \Omega^0(T^*Y \otimes \nu)$. More specifically, if we decompose $s^{-1} ds(v)$ as an element of $so(7)$ on $T_{f(y)}(M) = T_y Y \oplus \nu_y(Y)$ in block matrices we can write:

$$(43) \quad s^{-1} ds(v) |_{f(y)} = \begin{pmatrix} * & -\alpha(v)^t \\ \alpha(v) & * \end{pmatrix}$$

Because $\alpha(v)$ is linear in v , we can view $\alpha \in \Omega^1(Y, ad \nu)$, therefore we can express $s^{-1} \Pi \circ d\tilde{\Phi}_s(v) = \nabla_{A_0}(v) + \alpha(v) = \nabla_{\mathbb{A}}(v)$ with $\mathbb{A} = A_0 + \alpha$. So the transversality is measured by the cokernel of the twisted Dirac operator $c_\varphi(\nabla_{\mathbb{A}}) = \mathcal{D}_{\mathbb{A}}$, where c_φ is the Clifford multiplication. Now by choosing $\alpha(v)$ we show that we can make $\mathcal{D}_{\mathbb{A}}$ onto. This is because, if \mathcal{D}_{A_0} is not already onto, we choose $0 \neq w \in im(\mathcal{D}_{A_0})^\perp$. By self adjointness of the Dirac operator $0 = \langle \mathcal{D}_{A_0}(v), w \rangle = \langle v, \mathcal{D}_{A_0}(w) \rangle$, for all v . So $\mathcal{D}_{A_0}(w) = 0$, by analytic continuation $w \neq 0$ on an open set. Then $w \in im(\mathcal{D}_{\mathbb{A}})^\perp$ implies $\langle c_\varphi(\alpha(v)), w \rangle = 0$ and hence $w = 0$, which is a contradiction. The last implication follows from by choosing s in (43) we can get the full Lie algebra $so(7)$, and hence $v \mapsto a(v)$ is onto, and the Clifford multiplication c is onto (Lemma 3).

So we obtain a smooth manifold $\tilde{\Phi}^{-1} \mathcal{Z}_\varphi(Y)$, and by choosing a regular value s of the projection $\tilde{\Phi}^{-1} \mathcal{Z}_\varphi(Y) \rightarrow \mathcal{G}_P(M)$ we get $\tilde{\Phi}_s^{-1} \mathcal{Z}_\varphi(Y)$ smooth (note that the derivative of the projection is Fredholm). Clearly the condition that Φ_s transversal to $\mathcal{Z}_\varphi(Y)$ is equivalent to Φ being transversal to $\mathcal{Z}_{s^* \varphi}(Y)$. \square

Theorem 6 says that the space of $s^*\varphi$ associative deformations of an φ associative submanifold $Y \subset M$, where $s \in \mathcal{G}_P(M)$, is a smooth (infinite dimensional) manifold. Infinitesimally these deformations correspond to the kernel of the twisted Dirac operator, twisted by the connections in the normal bundle $\nu(Y)$. Define

$$(44) \quad \sigma : \mathcal{G}(P) \rightarrow \Omega^0(\tilde{M}, \Xi^* \otimes \lambda(\mathbb{V}))$$

by $\sigma(s)(L)(v) = \alpha_s(v, L) \in \Xi^* \otimes \mathbb{V}$, where $\alpha_s(v, L)$ is obtained by decomposing $s^{-1}ds(v) \in \Omega^0(M, ad(P))$ on $TM = L \oplus L^\perp$ as an element of $so(7)$

$$(45) \quad s^{-1}ds(v) |_{L} = \begin{pmatrix} * & -\alpha(v, L)^t \\ \alpha(v, L) & * \end{pmatrix}$$

We can think of $\Omega^0(\tilde{M}, \Xi^* \otimes \lambda(\mathbb{V}))$ as an universal space parameterizing connections on $\nu \rightarrow Y$. The Gauss map \tilde{f} of any imbedding $f : Y \hookrightarrow M$ pulls back $\Xi^* \otimes \lambda(\mathbb{V})$ to the parameter space $\Omega^1(Y, \lambda(\nu))$ of the connections on $\nu(Y)$.

$$(46) \quad \Omega^0(\tilde{M}, \Xi^* \otimes \lambda(\mathbb{V})) \xrightarrow{\tilde{f}^*} \Omega^1(Y, \lambda(\nu))$$

Clearly the set $\Omega^1(\tilde{M}, \lambda(\mathbb{V}))$ can also be used as the universal parameter space. As in Section 3.2, given any imbedding $f : Y \hookrightarrow M$, we can deform the background connection $A_0 \rightarrow A = \mathbb{A}(b, a)$ in the normal bundle $\nu(Y)$, with $b \in \Omega^1(Y, \lambda_+(\nu))$ and $a \in \Omega^1(Y, \lambda_-(\nu))$, and get a perturbed version of (35)

$$(47) \quad \Omega^0(\nu) \times \Omega^1(\lambda_\pm(\nu)) \xrightarrow{\mathcal{D}_A} \Omega^0(\nu)$$

with the twisted Dirac equation $\mathcal{D}_\mathbb{A}(v) = c(\nabla_\mathbb{A}(v)) = \mathcal{D}_{A_0}(v) + \alpha v = 0$, where $\alpha = (b, a)$. Here we prefer perturbing by a (perturbing b has the effect of perturbing the metric on Y). A generic nonzero a makes the map $v \mapsto \mathcal{D}_{A_0+a}(v)$ surjective. We can choose this perturbation term a universally.

