

Corks, Plugs and exotic structures

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ABSTRACT. We discuss corks, and introduce new objects which we call plugs. Though plugs are fundamentally different objects, they also detect exotic smooth structures in 4-manifolds like corks. We discuss relation between corks, plugs and rational blow-downs. We show how to detect corks and plugs inside of some exotic manifolds. Furthermore, we construct knotted corks and plugs.

1. Introduction

Let W_n , $\overline{W}_n$ and $W_{m,n}$ be the smooth 4-manifolds given by Figure 1. Notice that W_1 is a version of the manifold defined in [22] by Mazur, and $\overline{W}_n$ is the “positron” introduced by the first author and Matveyev [8]. The first author [1] proved that $E(2)\#\overline{\mathbf{CP}}^2$ changes its diffeomorphism type by regluing an imbedded copy of W_1 inside via a natural involution on the boundary ∂W_1 . This was later generalized to $E(n)\#\overline{\mathbf{CP}}^2$ ($n \geq 2$) by Bižaca-Gompf [11]. The following general theorem was first proved independently by Matveyev [21], Curtis-Freedman-Hsiang-Stong [13], and later on strengthened by the first author and Matveyev [8]:

Theorem 1.1 ([21], [13], [8]). *For every homeomorphic but non-diffeomorphic pair of simply connected closed 4-manifolds, one is obtained from the other by removing a contractible 4-manifold and gluing it via an involution on the boundary. Such a contractible 4-manifold has since been called a Cork. Furthermore, corks and their complements can always be made compact Stein 4-manifolds.*

Also in [19] Kirby gave a description of the 5-dimensional h -cobordisms induced by corks. So, clearly corks are very important in 4-manifold topology, but unfortunately not much is known about them. Even though corks determine exotic copies of any manifold, we only know a few concrete examples. Sometime ago the first author posed the question: “Is W_1 a universal cork?”, i.e. whether W_1 is sufficient to detect every exotic structure? This paper came out of our searches for corks in some concrete examples of exotic manifolds.

Key words and phrases. 4-manifold, handlebody, Stein manifold, rational blow-down, h -cobordism.

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In this work we realized that besides a *Cork* there is another basic fundamental object in 4-manifolds which detects exoticity; we named it a *Plug*. Plugs naturally appear when we do surgeries, for example, rational blow-downs. So searching plug structures of 4-manifolds may be easier than investigating cork structures. Even though plugs are defined in a similar way to corks, they are different objects and they can't be explained by corks. However, in some cases, plug operations has the same effect as cork operations (Remark 5.3). Thus plugs might be helpful to study corks. Similarly to corks, enlarging plugs easily provide us exotic pairs of 4-manifolds. It turns out that just as corks generalize the Mazur manifold, plugs generalize the “Gluck construction”. The lack of understanding how different corks are related to each other have prevented us from defining 4-manifold invariants from the Stein decomposition theorem of 4-manifolds [8]. We hope that plugs will shed light on understanding corks (e.g. they might be deformations of corks).

This paper is organized as follows. In Section 2, we recall the definition of *Corks*, and then define new objects which we call *Plugs*. Plugs have a property similar to corks and they naturally appear in 0-log transform and rational blow-down operations. We prove for $n \geq 2$ and $m \geq 1$ that W_n and $\overline{W}_n$ are corks and that $W_{m,n}$ is a plug. It is an interesting question whether the sequences of corks W_n , $\overline{W}_n$ and the plugs $W_{m,n}$ are sufficient to detect exoticity of all 4-manifolds.

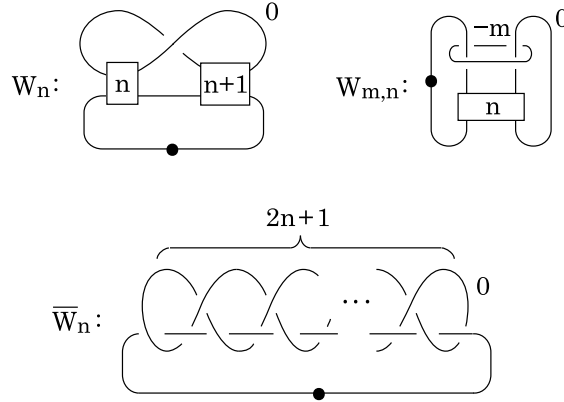


FIGURE 1

In Section 3, we give examples of 4-manifolds which contain W_n (resp. $W_{m,n}$) as a cork (resp. as a plug). For example, W_1 is a cork of elliptic surfaces $E(n)_{p,q} \# \overline{\mathbf{CP}}^2$ ($n \geq 2$) and knot surgered elliptic surfaces $E(n)_K \# \overline{\mathbf{CP}}^2$ ($n \geq 2$). In Section 4 we review the operations of rational blow-down and logarithmic transform, and later in Section 5 we relate them to corks and plugs. This shows that the cork and plug operations can be naturally identified under some conditions. As an interesting test case, in Section 6, we

draw a handlebody picture of the exotic $\mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2$ without 1- and 3-handles, which was recently constructed (and named E'_3) by the second author [25], [26], and demonstrate how to locate corks and plugs inside. Moreover, by constructions similar to [26], we show that corks and plugs can be knotted, namely, two different embeddings of the same cork (resp. plug) can produce two different smooth structures by cork (resp. plug) operations.

In a subsequent paper [9], we will show that by enlarging corks and plugs in two different ways we can construct pairs of small Stein manifolds that are exotic copies of each other (existence of exotic Stein manifold pairs has been established recently in [10]).

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2. Corks and Plugs

In this section, we define corks and plugs, and give examples of exotic smooth structures on Stein manifolds.

Definition 2.1. Let C be a compact Stein 4-manifold with boundary and $\tau : \partial C \rightarrow \partial C$ an involution on the boundary. We call (C, τ) a *Cork* if τ extends to a self-homeomorphism of C , but cannot extend to any self-diffeomorphism of C . A cork (C, τ) is called a cork of a smooth 4-manifold X , if $C \subset X$ and X changes its diffeomorphism type when removing C and re-gluing it via τ . Note that this operation does not change the homeomorphism type of X .

Definition 2.2. Let P be a compact Stein 4-manifold with boundary and $\tau : \partial P \rightarrow \partial P$ an involution on the boundary, which cannot extend to any self-homeomorphism of P . We call (P, τ) a *Plug* of X , if $P \subset X$ and X keeps its homeomorphism type and changes its diffeomorphism type when removing P and gluing it via τ . We call (P, τ) a *Plug* if there exists a smooth 4-manifold X such that (P, τ) is a plug of X .

Definition 2.3. Let W_n , $\overline{W}_n$ and $W_{m,n}$ be smooth 4-manifolds in Figure 1. Let $f_n : \partial W_n \rightarrow \partial W_n$, $\bar{f}_n : \partial \overline{W}_n \rightarrow \partial \overline{W}_n$ and $f_{m,n} : \partial W_{m,n} \rightarrow \partial W_{m,n}$ be the obvious involutions obtained by first surgering $S^1 \times B^3$ to $B^2 \times S^2$ in the interiors of W_n , $\overline{W}_n$ and $W_{m,n}$, then surgering the other imbedded $B^2 \times S^2$ back to $S^1 \times B^3$ (i.e. replacing the dots in Figure 1), respectively. Notice the diagrams of W_n , $\overline{W}_n$, and $W_{m,n}$ are symmetric links.

Remark 2.4. Note the following useful facts:

- (1) W_1 is a Mazur manifold, and (W_1, f_1) is equal to (W, f) in [1], [6].
- (2) $\overline{W}_n$ is the positron introduced by the first author and Matveyev in [8].
- (3) W_n and $\overline{W}_n$ are contractible, whereas $W_{m,n}$ is homotopy equivalent to S^2 .
- (4) W_n ($n \geq 0$), $\overline{W}_n$ ($n \geq 0$) and $W_{m,n}$ ($m, n \geq 1$) are simply connected compact Stein 4-manifolds. We check this by changing the 1-handle notations of W_n , $\overline{W}_n$ and $W_{m,n}$, and

putting the 2-handles in Legendrian positions as in Figure 2, and checking the Eliashberg criterion: the framings on the 2-handles are less than Thurston-Bennequin number.

(5) If $W_{1,0} \subset X$, removing and regluing $W_{1,0}$ to via $f_{1,0}$ has an affect of introducing a Gluck twist to X (i.e. cutting out an imbedded copy of $S^2 \times D^2$ from X , and then regluing by the nontrivial diffeomorphism of $S^2 \times S^1$). So far, we only know one example of smooth manifold (which is non-orientable) that becomes exotic by this Gluck operation, [3].