6. COMPLEX ASSOCIATIVE SUBMANIFOLDS

Let (M, φ) be a manifold with a G_2 structure. Here we will study an interesting class of associative submanifolds whose normal bundles come with an almost complex structure. The subgroups $U(2) \subset SO(4) \subset G_2 = G_2^\varphi$, more specifically

$$(S^1 \times SU(2))/\mathbb{Z}_2 \subset (SU(2) \times SU(2))/\mathbb{Z}_2 \subset G_2$$

give a $U(2)$ -principal bundle $\mathcal{P}_{G_2}(M) \rightarrow \bar{M}_\varphi = \mathcal{P}_{G_2}(M)/U(2)$. Also \bar{M}_φ is the total space of an S^2 bundle $\bar{M}_\varphi \rightarrow \tilde{M}_\varphi = \mathcal{P}_{G_2}(M)/SO(4)$, which is just the sphere bundle

$$(48) \quad \bar{M}_\varphi = S(\Xi) \rightarrow \tilde{M}_\varphi$$

of the \mathbb{R}^3 -bundle: $\lambda_+(\mathbb{V}) = \Xi \rightarrow \tilde{M}_\varphi$. We can identify the sections of (48) with almost complex structures on \mathbb{V} . Notice $\bar{M}_\varphi \rightarrow M$ is a bundle with fibers $G_2/U(2)$, which we can view as the complex version of the associative Grassmanns $G_\mathbb{C}^\varphi(3, 7)$.

In fact if $V_2(M)$ and $G_2(M)$ are the bundle of orthonormal 2-frames and oriented 2-planes in M respectively, the fibration $V_2(M) \rightarrow G_2(M)$ can be identified by:

$$\mathcal{P}_{G_2}(M)/SU(2) \rightarrow \mathcal{P}_{G_2}(M)/U(2)$$

with its projection $\{u, v\} \mapsto (\langle u, v, u \times v \rangle, u \times v)$, and also the projection map $\mathcal{P}_{G_2}(M) \rightarrow \mathcal{P}_{G_2}(M)/SU(2)$ on the fibers is given by the map $G_2 \rightarrow V_2(\mathbb{R}^7)$ defined by $\{v_1, v_2, v_3\} \mapsto \{v_1, v_2\}$ (recall the definition of G_2 in (1)). Put another way, G_2 acts transitively on $V_2(\mathbb{R}^7)$ with stabilizer $SU(2)$. By summing up above:

Proposition 7. $\bar{M}_\varphi = S(\Xi) = G_2(M)$

More generally, for the Riemannian manifold (M^7, g_φ) we can take the sphere bundle $\bar{M} \rightarrow \tilde{M}$ of $\lambda_+(\mathbb{V}) \rightarrow \tilde{M}$, and get codimension 4 inclusion of the smooth manifolds $\bar{M}_\varphi \subset \bar{M}$ (of dimensions 17 and 21). The sections of the bundle $\bar{M} \rightarrow \tilde{M}$ gives the parametrization of the almost complex structures on \mathbb{V} , and $\bar{M} \rightarrow M$ is a bundle with fibers $G_{\mathbb{C}}(3, 7) := SO(7)/U(2) \times SO(3)$. For all G_2 structures φ inducing the same metric on M , we have the inclusions $G_2^\varphi \hookrightarrow SO(7)$ inducing imbeddings $G_2(M) = \bar{M}_\varphi \hookrightarrow \bar{M}$, which is fiberwise $\langle u, v \rangle \mapsto \langle u, v, u \times v \rangle$

$$G(2, 7) = G_2^\varphi/U(2) \hookrightarrow SO(7)/U(2) \times SO(3)$$

By [T] the bundle $V_2(M) \rightarrow M$ has always a section $\Lambda = \{u, v\}$, which induces sections of the bundles $\bar{M}_\varphi \rightarrow \tilde{M}_\varphi$ and $\tilde{M}_\varphi \rightarrow M$ (for simplicity we will abuse notation and denote all these sections by Λ also). So Λ gives an almost complex structure on $\mathbb{V} \rightarrow \tilde{M}_\varphi$. By Λ , we can pull back Ξ and \mathbb{V} to bundles \mathbf{E} and \mathbf{V} on M , respectively, and \mathbf{V} has an almost complex structure (by the discussion following Lemma 2 we can describe this complex structure with the cross product with $u \times v$).

Definition 7. *From now on, we will denote a 7-manifold with a G_2 structure and a nonvanishing 2-frame field Λ with (M, φ, Λ) .*

Given (M, φ, Λ) , then the induced $U(2)$ structure on $\mathbb{V} \rightarrow \tilde{M}_\varphi$ canonically lifts to a $Spin^c(4)$ structure by the diagram:

$$(49) \quad \begin{array}{ccc} & Spin^c(4) & \\ & \nearrow & \downarrow \\ U(2) & \rightarrow & SO(4) \times S^1 \end{array}$$

where $U(2) = (S^1 \times S^3)/\mathbb{Z}_2$, $SO(4) = (S^3 \times S^3)/\mathbb{Z}_2$, $Spin^c(4) = (S^3 \times S^3 \times S^1)/\mathbb{Z}_2$, where the horizontal map $[\lambda, A] \mapsto ([\lambda, A], \lambda^2)$ lifts to the map $[\lambda, A] \mapsto (\lambda, A, \lambda)$. This means there is a \mathbb{C}^2 -bundle $\mathbb{W} \rightarrow \tilde{M}_\varphi$ with $\mathbb{V}_{\mathbb{C}} = \mathbb{W} \oplus \bar{\mathbb{W}}$, and transition function λ^2 gives the determinant line bundle $K = \Lambda^2 \bar{\mathbb{W}} \rightarrow \tilde{M}_\varphi$. Also we can write $\mathbb{V}_{\mathbb{C}} = \mathbb{W}^+ \oplus \mathbb{W}^-$ with $\mathbb{W}^+ = K^{-1} + \mathbb{C}$ and $\mathbb{W}^- = \bar{\mathbb{W}}$. Recall $\Xi^* = \lambda_+(\mathbb{V}) = \Lambda_+^2(\mathbb{V}) = K + \mathbb{R}$. We have a Clifford action $\mathbb{V} \otimes \mathbb{W}^+ \rightarrow \mathbb{W}^-$ which extends to Clifford action

$$\Xi^* \otimes \mathbb{W}^+ \rightarrow \mathbb{W}^+$$

The identifications $\mathbb{W}^+ \otimes \bar{\mathbb{W}}^- = \mathbb{C} \oplus \mathbb{C} \oplus K \oplus \bar{K} = (K \oplus \mathbb{R})_{\mathbb{C}} \oplus \mathbb{C} = \Xi_{\mathbb{C}}^* \oplus \mathbb{C}$ gives the usual quadratic bundle map of the Seiberg-Witten theory (c.f. [A]):

$$(50) \quad \sigma : \mathbb{W}^+ \otimes \mathbb{W}^+ \rightarrow \Xi_{\mathbb{C}}^*$$

$$\sigma(x, x) = \left(\frac{|z|^2 - |w|^2}{2}, \bar{z}w \right), \text{ where } x = (z, w)$$