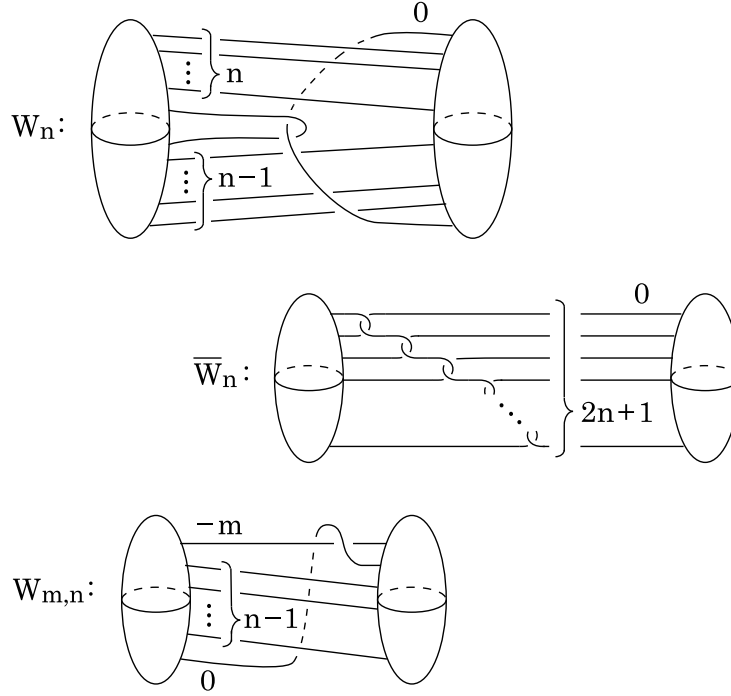


FIGURE 2

We quickly prove the following theorem.

Theorem 2.5. (1) For $n \geq 1$, the pairs (W_n, f_n) and $(\overline{W}_n, \bar{f}_n)$ are corks.
 (2) For $m \geq 1$ and $n \geq 2$, the pair $(W_{m,n}, f_{m,n})$ is a plug.

Remark 2.6. (1) In the case of (W_1, f_1) this theorem was proved by the first author [1].
 (2) (W_0, f_0) and $(\overline{W}_0, \bar{f}_0)$ are not a corks, because each of the involutions f_0 and $\bar{f}_0$ comes from an involution on the boundary of 4-ball B^4 , and every diffeomorphism of S^3 extends to a self-diffeomorphism of B^4 . The pair $(W_{m,1}, f_{m,1})$ (m : arbitrary integer) is not a plug by the same reason.

(3) $(W_{m,0}, f_{m,0})$ (m : arbitrary integer) is not a plug because $W_{m,0} = S^2 \times B^2$ is not a Stein manifold.

This theorem clearly follows from the following Lemmas 2.7 and 2.8.

Lemma 2.7. (1) For $n \geq 1$, the involution $f_n : \partial W_n \rightarrow \partial W_n$ extends to a self homeomorphism of W_n .
 (2) For $n \geq 1$, the involution $\bar{f}_n : \partial \bar{W}_n \rightarrow \partial \bar{W}_n$ extends to a self homeomorphism of $\bar{W}_n$.
 (3) For $m \geq 1$ and $n \geq 2$, the involution $f_{m,n} : \partial W_{m,n} \rightarrow \partial W_{m,n}$ cannot extend to any self homeomorphism of $W_{m,n}$.

Proof. (1) (resp. (2)) is immediate since the boundary of W_n (resp. $\bar{W}_n$) is a homology sphere and W_n (resp. $\bar{W}_n$) is contractible (see [12] for a general discussion). To see (3), suppose that $f_{m,n} : \partial W_{m,n} \rightarrow \partial W_{m,n}$ extends to a self homeomorphism of $W_{m,n}$ and $m \geq 1$ and $n \geq 2$. Then the two smooth 4-manifolds in the Figure 3 have the same intersection form, because one is obtained from the other by removing $W_{m,n}$ and regluing it via $f_{m,n}$. The intersection forms of the left and right 4-manifolds in Figure 3 are

$$\begin{pmatrix} -2n - mn^2 & 1 \\ 1 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} -2n - mn^2 & -1 - mn \\ -1 - mn & -1 - m \end{pmatrix},$$

respectively. However, we can easily prove that they are not isomorphic. (The right intersection form does not contain any element with self intersection number -1 .) This is a contradiction. $\square$

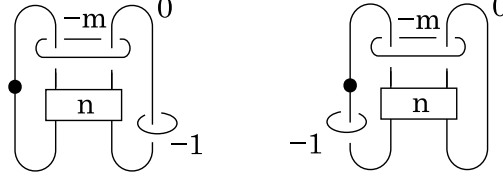


FIGURE 3

The following is a generalization of [2] and [7], it basically says that by enlarging W_n , $\bar{W}_n$ and $W_{m,n}$ we can obtain exotic manifolds (hence these manifolds are corks and plugs of their enlargements).

Lemma 2.8. (1) Let W_n^1 and W_n^2 be simply connected compact smooth 4-manifolds defined in Figure 4. Then W_n^1 and W_n^2 ($n \geq 1$) are homeomorphic but not diffeomorphic. In particular, the involution $f_n : \partial W_n \rightarrow \partial W_n$ cannot extend to any self diffeomorphism of W_n . Note that W_n^2 is obtained from W_n^1 by regluing W_n via f_n .

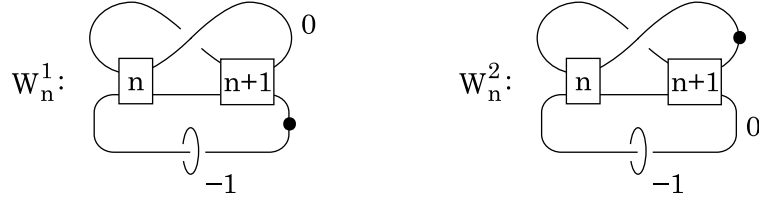


FIGURE 4

(2) Let $\overline{W}_n^1$ and $\overline{W}_n^2$ be simply connected compact smooth 4-manifolds defined in Figure 5. Then $\overline{W}_n^1$ and $\overline{W}_n^2$ ($n \geq 1$) are homeomorphic but not diffeomorphic. In particular, the involution $\overline{f}_n : \partial \overline{W}_n \rightarrow \partial \overline{W}_n$ cannot extend to any self diffeomorphism of $\overline{W}_n$. Note that $\overline{W}_n^2$ is obtained from $\overline{W}_n^1$ by regluing $\overline{W}_n$ via $\overline{f}_n$.

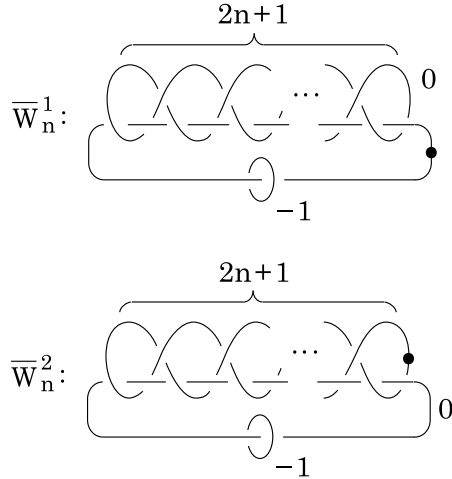


FIGURE 5

(3) Let $W_{m,n}^1$ and $W_{m,n}^2$ be simply connected compact smooth 4-manifolds defined in Figure 6. Then $W_{m,n}^1$ and $W_{m,n}^2$ ($m \geq 1, n \geq 2$) are homeomorphic but not diffeomorphic. Note that $W_{m,n}^2$ is obtained from $W_{m,n}^1$ by regluing $W_{m,n}$ via $f_{m,n}$.

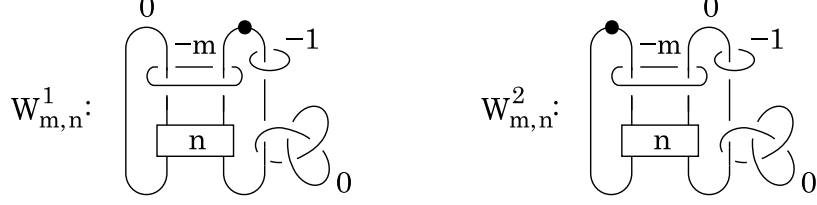


FIGURE 6

Proof. (1) Lemma 2.7.(1) implies that W_n^1 and W_n^2 are homeomorphic. We can easily check that W_n^1 is a compact Stein manifold by an isotopy in Figure 4 (it follows from the Eliashberg criterion: The framing on the 2-handles are less or equal to the Thurston-Benequin framing). Since every compact Stein manifold can be embedded into a minimal symplectic 4-manifold (e.g. [20]), W_n^1 has no 2-sphere with self intersection number -1 . But clearly W_n^2 contains a 2-sphere with self intersection number -1 . In particular, W_n^2 is not a compact Stein manifold. Hence W_n^1 and W_n^2 ($n \geq 1$) are not diffeomorphic.

(2) Similar to (1), we can show that $\overline{W}_n^1$ is a compact Stein manifold and that $\overline{W}_n^2$ is not a compact Stein manifold. Therefore $\overline{W}_n^1$ and $\overline{W}_n^2$ are not diffeomorphic.

(3) As before, similar to (1), we can show that $W_{m,n}^1$ is a compact Stein manifold and that $W_{m,n}^2$ is not a compact Stein manifold. Therefore, $W_{m,n}^1$ and $W_{m,n}^2$ ($n \geq 1$) are not diffeomorphic. We check they are homeomorphic. The intersection forms of $W_{m,n}^1$ and $W_{m,n}^2$ are

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & -m & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -2n - n^2m & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix},$$

respectively. We can easily show that they are isomorphic to $\langle 1 \rangle \oplus \langle -1 \rangle \oplus \langle -1 \rangle$. It thus follows from Boyer's theorem [12] that $W_{m,n}^1$ and $W_{m,n}^2$ are homeomorphic. Note that their boundaries are diffeomorphic. $\square$

Remark 2.9. Here we showed W_n , $\overline{W}_n$ and $W_{m,n}$ are corks and plugs by using a quick argument of the first author and Matveyev [8] based on property of Stein manifolds. Notice that this technique gives W_n^i are $W_{m,n}^i$ are Stein only when $i = 1$, whereas when $i = 2$ they are not Stein (they have an imbedded -1 sphere). In a subsequent paper [9], by a more sophisticated argument, by enlarging corks and plugs we construct simply connected exotic compact Stein manifold pairs realizing any Betti number $b_2 \geq 1$.

The technical proposition below makes arguments of stabilizations easy. We use this proposition in Section 5.

Proposition 2.10. (1) *Let X be a simply connected compact smooth 4-manifold which contains W_n (resp. $\overline{W}_n$). Let Y be the simply connected compact smooth 4-manifold obtained from X by removing W_n (resp. $\overline{W}_n$) and regluing it via f_n (resp. $\overline{f}_n$). Then $X\#S^2 \times S^2$ is diffeomorphic to $Y\#S^2 \times S^2$.*

(2) *Let X be a compact smooth 4-manifold which contains $W_{m,n}$. Let Y be the compact smooth 4-manifold obtained from X by removing $W_{m,n}$ and regluing it via $f_{m,n}$. If m is even (resp. odd), then $X\#S^2 \times S^2$ (resp. $X\#\mathbf{CP}^2\#\overline{\mathbf{CP}}^2$) is diffeomorphic to $Y\#S^2 \times S^2$ (resp. $Y\#\mathbf{CP}^2\#\overline{\mathbf{CP}}^2$).*

Proof. (1) $X\#S^2 \times S^2$ is obtained from X by surgering a homotopically trivial loop (with the correct framing). We choose to surger X along a meridian of the unique dotted circle of W_n (resp. $\overline{W}_n$). This corresponds to turning the unique 1-handle of W_n (resp. $\overline{W}_n$) into a 0-framed 2-handle (i.e. replace the dot with 0-framing). Similarly, $Y\#S^2 \times S^2$ is obtained from Y by replacing the dot of the dotted circle of W_n (resp. $\overline{W}_n$) with 0. It thus follows from the definition of Y that $X\#S^2 \times S^2$ and $Y\#S^2 \times S^2$ have the same handlebody diagrams.

(2) Since the dotted circle of $W_{m,n}$ has $-m$ -framed meridian, changing the dotted circle into 0-framed circle corresponds to a connected sum with $S^2 \times S^2$. This fact implies the required claim. $\square$

3. Examples

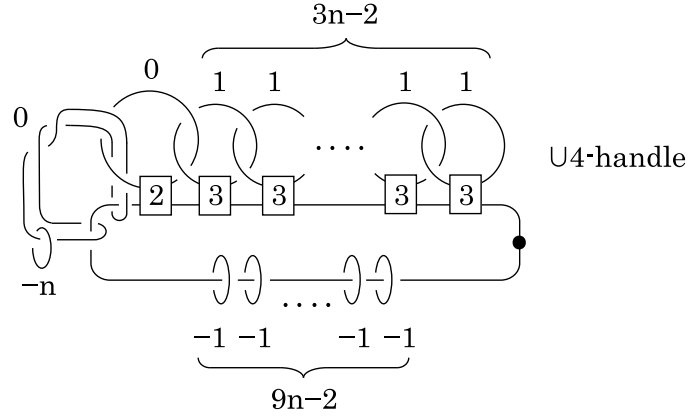
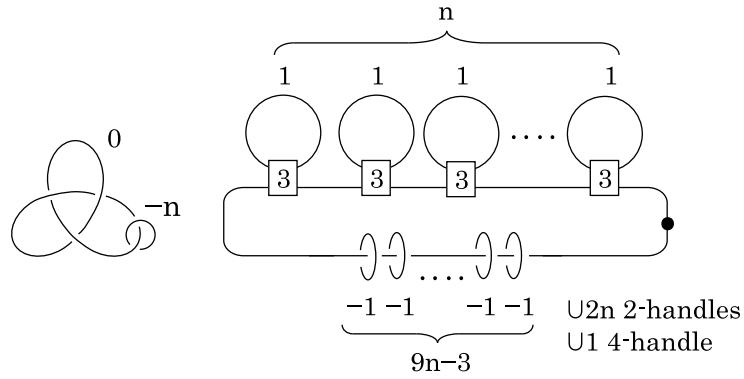
Here, by improving an argument in Gompf-Stipsicz [17, Section 9.3], we give examples of closed manifolds containing W_n and $\overline{W}_n$ (resp. $W_{m,n}$) as corks (resp. plugs).

Let $E(n)$ be the relatively minimal simply connected elliptic surface with Euler characteristic $12n$ and with no multiple fibers, and $E(n)_{p_1, \dots, p_k}$ the elliptic surface obtained from $E(n)$ by performing logarithmic transformations of multiplicities $p_1, \dots, p_k$. Now recall.

Theorem 3.1 (Gompf-Stipsicz [17]). *For $n \geq 1$, the elliptic surface $E(n)$ has the handle decomposition as in Figure 7. The obvious cusp neighborhood (i.e. the dotted circle, -1 -framed meridian of the dotted circle, and the left most 0-framed unknot) is isotopic to the regular neighborhood of a cusp fiber of $E(n)$.*

Corollary 3.2. *For $n \geq 1$, the elliptic surface $E(n)$ has the handle decomposition as in Figure 8. The obvious cusp neighborhood (i.e. 0-framed trefoil knot) is isotopic to the regular neighborhood of a cusp fiber of $E(n)$.*

Proof. In Figure 7, pull off the leftmost 0-framed unknot from the dotted circle by sliding over -1 -framed knot. $\square$


 FIGURE 7. $E(n)$

 FIGURE 8. $E(n)$

Proposition 3.3. *For $n \geq 1$ and $m \geq 1$ we have*

- (1) $E(2n) \# \overline{\mathbf{CP}}^2$ has the handle decompositions as in Figure 9.
- (2) $E(2n) \# m \overline{\mathbf{CP}}^2$ has the handle decompositions as in Figure 10.
- (3) $E(2n+1) \# \overline{\mathbf{CP}}^2$ has the handle decomposition as in Figure 11.