Definition 8. A submanifold $f : Y^3 \hookrightarrow M$ of (M^7, φ, Λ) is called Λ -associative if $\tilde{f} = \Lambda \circ f$ where \tilde{f} is the Gauss map, and it is called almost Λ -associative if it comes from transverse section of the bundle $\mathbf{V} \rightarrow M$ (recall \mathbf{V} is obtained from Λ).

An Λ -associative, more generally almost Λ -associative submanifold Y of (M, φ, Λ) induce canonical isomorphisms $TY \cong \tilde{f}^*(\Xi)$ and $\nu(Y) \cong \tilde{f}^*(\mathbb{V})$ (by transversality).

$$\begin{array}{ccc} & & \tilde{M}_{\varphi} \\ & \tilde{f} \nearrow & \downarrow \uparrow \Lambda \\ Y & \xrightarrow{f} & M \end{array}$$

So normal bundle of any almost Λ -associative submanifold $Y^3 \subset M^7$ has a $U(2)$ structure, therefore it has a $Spin^c(4)$ structure, with induced \mathbb{C}^2 -bundle $W \rightarrow Y$ and its determinant line bundle $K \rightarrow Y$, and a Clifford action of $T^*Y \otimes W \rightarrow W$ (induced from the cross product). An example of a Λ associative submanifold is the zero section of the spinor bundle $\mathcal{S} \rightarrow Y^3$ (the G_2 manifold constructed in [BSa]).

In general, the background $SO(4)$ connection A_0 on the normal bundle $\nu(Y)$ of a Λ -associative submanifold Y may not reduce to a $U(2)$ connection if the 1-form whose dual gives the splitting $TY \cong K \oplus \mathbb{R}$ is not parallel. Nevertheless from the $Spin^c(4)$ structure on $\nu(Y)$ we do get a connection on the complex bundle $W \rightarrow Y$ provided we pick a connection on the line bundle $K \rightarrow Y$ (from (49)). In the next section we will study the local deformation space of Λ -associative manifolds, by deforming them in the complex bundle W , with the help of the connections on K .

Remark 4. Associative submanifolds with $Spin^c(3)$ structure $(Y, c) \hookrightarrow (M, \varphi)$ come equipped with an $U(2)$ structure (hence $Spin^c(4)$ structure) on their normal bundles (Lemma 2). We can free the deformations space these manifolds from the extra parameter c , by picking up a generic Λ , and studying the deformations of more relax almost Λ -associative submanifolds $Y \subset (M, \varphi, \Lambda)$. In this case the $Spin^c(3)$ structure on TY comes from the pull-back. Also, by further deforming the 2-plane field Λ on M deforms the the $Spin^c$ structure on Y .

Remark 5. We could have considered complex structures on \mathbb{V} corresponding to the right reduction, i.e. the subgroup $(SU(2) \times S^1)/\mathbb{Z}_2 \subset (SU(2) \times SU(2))/\mathbb{Z}_2 \subset G_2$. In this case, they correspond to the sections of the S^2 bundle $\lambda_-(\mathbb{V}) \rightarrow \tilde{M}_{\varphi}$. Here we opted to the left reductions since they concretely relate to Ξ by $\lambda_+(\mathbb{V}) = \Xi$.

7. DEFORMING Λ -ASSOCIATIVE SUBMANIFOLDS

Let (M, φ, Λ) be a manifold with G_2 structure and a non-vanishing 2-plane field, $\mathcal{M}(M, \varphi, \Lambda)$ be the space of Λ -associative submanifolds. Here we will study the local “complex” deformations of $\mathcal{M}(M, \varphi, \Lambda)$ near a particular $f : Y \hookrightarrow M$. These are the deformations of Y inside its complex normal bundle W , with the help of the connections $\mathcal{A}(K)$ on the line bundle $K = \det W$. These deformations are identified with the kernel of a twisted Dirac operator twisted by the connections in $\mathcal{A}(K)$. Introducing new variables $\mathcal{A}(K)$ makes the deformation space smooth. Up to this point this section can be viewed as a version of Theorem 4 for the Λ -associative submanifolds. But now the connection parameter can be constraint with the natural map (50) to obtain Seiberg-Witten like equations, which gives a compactness result for this more restricted local deformation space of Y . Reader should note that these equations are $Spin^c(4)$ Seiberg-Witten equations on Y^3 (which are usually associated to 4-manifolds), as opposed to the usual $Spin^c(3)$ Seiberg-Witten equations. The Clifford action $T^*Y \otimes \mathbb{W} \rightarrow W$ is induced via the identification $\Lambda_+^2 W = T^*Y$, it is also induced by the cross product operation on M .