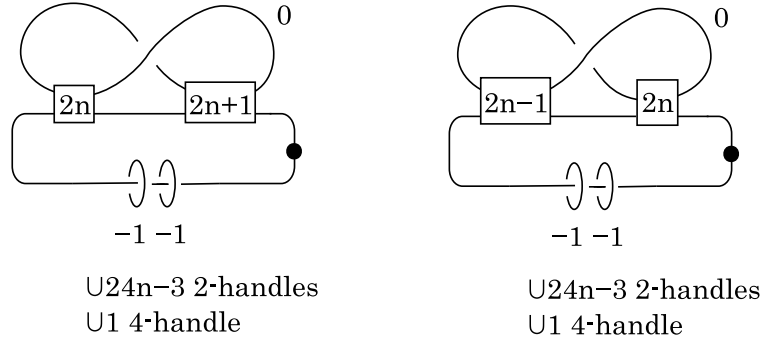


FIGURE 9. Two decompositions of $E(2n) \# \overline{\mathbf{CP}}^2$

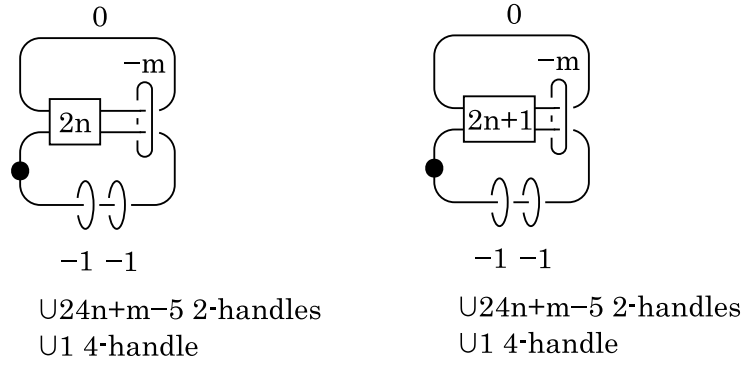


FIGURE 10. Two decompositions of $E(2n) \# m \overline{\mathbf{CP}}^2$ ($m \geq 1$)

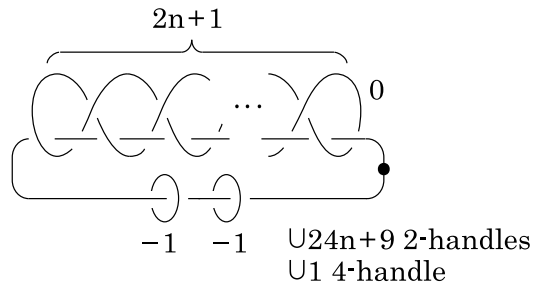


FIGURE 11. $E(2n+1) \# \overline{\mathbf{CP}}^2$ ($n \geq 1$)

Proof. (1) Starting with Figure 8, the handle slides as in Figure 26 give Figure 27. We easily get Figure 28 by handle slides and isotopies to the $E(2n)$ case of Figure 27. By blowing up, we get Figure 29. We have the left diagram in Figure 9 by a slide. We also get Figure 30 by a slide in Figure 29. Notice that the Figure 30 is isotopic to the right diagram in Figure 9.

(2) In Figure 29, blowing up the left (resp. right) middle -1 -framed circle $m - 1$ times gives the left (resp. right) diagram in Figure 10.

(3) It follows from a diagram of $E(n)$ in Figure 27 that $E(2n + 1)$ admits a diagram as in Figure 31. We obtain a diagram of $E(2n + 1)$ in Figure 36 by handle slides and isotopies as in Figure 31 $\sim$ 36. By blowing up, we get Figure 11. $\square$

This proposition implies the following theorem which says that many exotic 4-manifolds admit W_n and $\overline{W}_n$ (resp. $W_{m,n}$) as corks (resp. plugs).

Theorem 3.4. *For each $n \geq 1$ and $m \geq 1$ we have*

- (1) (W_{2n-1}, f_{2n-1}) and (W_{2n}, f_{2n}) are corks of $E(2n) \# \overline{\mathbf{CP}}^2$.
- (2) $(W_{m,2n}, f_{m,2n})$ and $(W_{m,2n+1}, f_{m,2n+1})$ are plugs of $E(2n) \# m\overline{\mathbf{CP}}^2$.
- (3) $(\overline{W}_n, \overline{f}_n)$ is a cork of $E(2n + 1) \# \overline{\mathbf{CP}}^2$.

Proof. (1) Figure 9 implies that $E(2n) \# \overline{\mathbf{CP}}^2$ splits off $2\overline{\mathbf{CP}}^2$ by regluing W_{2n} (resp. W_{2n-1}) via f_{2n} (resp. f_{2n-1}). The Seiberg-Witten invariant and the blow-up formula imply that $E(2n) \# \overline{\mathbf{CP}}^2$ cannot split off $2\overline{\mathbf{CP}}^2$, therefore (W_{2n-1}, f_{2n-1}) and (W_{2n}, f_{2n}) are corks of $E(2n) \# \overline{\mathbf{CP}}^2$. (2) and (3) are similar to (1). $\square$

Let $E(n)_{p,q}$ denote the 4-manifold obtained from $E(n)$ by the logarithmic transform on two disjoint torus fibers. Also, for a knot K in S^3 with a nontrivial Alexander polynomial, let $E(n)_K$ be the 4-manifold obtained from $E(n)$ by Fintushel-Stern's knot surgery ([15]) with K in the regular neighborhood of a cusp fiber. Then we have:

Theorem 3.5. *For each $p, q \geq 1$ and $\gcd(p, q) = 1$ we have*

- (1) W_1 (resp. $\overline{W}_1$) is a cork of $E(n)_{p,q} \# \overline{\mathbf{CP}}^2$ ($n \geq 2$) [resp. $n \geq 3$].
- (2) $W_{1,3}$ is a plug of $E(n)_{p,q} \# \overline{\mathbf{CP}}^2$ ($n \geq 2$) (also valid for $n = 1$ and $p, q \geq 2$).
- (3) W_1 (resp. $\overline{W}_1$) is a cork of $E(n)_K \# \overline{\mathbf{CP}}^2$ ($n \geq 2$) [resp. $n \geq 3$].
- (4) $W_{1,3}$ is a plug of $E(n)_K \# \overline{\mathbf{CP}}^2$ ($n \geq 1$).

Proof. Start from Figure 27. We can easily construct W_1 (resp. $W_{1,3}$) in $E(n) \# \overline{\mathbf{CP}}^2$ when $n \geq 2$ (resp. $n \geq 1$) away from the regular neighborhood of a cusp fiber of $E(n)$, by an argument similar to the proof of Proposition 3.3. Perform logarithmic transformations of multiplicities p and q (resp. knot surgery along K) in the cusp neighborhood. Then an argument similar to the proof of Theorem 3.4 shows the required claims. The case of $\overline{W}_1$ is almost the same. $\square$

4. Rational blow-down and logarithmic transform

In this section, we review the rational blow-down introduced by Fintushel-Stern [14] and logarithmic transform (see also Gompf-Stipsicz [17]). We also give a new procedure to change handlebody diagrams by rational blow-downs and logarithmic transforms.

Let C_p and B_p ($p \geq 2$) be the smooth 4-manifolds in Figure 12. The boundary ∂C_p of C_p is diffeomorphic to the lens space $L(p^2, p-1)$ and to the boundary ∂B_p of B_p .

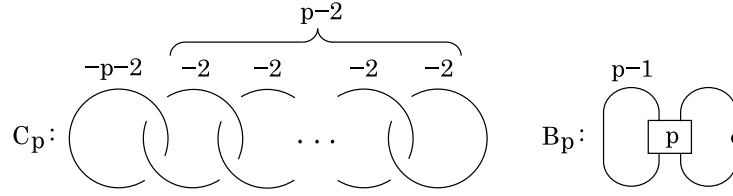
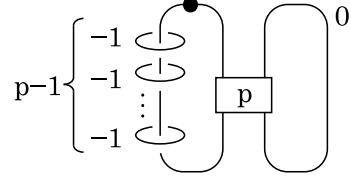
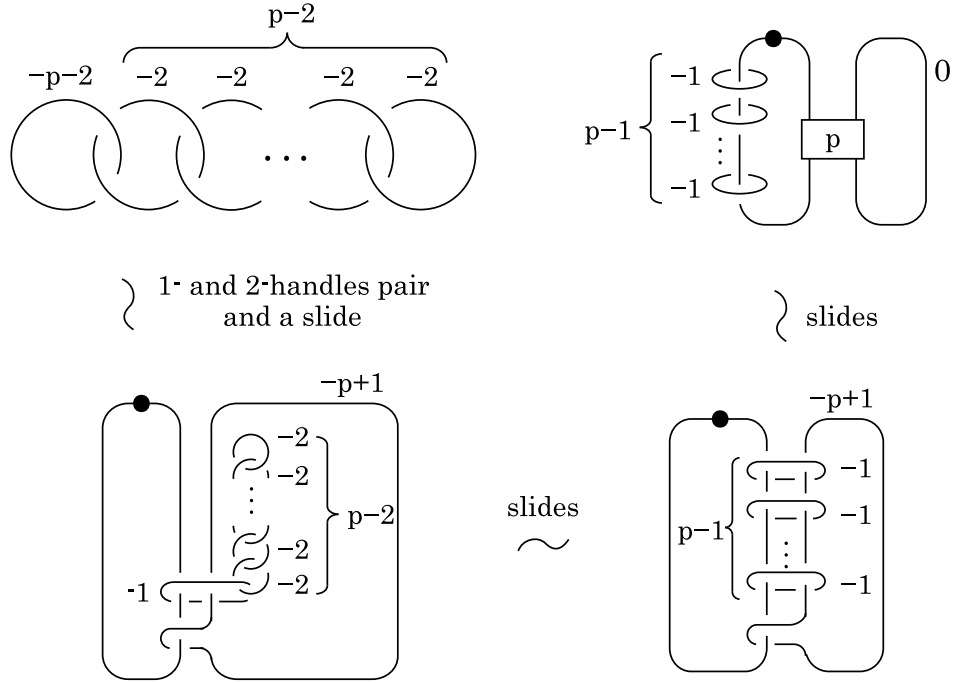