Let $Y \in \mathcal{M}(M, \varphi, \Lambda)$. Let $W \rightarrow Y$ be the complex bundle associated to $\nu(Y)$, and $K \rightarrow Y$ be its determinant line bundle. Let B_0 be the background connection on $\nu(Y)$ (induced by φ), then as discussed in last section B_0 along with $A \in \mathcal{A}(K)$ defines a connection on $W \rightarrow Y$, denote by $\mathbb{A} = B_0 \oplus A$. We can write $A = A_0 + a$ with $a \in \Omega^1(Y) = T_{A_0}\mathcal{A}(K)$ (tangent space of connections) and $\mathbb{A} = \mathbb{A}(a)$. Then we get a complex version of the map (47) $(v, a) \mapsto \mathcal{D}_{\mathbb{A}}(v) = \mathcal{D}_{\mathbb{A}(0)}(v) + a.v$

$$(51) \quad \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R}) \xrightarrow{\mathcal{D}_{\mathbb{A}}} \Omega^0(Y, W)$$

which is the derivative of a similarly defined map

$$(52) \quad \Omega^0(Y, W) \times \mathcal{A}(K) \rightarrow \Omega^0(Y, W)$$

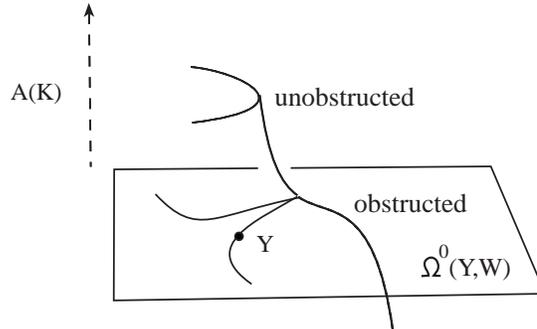


FIGURE 3

In each slice $\mathbb{A}(a)$, we are deforming along normal vector fields by the connection $\mathbb{A}(a)$, which is a perturbation of the background connection $\mathbb{A}(0)$. To get compactness we can cut down this parametrized moduli space with an additional equation (induced from the map (50)) of the Seiberg-Witten theory $\Psi^{-1}(0)$, where

$$\begin{aligned} \Psi : \Omega^0(Y, W) \times \mathcal{A}(K) &\rightarrow \Omega^0(Y, W) \times \Omega^2(Y, i\mathbb{R}) \\ (53) \quad \mathcal{D}_{\mathbb{A}}(v) &= 0 \\ *F_A &= \sigma(v, v) \end{aligned}$$

where F_A is the curvature of the connection $A = A_0 + a$ in K , and $*$ is the star operator on Y . Note that Y comes equipped with the natural submanifold metric. Now we proceed exactly as in the Seiberg-Witten theory of 3-manifolds (e.g. [C], [Lim], [Ma], [W]). To obtain smoothness of $\Psi^{-1}(0)$, we perturb the equations by 1-forms $\delta \in \Omega^1(Y)$ and get a new equation $\Phi = 0$, where

$$\begin{aligned} \Phi : \Omega^0(Y, W) \times \mathcal{A}(K) \times \Omega^1(Y) &\rightarrow \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R}) \\ \mathcal{D}_{\mathbb{A}}(v) &= 0 \\ *F_A + i\delta &= \sigma(v, v) \end{aligned}$$

We can choose the perturbation term universally $\delta = f^*(\Delta)$, where $\Delta \in \Omega^1(M)$. Then Φ has a linearization:

$$\begin{aligned} D\Phi_{(v_0, A_0, 0)} : \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R}) \times \Omega^1(Y) &\rightarrow \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R}) \\ D\Phi_{(v_0, A_0, 0)}(v, a, \delta) &= (\mathcal{D}_{A_0}(v) + a.v_0, *da + i\delta - 2\sigma(v_0, v)) \end{aligned}$$

We see that $\Phi^{-1}(0)$ is smooth and the projection $\Phi^{-1}(0) \rightarrow \Omega^1(Y)$ is onto, so by Sard's theorem for a generic choice of δ we can make $\Phi_{\delta}^{-1}(0)$ smooth, where $\Phi_{\delta}(v, A) = \Phi(v, A, \delta)$. The bundle W of Y has a complex structure, so the gauge group $\mathcal{G}(K) = \text{Map}(Y, S^1)$ acts on the solution set $\Phi_{\delta}^{-1}(0)$, and makes the quotient $\Phi_{\delta}^{-1}(0)/\mathcal{G}(K)$ a smooth zero-dimensional manifold. This is because the infinitesimal action of $\mathcal{G}(K)$ on the complex $\Phi_{\delta} : \Omega^0(Y, W) \times \mathcal{A}(L) \rightarrow \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R})$ is given by the map

$$\Omega^0(Y, i\mathbb{R}) \xrightarrow{G} \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R})$$

where $G(f) = (fv_0, df)$. So after dividing by \mathcal{G} , tangentially the complex Φ_{δ} becomes

$$\Omega^0(Y; i\mathbb{R}) \xrightarrow{G} \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R}) \rightarrow \Omega^0(Y, W) \times \Omega^1(Y, i\mathbb{R})/G$$

Hence the index of this complex is the sum of the indices of the Dirac operator $\mathcal{D}_{A_0} : \Omega^0(Y, W) \rightarrow \Omega^0(Y, W)$ (which is zero), and the index of the following complex

$$\Omega^0(Y, i\mathbb{R}) \times \Omega^1(Y, i\mathbb{R}) \rightarrow \Omega^0(Y, i\mathbb{R}) \times \Omega^1(Y; i\mathbb{R})$$

given by $(f, a) \mapsto (d^*(a), df + *da)$, which is also zero since Y^3 has zero Euler characteristic. Furthermore, $\Phi_{\delta}^{-1}(0)/\mathcal{G}(K)$ is compact and oriented (the same proof as in the Seiberg-Witten theory). Hence we get a number $SW_Y(M)$. Here we don't worry about metric dependence of $SW_Y(M)$ since we have a fixed background metric

induced from the G_2 structure. Hence we associated a number to a Λ -associative submanifold Y of (M, φ, Λ) . In particular, Y moves in an unobstructed way along the parametrized sections the complex normal bundle $\Omega^0(Y, W) \times \mathcal{A}(L)$. Furthermore all these constructions work for almost Λ -associative submanifolds. So we have:

Theorem 8. *Let Y be an almost Λ -associative submanifold of (M, φ, Λ) . By cutting down the space of parametrized complex deformations of Y with an additional equation as in (53) we obtain a zero dimensional compact smooth oriented manifold, hence we can associate a number $\Lambda_\varphi(Y) \in \mathbb{Z}$.*

Remark 6. *Clearly $\Lambda_\varphi(Y)$ is invariant under small isotopies through almost Λ -associative submanifolds $Y \subset (M, \varphi, \Lambda)$.*