FIGURE 12

Suppose that C_p embeds in a smooth 4-manifold X . Let $X_{(p)}$ be a smooth 4-manifold obtained from X by removing C_p and gluing B_p . The 4-manifold $X_{(p)}$ is called the rational blow-down of X along C_p . Note that $X_{(p)}$ is uniquely determined up to diffeomorphism by a fixed pair (X, C_p) . We briefly recall the procedure given in [17, Section 8.5] to obtain a handlebody diagram of $X_{(p)}$ from a diagram of X . Change the diagram of C_p in Figure 12 into the diagram in Figure 13 (introduce a 1-handle/2-handle pair and slide handles as shown in Figure 14). Then surger the obvious $S^1 \times B^3$ to $B^2 \times S^2$ and surger the other $B^2 \times S^2$ to $S^1 \times B^3$ (i.e. change the dot and 0 in Figure 13). Finally blow down -1 -framed unknots. See Figure 15 for this procedure.

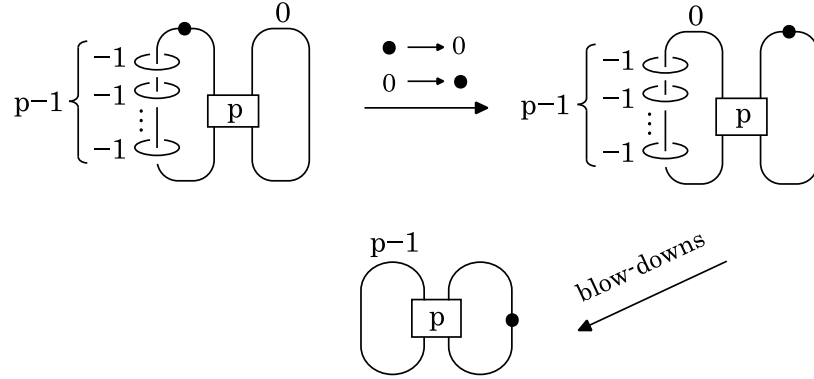
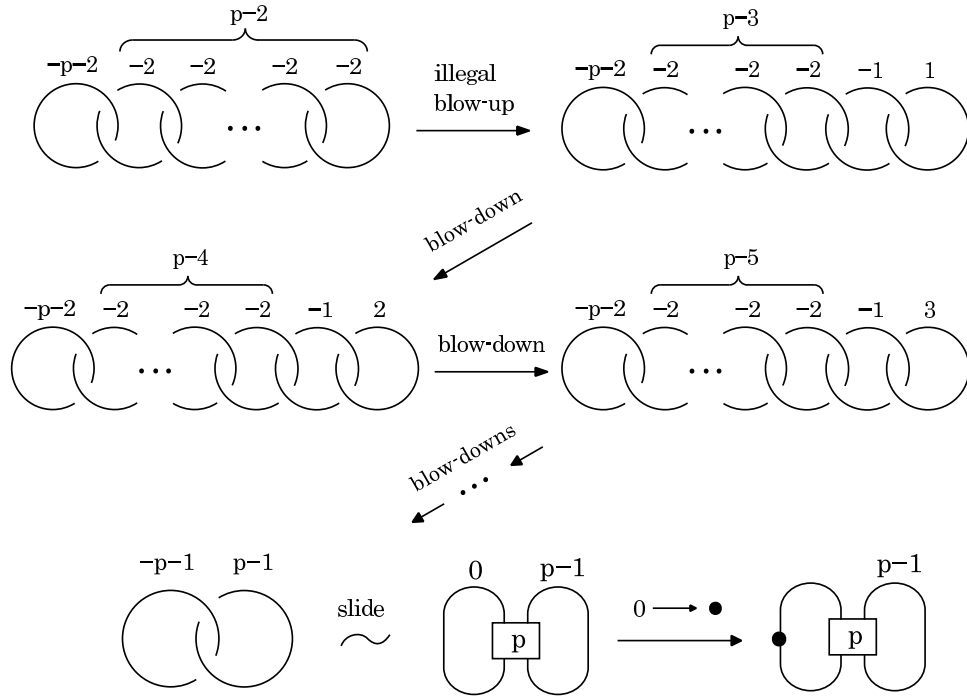
We now give a new way to draw a handlebody diagram of $X_{(p)}$, as indicated in Figure 16: First we connected sum with $\mathbf{CP}^2$ and get the second diagram. Now we have a -1 -framed unknot, then blow down this -1 -framed unknot. Now again we have a -1 -framed unknot, by repeating blow-downs as before, we get the fifth diagram. A handle slide gives the sixth diagram. Finally we replace 0 with a dot. Since this procedure changes C_p into B_p , this is the rational blow-down operation.


 FIGURE 13. C_p

 FIGURE 14. diagrams of C_p

Next, we discuss the logarithmic transform. Let φ_p ($p \geq 0$) be the self-diffeomorphism of T^3 induced by the automorphism

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & p \end{pmatrix}$$

of $H_1(S^1; \mathbf{Z}) \oplus H_1(S^1; \mathbf{Z}) \oplus H_1(S^1; \mathbf{Z})$ with the obvious basis. Suppose that a smooth 4-manifold X contains a torus T with trivial normal bundle $\nu(T) \approx T \times B^2$.


 FIGURE 15. Rational blow-down along C_p

 FIGURE 16. Rational blow-down along C_p

Now we remove $\text{int } \nu(T)$ from X and glue $T^2 \times B^2$ back via $\varphi_p : T^2 \times S^1 \rightarrow \partial\nu(T)$. We call this operation a logarithmic transform of X with multiplicity p . Note that this definition is consistent with the one given in [17] when T is in a cusp neighborhood (i.e. B^4 with a 2-handle attached along a 0-framed right trefoil knot).

Let X_p be a logarithmic transform of X with multiplicity p . We can draw a handlebody diagram of X_p as indicated in Figure 17. The first diagram denotes $\nu(T) \approx T \times B^2$ in X . We will keep track of circles α, β, γ on the boundary $\partial\nu(T)$ during the diffeomorphisms we describe. Replace the dot of the lower circle with 0 of the middle circle. Then we get the third diagram by an isotopy. This corresponds to a logarithmic transform of X with multiplicity 0. We next change the dot of the lower circle into 0 and blow up as shown in the figure. Then we blow down the upper -1 -framed circle. We again blow up slide and blow down as shown in the Figure 14. By repeating blow-ups and blow-downs similarly, we get the eighth diagram. Finally change the lower 0-framed unknot into a dotted circle. This gives the last diagram, which is the handlebody of X_p . By inspecting the positions of α, β, γ in the last diagram, we can easily verify that this gluing map corresponds to the map $\varphi_p^{-1} : \partial\nu(T) \rightarrow T^2 \times S^1$.