The equations (53) can be induced universally from equations on (M^7, φ, Λ) by restriction: The 2-frame field $\langle u, v \rangle$ gives a splitting of the tangent bundle $TM = \mathbf{E} \oplus \mathbf{V}$ with an $SO(3)$ bundle $\mathbf{E} = \langle u, v, u \times v \rangle$ and a $U(2)$ -bundle $\mathbf{V} = \mathbf{E}^\perp$, such that $\lambda_+(\mathbf{V}) = \mathbf{E}$. Let $\mathbf{W} \rightarrow M$ be the induced \mathbb{C}^2 -bundle, and $\mathbf{K} \rightarrow M$ be the determinant line bundle of \mathbf{W} . We can define an action $T^*(M) \otimes \mathbf{W} \rightarrow \mathbf{W}$: For $w = x + y \in TM$, with $x \in \mathbf{E}$, $y \in \mathbf{V}$ and $z \in \mathbf{W}$ with $w.z = xz$. It is easy to check that this is a partial Clifford action, i.e. $w.(w.z) = -|x|^2 z \text{ Id}$, and it extends to an action $\Lambda^2(T^*M) \otimes \mathbf{W} \rightarrow \mathbf{W}$, and we have the map $\sigma : \mathbf{W} \otimes \mathbf{W} \rightarrow \mathbf{E}_\mathbb{C}$ of (50).

These bundles inherit connections from the Levi-Civita connection of (M, g_φ) . Let $\mathcal{A}(\mathbf{K})$ be the connections on \mathbf{K} . Let A_0 denote the background connections. Then any $A \in \mathcal{A}(\mathbf{K})$ along with A_0 determines a connection on \mathbf{W} . Write $A = A_0 + a$ with $a \in \Omega^1(M)$. Hence for $A \in \mathcal{A}(\mathbf{K})$ we can define a partial Dirac operator $\mathcal{D}_A(v) = \mathcal{D}_{A_0}(v) + a.v$ on $\mathbf{W} \rightarrow M$, which is the composition:

$$\Omega^0(M, \mathbf{W}) \xrightarrow{\nabla_A} \Omega^0(M, T^*M \otimes \mathbf{W}) \xrightarrow{c_\varphi} \Omega^0(M, \mathbf{W})$$

We can now write the global version of the equations (54) on M in the usual way

$$\phi : \Omega^0(M, \mathbf{W}) \times \mathcal{A}(\mathbf{L}) \rightarrow \Omega^0(M, \mathbf{W}) \times \Omega^1(M) \quad \text{which is}$$

$$(54) \quad \begin{aligned} \mathcal{D}_A(v) &= 0 \\ *F_A &= \sigma(v, v) \end{aligned}$$

where $*$: $TM \rightarrow TM$ is the star operator on \mathbf{E} and zero on \mathbf{V} . We can perturb these equations by 1-forms to $\Phi = 0$, and proceed as before. \mathbf{W} has a complex structure. The gauge group $\mathcal{G}(\mathbf{L}) = \text{Map}(M, S^1)$ acts on the solution set $\Phi^{-1}(0)$, and the quotient $\Phi^{-1}(0)/\mathcal{G}(\mathbf{L})$ can be formed. To sum up we have:

Proposition 9. *Any almost Λ -associative submanifold $f : Y^3 \hookrightarrow (M, \varphi, \Lambda)$ pulls back the equations (54) to the Seiberg-Witten equations (53) on Y .*

8. ASSOCIATIVE 3-PLANE FIELDS OF G_2 MANIFOLDS

Recall that, any non-vanishing oriented 2-plane field $\Lambda = \langle u, v \rangle$ on (M, φ) determines a section $\Lambda_\varphi : M \rightarrow \tilde{M}_\varphi \subset \tilde{M}$. In particular, it gives a non-vanishing associative 3-plane field $\mathbf{E} = \mathbf{E}_{\Lambda, \varphi} \rightarrow M$ on M , and a complex structure on the complementary 4-plane field $\mathbf{V} = \mathbf{V}_{\Lambda, \varphi} \rightarrow M$, and a splitting $TM = \mathbf{E} \oplus \mathbf{V}$, with $\lambda_+(\mathbf{V}) = \mathbf{E}$. From the construction we get a further splitting $\mathbf{E} = \mathbf{\Lambda} \oplus \xi$, corresponding to $\langle u, v \rangle \oplus \langle u \times v \rangle$. The orientation of the 2-dimensional bundle $\mathbf{\Lambda}$ gives it a complex structure, and we have

$$(55) \quad TM = \bar{\mathbf{E}} \oplus \xi$$

where $\bar{\mathbf{E}} = \mathbf{\Lambda} \oplus \mathbf{V}$ is a 6-plane bundle with a complex structure and ξ is the line bundle $\langle u \times v \rangle$. Note that if φ is integrable and the vector field $u \times v$ is parallel then M would be a Calabi-Yau $\times S^1$ (since G_2 holonomy would reduce to $SU(3)$). So non-vanishing oriented 2-plane fields may be thought of objects taming the G_2 structure. Any integral submanifold of the corresponding distribution \mathbf{E} is an associative submanifold $Y^3 \subset M$ with a $Spin^c$ -structure (i.e. the 2-plane field $\xi = \Lambda|_Y$).

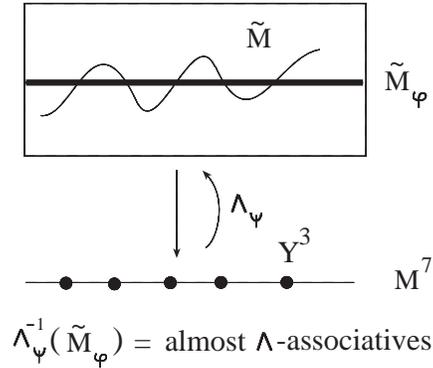


FIGURE 4

By fixing the plane field Λ , and varying $\varphi \in \tilde{\Omega}_+^3(M)$ (the set of G_2 structures inducing the same metric on M) has the effect of varying $\xi \in \Lambda^\perp$ (the cross product operation on Λ) and varying the complex structure on $\mathbf{V} = (\Lambda \oplus \xi)^\perp$. These ξ 's are the sections of the S^4 -sphere bundle of $\Lambda^\perp \rightarrow M^7$, hence generically any other section will agree with Λ_φ on some 3-manifold $Y \subset M$. We will show that this 3-manifold is almost Λ -associative. First consider the parametrized section:

$$(56) \quad \mathbf{\Lambda} : \tilde{\Omega}_+^3(M) \times M \rightarrow \tilde{M}$$

$(\lambda, x) \mapsto \Lambda_\lambda(x)$. By Lemma 5 there is an identification $\tilde{\Omega}_+^3(M) = \{s^*(\varphi) \mid s \in \mathcal{G}(P)\}$ (the sections of an $\mathbb{R}P^7$ bundle over M). We claim $\mathbf{\Lambda}$ is transversal to \tilde{M}_φ .