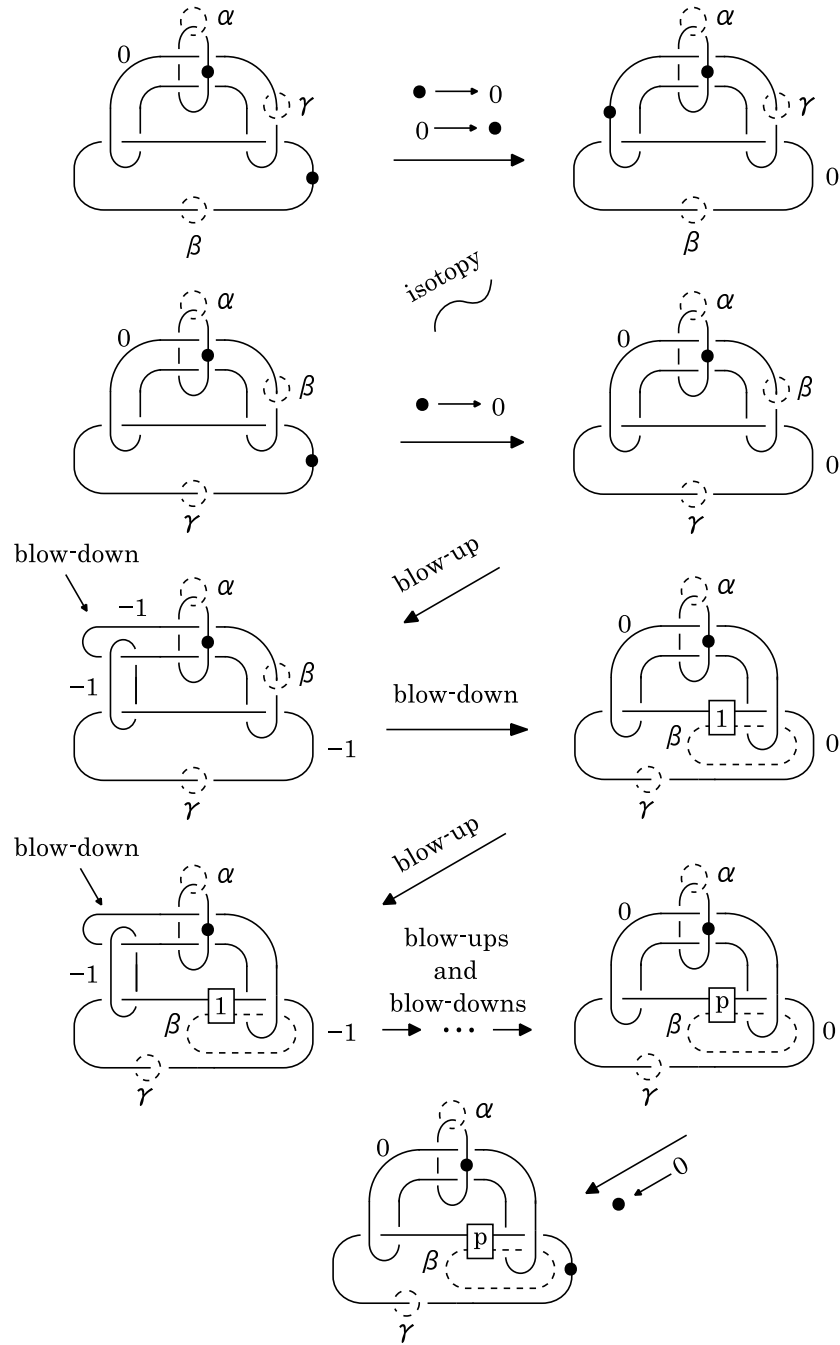
Remark 4.1. Note that the 0-log transform operation removes a Stein manifold $T^2 \times B^2$ and reglues it via an involution on the boundary. It follows from Gompf [16] that 0-log transforms can be plug operations.

The rational blow-down has the following relation with the logarithmic transform.

Theorem 4.2 (Fintushel-Stern [14], see also Gompf-Stipsicz [17]). *Suppose that a smooth 4-manifold X contains a fishtail neighborhood, that is, the smooth 4-manifold in Figure 18. Let X_p ($p \geq 2$) be the smooth 4-manifold obtained from X by a logarithmic transform with multiplicity p in the fishtail neighborhood. Then there exists a copy of C_p in $X \# (p-1) \overline{\mathbf{CP}}^2$ such that the rational blow-down of $X \# (p-1) \overline{\mathbf{CP}}^2$ along the copy of C_p is diffeomorphic to X_p .*

Proof. Fintushel-Stern constructed C_p in $X \# (p-1) \overline{\mathbf{CP}}^2$ as indicated in Figure 19. We keep track of circles α, β, γ on the boundary $\partial(T^2 \times B^2)$. We cancel a 1-handle of the fishtail neighborhood as in the second diagram. By blowing up $(p-1)$ times, we get C_p as in the last diagram.

We draw a diagram of the rational blow down of $X \# (p-1) \overline{\mathbf{CP}}^2$ along this C_p by the procedure in Figure 16. Then we can easily show that this is the same operation as logarithmic transform of X with multiplicity p , by checking positions of α, β, γ . $\square$


 FIGURE 17. Logarithmic transform with multiplicity p

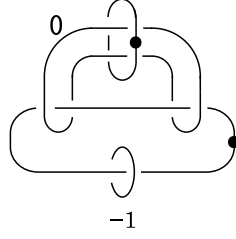
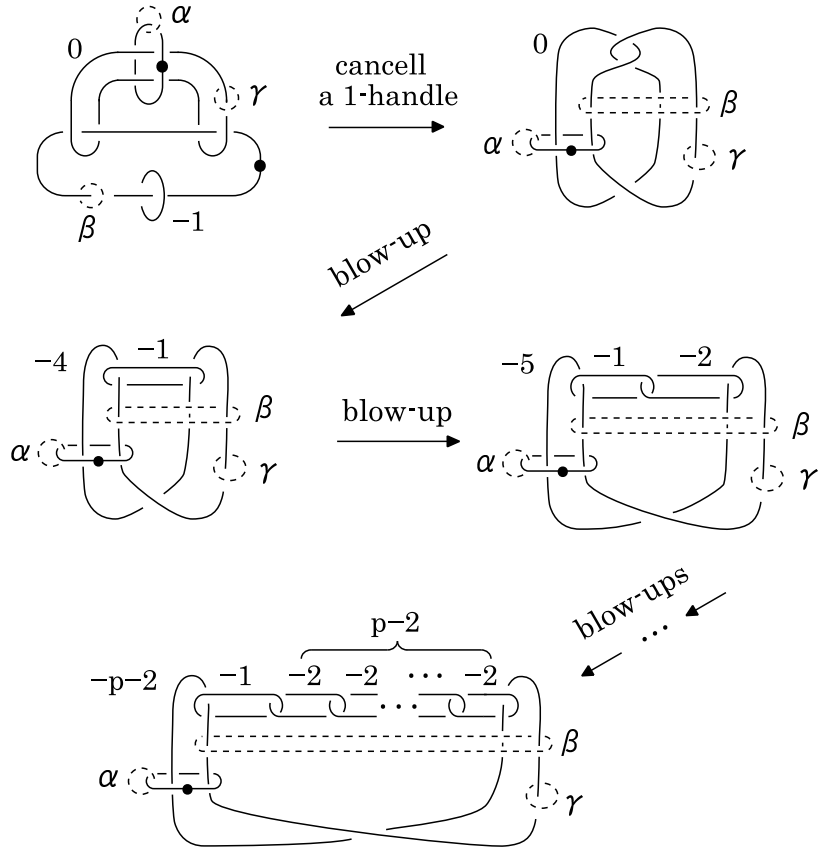


FIGURE 18. fishtail neighborhood


 FIGURE 19. Construction of C_p in $X\#(p-1)\overline{\mathbf{CP}}^2$

5. Corks, Plugs and rational blow-down

In this section we relate the rational blow-down operation with regluing corks and plugs. Recall that the rational blow-down operation can change the first and second homology groups. Since regluing corks W_n and plugs $W_{m,n}$ does not change these groups, we need suitable assumptions to relate these operations.

Let $T_{p,q}$, $U_{p,q}$ and $V_{p,q}$ be the smooth 4-manifolds in Figure 20. Note that 4-manifolds $T_{p,q}$, $U_{p,q}$ and $V_{p,q}$ contain C_p .