First we need to recall a few facts: By [B2], the deformations of the G_2 structure φ fixing the metric $g = g_\varphi$, are parametrized by φ_λ below, where $\lambda = [a, \alpha]$ are the sections $\tilde{\Omega}_+^3(M)$ of the $\mathbb{R}\mathbb{P}^7$ -bundle, which is the projectivization $P(\mathbb{R} \oplus T^*M) \rightarrow M$

$$\varphi_\lambda = (a^2 - |\alpha|^2)\varphi + 2a * (\alpha \wedge \varphi) + 2\alpha \wedge *(\alpha \wedge * \varphi)$$

where $a^2 + |\alpha|^2 = 1$. From the identities $*(\alpha \wedge \varphi) = -\alpha^\# \lrcorner * \varphi$ and $*(\alpha \wedge * \varphi) = \alpha^\# \lrcorner \varphi$, where $\alpha^\#$ is the metric dual of α , we can also express

$$(57) \quad \varphi_\lambda = \varphi - 2\alpha^\# \lrcorner [a(*\varphi) + \alpha \wedge \varphi]$$

$$*\varphi_\lambda = *\varphi + 2\alpha \wedge [a\varphi - (\alpha^\# \lrcorner * \varphi)]$$

Not to clutter notations, we denote $\Lambda_\lambda = \Lambda_{\varphi_\lambda}$ and use the metric to identify $T^*(M) = \mathbf{E} \oplus \mathbf{V}$, and identify M with the zero section of the bundle $\mathbf{V} \rightarrow M$.

Theorem 10. *For $\alpha \in \Omega^1(M)$ which is a transverse section of $\mathbf{V} \rightarrow M$, the map Λ_λ , where $\lambda = [a, \alpha]$ and $a \neq 0$, is transversal to \tilde{M}_φ , and $\Lambda_\lambda^{-1}(\tilde{M}_\varphi) = \alpha^{-1}(M)$.*

Proof. The set $\Lambda_\lambda^{-1}(\tilde{M}_\varphi)$ is given by the solutions of the equation $\Lambda_{\varphi_\lambda}(x) = \Lambda_\varphi(x)$, where $\varphi \mapsto \varphi_\lambda$ is a deformation of φ . Since \mathbf{E} is obtained from the oriented 2-plane field $\Lambda = \langle u, v \rangle$ by association $\langle u, v \rangle \mapsto \langle u, v, u \times_\varphi v \rangle$, this equation is equivalent to $(u \times v)_\lambda(x) = (u \times v)(x)$ (up to positive scalar multiple), where $(u \times v)_\lambda$ denotes the cross product corresponding to φ_λ . By using $(u \times v)^\# = u \lrcorner v \lrcorner \varphi$, we can calculate the deviation of the cross product operation under the deformation

$$\begin{aligned} (u \times v)_\lambda &= (1 - 2|\alpha|^2)(u \times v) + 2[-a\chi(u, v, \alpha^\#) \\ &\quad + \alpha(v)(u \times \alpha^\#) - \alpha(u)(v \times \alpha^\#) + \varphi(u, v, \alpha^\#)\alpha^\#] \end{aligned}$$

So the equation $(u \times v)_\lambda(x) = (u \times v)(x)$ is given by the equation $F = 0$ where:

$$F = a\chi(u, v, \alpha^\#) - \alpha(v)(u \times \alpha^\#) + \alpha(u)(v \times \alpha^\#) - \varphi(u, v, \alpha^\#)\alpha^\# + |\alpha|^2(u \times v)$$

Note that when $\alpha^\# \in \mathbf{E}$, the equation $F(x) = 0$ holds for all $x \in M$. Let us choose our deformation $\alpha^\# \in \mathbf{V}$, which is a transverse section of $\mathbf{V} \rightarrow M$. In this case by (4), Lemma 1, and Lemma 2 the equation $F(x) = 0$ is equivalent to

$$aJ(\alpha^\#) = -|\alpha|^2(u \times v)$$

where J is the complex structure defined in Lemma 2. Since $J(\alpha^\#) \in \mathbf{V}$ and $u \times v \in \mathbf{E}$, this equation holds only at points satisfying $\alpha^\#(x) = 0$. By taking derivative of $F(a, \alpha)$ we see that F is transversal to \tilde{M}_φ when $a \neq 0$. \square

9. CAYLEY SUBMANIFOLDS OF $Spin(7)$

Much of what we have discussed for associative submanifolds of a G_2 manifold holds for Cayley submanifolds of a $Spin(7)$ manifold. Let (N^8, Ψ) be a $Spin(7)$ manifold, and $\mathcal{P}_{Spin(7)}(N) \rightarrow N$ be its $Spin(7)$ frame bundle, and $G(4, 8)$ be the Grassmannian of oriented 4 planes in \mathbb{R}^8 . As in the G_2 case we can form the bundle