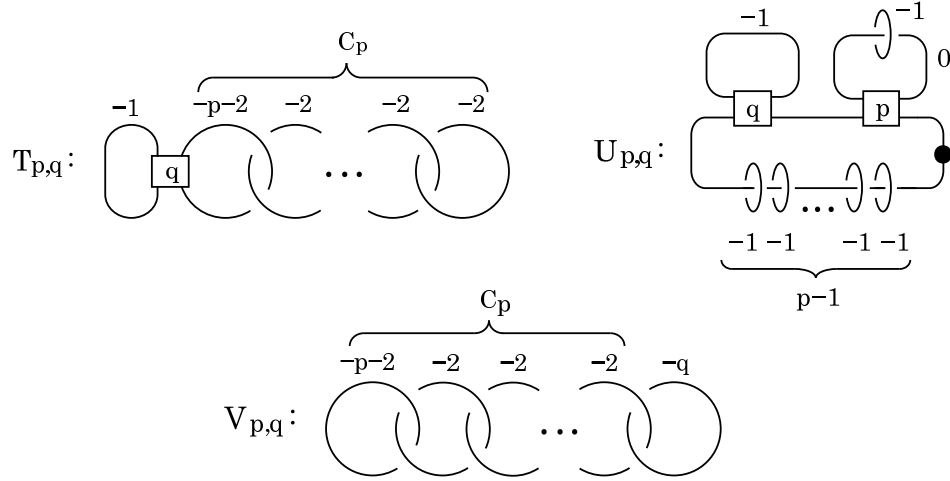


FIGURE 20

Theorem 5.1. (1) Suppose that a smooth 4-manifold X contains $T_{p,p-1}$ (resp. $T_{p,p+1}$). Let $X_{(p)}$ be the rational blow-down of X along C_p . Then X contains W_{p-2} (resp. W_{p-1}) such that $X_{(p)} \# (p-1) \overline{\mathbf{CP}}^2$ is obtained from X by removing W_{p-2} (resp. W_{p-1}) and regluing it via f_{p-2} (resp. f_{p-1}).

(2) Suppose that a smooth 4-manifold X contains $U_{p,p-1}$ (resp. $U_{p,p+1}$). Let $X_{(p)}$ be the rational blow-down of X along C_p . Then X contains W_{p-2} (resp. W_{p-1}) such that $X_{(p)} \# (p-1) \overline{\mathbf{CP}}^2$ is obtained from X by removing W_{p-2} (resp. W_{p-1}) and regluing it via f_{p-2} (resp. f_{p-1}).

(3) Suppose that a smooth 4-manifold X contains $V_{p,q}$. Let $X_{(p)}$ be the rational blow-down of X along C_p . Then X contains $W_{q-1,p}$ such that $X_{(p)} \# (p-1) \overline{\mathbf{CP}}^2$ is obtained from X by removing $W_{q-1,p}$ and regluing it via $f_{q-1,p}$.

Proof. (1) The $q = p + 1$ case: We can easily get the handle decomposition of $T_{p,p+1}$ in Figure 37 by handle slides of C_p as in Section 4 (See Figure 14.). Slide handles as shown in Figure 37. In the third diagram of Figure 37, we can find a 0-framed unknot which links the dotted circle geometrically once. Replace this dot and 0 as shown in the first row of Figure 38. This operation keeps the diffeomorphism type of X because this corresponds to removing B^4 and regluing B^4 . As a consequence, we can easily find W_{p-1} in X . Note that Figure 39 is isotopic to the standard diagram of W_{p-1} . By removing W_{p-1} in X and regluing it via f_{p-1} , we get the lower diagram of Figure 38.

We can easily check that $X_p \# (p-1) \overline{\mathbf{CP}}^2$ is obtained by replacing the dot and 0 as shown in the left side of Figure 38. Hence, we obtain $X_p \# (p-1) \overline{\mathbf{CP}}^2$ from X by removing W_{p-1} and regluing it via f_{p-1} .

The $q = p - 1$ case is similar. (2) is similar to the (1) case. For (3): handle slides in Figure 20 give Figure 40 (See Figure 14). Then the required claim easily follows from this figure. $\square$

Applying the theorem above, we easily get the following examples. Let E_3 be the smooth 4-manifold constructed by the second author [26]. Note that E_3 is homeomorphic but not diffeomorphic to $\mathbf{CP}^2 \# 9 \overline{\mathbf{CP}}^2$.

Example 5.2. (1) $E_3 \# 2 \overline{\mathbf{CP}}^2$ is obtained from $\mathbf{CP}^2 \# 11 \overline{\mathbf{CP}}^2$ by removing a cork W_1 and regluing it via f_1 . Note that $E_3 \# 2 \overline{\mathbf{CP}}^2$ is homeomorphic but not diffeomorphic to $\mathbf{CP}^2 \# 11 \overline{\mathbf{CP}}^2$.

(2) $E_3 \# 2 \overline{\mathbf{CP}}^2$ is obtained from $\mathbf{CP}^2 \# 11 \overline{\mathbf{CP}}^2$ by removing a plug $W_{1,3}$ and regluing it via $f_{1,3}$.

(3) $E_3 \# \mathbf{CP}^2 \# 3 \overline{\mathbf{CP}}^2$ is diffeomorphic to $2 \mathbf{CP}^2 \# 12 \overline{\mathbf{CP}}^2$.

Proof. (1) See [26, Definition 3.4 and Figure 25]. The 4-manifold E_3 is obtained by rationally blowing down C_3 in a submanifold $T_{3,2}$ of $\mathbf{CP}^2 \# 11 \overline{\mathbf{CP}}^2$. Thus Theorem 5.1 implies the required claim.

(2) See [26, Proposition 3.2.(1) and Definition 3.4]. The 4-manifold E_3 is obtained by rationally blowing down C_3 in a submanifold $V_{3,2}$ of $\mathbf{CP}^2 \# 11 \overline{\mathbf{CP}}^2$. Thus Theorem 5.1 implies the required claim.

(3) This clearly follows from (2) and Proposition 2.10. $\square$

Remark 5.3. Example 5.2.(1) and (2) show that regluing a cork W_3 of $\mathbf{CP}^2 \# 11 \overline{\mathbf{CP}}^2$ is the same operation as regluing a plug $W_{1,3}$, in this case. So, investigating plug structures of 4-manifolds might be helpful to study cork structures of 4-manifolds.

Example 5.4. For $n \geq 1$, the elliptic surface $E(2n) \# 2 \overline{\mathbf{CP}}^2$ contains C_{2n} such that $(E(2n) \# 2 \overline{\mathbf{CP}}^2)_{(2n)} \# (2n-1) \overline{\mathbf{CP}}^2$ is obtained from $E(2n) \# (2n+1) \overline{\mathbf{CP}}^2$ by removing

W_{2n-1} and regluing it via f_{2n-1} . Here $(E(2n)\#2\overline{\mathbf{CP}}^2)_{(2n)}$ denotes the rational blow-down of $E(2n)\#2\overline{\mathbf{CP}}^2$ along the copy of C_{2n} . Note that $(E(2n)\#2\overline{\mathbf{CP}}^2)_{(2n)}\#(2n-1)\overline{\mathbf{CP}}^2$ is homeomorphic but not diffeomorphic to $E(2n)\#(2n+1)\overline{\mathbf{CP}}^2$.

Proof. Start from Figure 28. By blowing ups, we can find $U_{2n,2n+1}$ in $E(2n)\#2\overline{\mathbf{CP}}^2$. Then the required claim follows from Theorem 5.1. $\square$

Example 5.5. For $n, p \geq 2$, the elliptic surface $E(n)_p\#(p-1)\overline{\mathbf{CP}}^2$ is obtained from $E(n)\#(p-1)\overline{\mathbf{CP}}^2$ by removing a plug $W_{n,p}$ and regluing it via $f_{n,p}$.

Proof. Start from Figure 8. Construct C_p in $E(n)\#(p-1)\overline{\mathbf{CP}}^2$ by blowing ups the cusp neighborhood, following the procedure in Figure 19. Note that a cusp neighborhood naturally contains a fishtail neighborhood. Then we can easily find $V_{p,n+1}$. In this case in the last diagram of Figure 19, we have γ with $-n$ framing. In the last diagram of Figure 19, we also have a -1 -framed unknot. Slide γ over the -1 -framed knot. Then we have $V_{p,n+1}$. Now the required claim follows from Theorem 4.2 and 5.1. $\square$

6. Exotic $\mathbf{CP}^2\#9\overline{\mathbf{CP}}^2$

The second author ([25], [26]) constructed the minimal smooth 4-manifold E'_3 which is homeomorphic but not diffeomorphic to $\mathbf{CP}^2\#9\overline{\mathbf{CP}}^2$, and it has neither 1- nor 3-handles. He also gave the way to draw the handlebody diagram of E'_3 , but he did not draw it. In this section, we draw a whole diagram of E'_3 , by slightly changing the construction in [26] to obtain a simple diagram. From this we find a cork and a plug of E'_3 . As far as the authors know, this is the first example of a cork and a plug in minimal 4-manifolds.