$$\tilde{N} = \mathcal{P}(N) \times_{SO(8)} G(4, 8) \rightarrow N.$$

Similarly we have the universal bundles $\Xi, \mathbb{V} \rightarrow \tilde{N}$ which are fiberwise extensions of the canonical bundle $\xi \rightarrow G(4, 8)$ and its dual $\nu = \xi^\perp \rightarrow G(4, 8)$, respectively. $Hom(\Xi, \mathbb{V}) = \Xi^* \otimes \mathbb{V} \rightarrow \tilde{N}$ is the vertical subbundle of $T(\tilde{N}) \rightarrow N$ with fibers $TG(4, 8)$. Let $G^\Psi(4, 8)$ be the Grassmannian of Cayley 4-planes in $G(4, 8)$ consisting of elements $L \in G(4, 8)$ satisfying $\Psi|L = vol(L)$. The group $Spin(7)$ acts transitively on $G^\Psi(4, 8)$ with the stabilizer $(SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$. Therefore, $G^\Psi(4, 8)$ can be identified by the quotient of $Spin(7)$ with the subgroup

$$(SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2 \subset Spin(7).$$

The action of $[q_+, q_-, \lambda] \in (SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$ on $\mathbb{R}^8 = \mathbb{H} \oplus \mathbb{H}$ is given by $(x, y) \rightarrow (q_+xq_-^{-1}, q_+y\lambda^{-1})$. As in G_2 case there is the Cayley Grassmannian bundle

$$\tilde{N}_\Psi = \mathcal{P}_{Spin(7)}(N) \times_{Spin(7)} G^\Psi(4, 8) \rightarrow N$$

which is $\tilde{N}_\Psi = \mathcal{P}(N)/(SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2 \rightarrow \mathcal{P}(N)/Spin(7) = N$. We have restriction of the bundles $\Xi^*, \mathbb{V} \rightarrow \tilde{N}_\Psi \subset \tilde{N}$. Furthermore, the principal $(SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$ bundle $\mathcal{P}(N) \rightarrow \tilde{N}_\Psi$ gives the following associated vector bundles over \tilde{N}_Ψ via the representations (see [HL], [M]).

$$(58) \quad \begin{aligned} \mathbb{W}^+ = \mathbb{V} : & \quad y \mapsto q_+y\lambda^{-1} \\ \mathbb{W}^- : & \quad y \mapsto q_-y\lambda^{-1} \\ \Xi^* : & \quad x \mapsto q_+xq_-^{-1} \\ \lambda_+(\Xi^*) : & \quad x \mapsto q_+xq_-^{-1} \\ \lambda_-(\Xi^*) : & \quad x \mapsto q_-xq_-^{-1} \\ \lambda_-(\mathbb{W}^+) : & \quad x \mapsto \lambda x\lambda^{-1} \end{aligned}$$

where $[q_+, q_-, \lambda] \in (SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$. We can identify: $\lambda_+(\mathbb{W}^+) = \lambda_+(\Xi^*)$, and we have the usual decomposition $\Lambda^2(\Xi^*) = \lambda_+(\Xi^*) \oplus \lambda_-(\Xi^*)$. We have the Clifford multiplications $\Xi^* \otimes \mathbb{W}^\pm \rightarrow \mathbb{W}^\mp$ given by: $x \otimes y \mapsto -\bar{x}y$ and $x \otimes y \mapsto xy$, on \mathbb{W}^+ and \mathbb{W}^- respectively, which extends to $\Lambda^2(\Xi^*) \otimes \mathbb{W}^+ \rightarrow \mathbb{W}^+$.

The Gauss map of an imbedding $f : X^4 \hookrightarrow N^8$ of any 4-manifold canonically lifts to an imbedding $\tilde{f} : X^4 \hookrightarrow \tilde{N}$, and the pull backs $\tilde{f}^*\Xi^* = T^*(X)$ and $\tilde{f}^*\mathbb{W}^+ = \nu(X)$ give cotangent and normal bundles of X . Furthermore, if X is a Cayley submanifold of N then the image of \tilde{f} lands in \tilde{N}_Ψ ; in this case pulling back the principal $Spin(7)$ frame bundle $\mathcal{P}(N) \rightarrow N$ induces an $(SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$ bundle $\mathcal{P}(X) \rightarrow X$. So by the representations (58) we get associated vector bundles

$W^+ = \nu(X), W^-, T^*(X)$ over X , i.e. the pull-backs of $\mathbb{W}^+, \mathbb{W}^-, \Xi^*$. So we have the actions $W^+ \otimes \lambda_-(W^+) \rightarrow W^+$ and $T^*X \otimes W^\pm \rightarrow W^\mp$ and $\Lambda^2(T^*X) \otimes W^+ \rightarrow W^+$.

The Levi-Civita connection induced by the $Spin(7)$ metric on N , induces connections on tangent and normal bundle of any submanifold $X^4 \subset N$. Call these connections background connections. Let \mathbb{A}_0 be the induced connection on $\nu(X) = W^+$. Using the Lie algebra decomposition $so(4) = so(3) \oplus so(3)$, we can decompose $\mathbb{A}_0 = S_0 \oplus A_0$, where S_0 and A_0 are connections on $\lambda_+(T^*X)$ and $\lambda_-(W^+)$, respectively. Any connection A of $\lambda_-(W^+)$ is in the form $A = A_0 + a$ where $a \in \Omega^1(X, \lambda_-(W^+))$, and by the association $A \mapsto S_0 \oplus A$ it induces a connection on $\nu(X)$. We will denote this connection by $\mathbb{A} = \mathbb{A}(a)$, and $\mathbb{A}_0 = \mathbb{A}(0)$. Later we will consider deformations

$$(59) \quad \mathbb{A}_0 \mapsto \mathbb{A}.$$

Let $\nabla_{\mathbb{A}} : \Omega^0(X, W^+) \rightarrow \Omega^1(X, W^+)$ by $\nabla_A = \sum e^i \otimes \nabla_{e_i}$, where $\{e_i\}$ and $\{e^i\}$ are orthonormal tangent and cotangent frame fields of X , respectively. When X is a Cayley manifold, the Clifford multiplication gives the twisted Dirac operator:

$$(60) \quad \mathcal{D}_{\mathbb{A}} : \Omega^0(X, W^+) \rightarrow \Omega^0(X, W^-)$$

The kernel of $\mathcal{D}_{\mathbb{A}_0}$ gives the infinitesimal deformations of Cayley submanifolds ([M]). As in the associative case by deforming $\mathbb{A}_0 \rightarrow \mathbb{A}$ we can make cokernel of $\mathcal{D}_{\mathbb{A}}$ zero.

Similar to the case of Λ -associative submanifolds in G_2 manifolds, we can study the Cayley submanifolds in $Spin(7)$ manifolds with complex normal bundles. There are several ways of lifting various subbundles to complex bundles, for example

$$Spin^c(4) = (SU(2) \times SU(2) \times S^1)/\mathbb{Z}_2 \subset (SU(2) \times SU(2) \times SU(2))/\mathbb{Z}_2$$

gives a $Spin^c(4)$ bundle $\mathcal{P}(N) \rightarrow \bar{N}_\Psi = \mathcal{P}(N)/Spin^c(4)$, and we have all the corresponding bundles of (58) over \bar{N}_Ψ (except in this case we have $\lambda \in S^1$). The S^2 -bundle $\bar{N}_\Psi \rightarrow \tilde{N}_\Psi$ can be identified with the sphere bundle of $\lambda_-(\mathbb{W}^+) \rightarrow \tilde{N}_\Psi$, and the sections of this bundle correspond to almost complex structures on \mathbb{W}^\pm . Previously, in the case of 7-manifolds, existence of such sections followed from the existence of 2-frame field [T], in the 8-dimensional $Spin(7)$ case we don't have a clean analogue of [T], so in this case we will make this an assumption and proceed. So consider a $Spin(7)$ manifold (N^8, Ψ, Λ) with a unit section $\Lambda : \bar{N}_\Psi \rightarrow \lambda_-(\mathbb{W}^+)$. Hence $\mathbb{W}^\pm \rightarrow \bar{N}_\Psi$ are $U(2)$ bundles, and $\lambda_-(W^+)$ is a line bundle $L \rightarrow \bar{N}_\Psi$. As in (50) there is a quadratic bundle map $\sigma : \mathbb{W}^+ \otimes \mathbb{W}^+ \rightarrow \lambda_+(\Xi^*)$

$$\sigma(x, x) = -\frac{1}{2}(xi\bar{x})i.$$

Now if $f : X^4 \hookrightarrow N^8$ is a Cayley submanifold, we can pull back these structures onto X by $\Lambda \circ \tilde{f}$. Then we can ‘‘perturb’’ the local Cayley deformations of X by deforming the connection as in (59), i.e. the kernel of the Dirac operator of (60).

Then if we can cut down the solution space $\mathcal{D}_{\mathbb{A}}^{-1}(0)$ by a second natural equation (by using “ a ” as a free variable) we arrive to the Seiberg-Witten equations:

$$(61) \quad \begin{aligned} \mathcal{D}_A(v) &= 0 \\ F_A^+ &= \sigma(v, v) \end{aligned}$$

As usual, by perturbing these equations by elements of $\Omega_+^2(X)$, i.e. by changing the second equation with $F_A^+ + \delta = \sigma(v, v)$ with $\delta \in \Omega_+^2(X)$ we get smoothness on the zero locus of the parameterized equation $F = 0$ where

$$F : \Omega^0(X, W^+) \times \mathcal{A}(L) \times \Omega_+^2(X) \rightarrow \Omega^0(X, W^-) \times \Omega_+^2(X)$$

and by generic choice of δ we can make the solution set $F_\delta^{-1}(0)$ smooth. The normal bundle W^+ of X has a complex structure, so the gauge group $\mathcal{G}(L) = \text{Map}(X, S^1)$ acts on $F_\delta^{-1}(0)$, and makes quotient $F_\delta^{-1}(0)/\mathcal{G}(L)$ a smooth manifold whose dimension d can be calculated from the index of the elliptic complex:

$$(62) \quad \Omega^0(X) \rightarrow \Omega^0(X, W^+) \times \Omega^1(X) \rightarrow \Omega^0(X, W^-) \times \Omega_+^2(X)$$

where the first map comes from gauge group action. As in Seiberg-Witten we get:

$$(63) \quad d = \frac{1}{4} [c_1^2(L) - (2e(X) + 3\sigma(X))].$$

Here e and σ denote the Euler characteristic and the signature. In particular, these parametrized deformations of complex Cayley submanifolds in $\Omega^0(X, W^+) \times \mathcal{A}(L)$ are unobstructed. Now by following the usual Seiberg-Witten theory we get:

Theorem 11. *Given (X, Ψ, Λ) , to any Cayley submanifold $f : X^4 \hookrightarrow N$ we can assign an invariant $\Lambda_\Psi(X)$, which is a polynomial in the homology classes of X . Also, the Seiberg-Witten equations (61) can be pulled back by f from global equations on N (analogue of Proposition 9).*

Note that $SU(3)$ and G_2 also act on the corresponding special Lagrangian and coassociative Grassmannians with $SO(3)$ and $SO(4)$ stabilizers, respectively [HL], giving the identifications $G^{SL}(3, 6) = SU(3)/SO(3)$ and $G^{coas}(4, 7) = G_2/SO(4)$. As before, one can study special Lagrangians in a Calabi-Yau manifold, and coassociative submanifolds in a G_2 manifold, by lifting their normal bundles to $SU(2)$. Their deformation spaces are unobstructed and can be identified with H^1 and H_+^2 , respectively. With a similar approach we can relate them to the reduced Donaldson invariants (as the Λ -associative and similarly defined Cayley’s are related to Seiberg-Witten invariants). Similarly one can treat the deformations of associative submanifolds whose boundaries lie on coassociative submanifolds, and the Cayley’s in $Spin(7)$ with associative boundaries in G_2 . Also asymptotically cylindrical associative submanifolds in a G_2 manifold with a Calabi-Yau boundary have similar local deformation spaces, their deformations are related to the corresponding holomorphic curves inside the Calabi-Yau boundary.

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DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, E. LANSING MI, 48824
E-mail address: akbulut@math.msu.edu

DEPARTMENT. OF MATHEMATICS, UNIVERSITY OF ROCHESTER, ROCHESTER NY, 14627
E-mail address: salur@math.rochester.edu