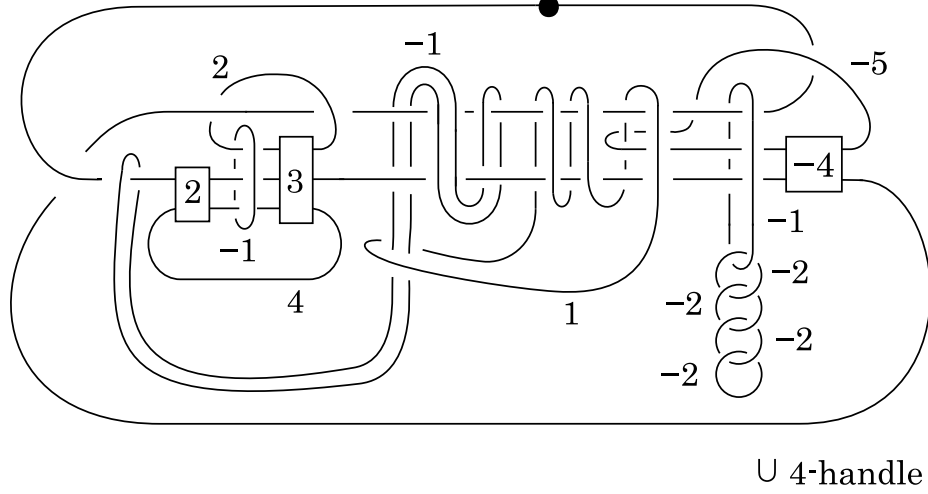
Theorem 6.1. (1) *Figure 21 is a handlebody diagram of a smooth 4-manifold which is homeomorphic but not diffeomorphic to $\mathbf{CP}^2\#9\overline{\mathbf{CP}}^2$ and has the same Seiberg-Witten invariant as the elliptic surface $E(1)_{2,3}$. Consequently, this 4-manifold is minimal, that is, it does not contain any 2-sphere with self intersection number -1 . It is clear from Figure 21 that this minimal 4-manifold has neither 1- nor 3-handles in that handle decomposition.*

(2) W_1 is a cork of the minimal 4-manifold above.

(3) $W_{1,3}$ is a plug of the minimal 4-manifold above.

Remark 6.2. (1) In [5] the first author proved that $\overline{W}_1$ is a cork of the elliptic surface $E(1)_{2,3}$.

(2) In [27], the second author constructed the smooth 4-manifold $X_{a,3}$ ($3 \leq a \leq 7$) which is homeomorphic but not diffeomorphic to $\mathbf{CP}^2\#(12-a)\overline{\mathbf{CP}}^2$. As in Theorem 6.1, we can prove that W_2 is a cork of $X_{4,3}$.

FIGURE 21. A minimal exotic $\mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2$

Here we also show that corks and plugs can be knotted, by similar constructions of exotic rational surfaces in [25] and [26].

Theorem 6.3 (Knotting corks). *There exist two copies of W_3 in $\mathbf{CP}^2 \# 14\overline{\mathbf{CP}}^2$ with the following two properties:*

- (1) *Two 4-manifolds obtained from $\mathbf{CP}^2 \# 14\overline{\mathbf{CP}}^2$ by removing each copy of W_3 and regluing it via f_3 are homeomorphic but not diffeomorphic to $\mathbf{CP}^2 \# 14\overline{\mathbf{CP}}^2$. In particular, each copy of W_3 is a cork of $\mathbf{CP}^2 \# 14\overline{\mathbf{CP}}^2$.*
- (2) *Two 4-manifolds obtained from $\mathbf{CP}^2 \# 14\overline{\mathbf{CP}}^2$ by removing each copy of W_3 and regluing it via f_3 are homeomorphic but not diffeomorphic to each other. In particular, the above two corks of $\mathbf{CP}^2 \# 14\overline{\mathbf{CP}}^2$ are not isotopic to each other.*

Theorem 6.4 (Knotting plugs). *There exist two copies of $W_{4,5}$ in $\mathbf{CP}^2 \# 16\overline{\mathbf{CP}}^2$ with the following two properties:*

- (1) *Two 4-manifolds obtained from $\mathbf{CP}^2 \# 16\overline{\mathbf{CP}}^2$ by removing each copy of $W_{4,5}$ and regluing it via $f_{4,5}$ are homeomorphic but not diffeomorphic to $\mathbf{CP}^2 \# 16\overline{\mathbf{CP}}^2$. In particular, each copy of $W_{4,5}$ is a plug of $\mathbf{CP}^2 \# 16\overline{\mathbf{CP}}^2$.*
- (2) *Two 4-manifolds obtained from $\mathbf{CP}^2 \# 16\overline{\mathbf{CP}}^2$ by removing each copy of $W_{4,5}$ and regluing it via $f_{4,5}$ are homeomorphic but not diffeomorphic to each other. In particular, the above two plugs of $\mathbf{CP}^2 \# 16\overline{\mathbf{CP}}^2$ are not isotopic to each other.*

Let $h, e_1, e_2, \dots, e_n$ be a canonical orthogonal basis of $H_2(\mathbf{CP}^2 \# n \overline{\mathbf{CP}}^2; \mathbf{Z})$ such that $h^2 = 1$ and $e_1^2 = e_2^2 = \dots = e_n^2 = -1$.

In handlebody diagrams, we often write the second homology classes given by 2-handles, instead of usual framings. Note that the square of the homology class given by a 2-handle is equal to the usual framing.

Proof of Theorem 6.1. (1) Recall the construction of E'_3 in [26]: Figure 41 is a standard diagram of $\mathbf{CP}^2 \# 2 \overline{\mathbf{CP}}^2$. By sliding handles as in [26, Figure 8 ~ 17], we obtain Figure 42. Blowing ups gives Figure 43. We have Figure 44 by an isotopy. We get Figure 46 by handle slides as indicated in Figure 44 and 45. By blowing up, we obtain Figure 47. Note that a copy of C_5 (see Figure 48) is inside Figure 47. By rationally blowing down this copy of C_5 , we get a smooth 4-manifold E'_3 . By using the rational blow-down procedure in Figure 15, we obtain Figure 21 of E'_3 . Though this construction of E'_3 is slightly different from [26], the same argument as in [26] shows that this E'_3 is homeomorphic to $\mathbf{CP}^2 \# 9 \overline{\mathbf{CP}}^2$ and has the same Seiberg-Witten invariant as the elliptic surface $E(1)_{2,3}$. Therefore this E'_3 is minimal.

(2) We can get Figure 49 from Figure 41 by introducing a 2-handle/3-handle pair and sliding the new 2-handle. We obtain Figure 50 of $\mathbf{CP}^2 \# 13 \overline{\mathbf{CP}}^2$ from Figure 49, similar to (1). Note that there is a copy of C_5 inside of Figure 50. This copy of C_5 is isotopic to the one in Figure 47, because we constructed the handle decomposition in Figure 50 without sliding any handles over the new 2-handle which comes from a 2-handle/3-handle pair. We thus get Figure 51 of E'_3 by rationally blowing down. The first diagram in Figure 52 is obtained from Figure 51 by ignoring handles. Handle slides give the last diagram in Figure 52. Note that there is W_1 inside the last diagram in Figure 52. By removing this W_1 from E'_3 and regluing it via f_1 , we get a 2-sphere with self intersection number -1 . Therefore W_1 is a cork of E'_3 .

(3) Similar to (2), we obtain the first diagram in Figure 53 from Figure 51 of E'_3 by ignoring handles. Handle slides give the last diagram in Figure 53. Notice the $W_{1,3}$ inside of the last diagram in Figure 53. By removing $W_{1,3}$ from E'_3 and regluing it via $f_{1,3}$, we get a 2-sphere with self intersection number -1 . Therefore $W_{1,3}$ is a plug of E'_3 . $\square$

Remark 6.5. (1) In the proof above, we gave one construction of E'_3 . We can give a lot of different constructions by using knotted bands in handle slides. We do not know if choices of bands in handle slides affect diffeomorphism types of E'_3 , though they do not affect homeomorphism types and Seiberg-Witten invariants. See also [26, Remark 6.1] and [28].

(2) In the proof above, we drew a digram of E'_3 by the rational blow-down procedure in Figure 15. Another rational blow-down procedure in Figure 16 gives a different diagram of E'_3 , though they are diffeomorphic.

6.1. Knotted corks and plugs in 4-manifolds

Proposition 6.6. (1) $\mathbf{CP}^2 \# 13\overline{\mathbf{CP}}^2$ admits a handle decomposition as in Figure 22. Note that Figure 22 contains C_5 .

(2) $\mathbf{CP}^2 \# 13\overline{\mathbf{CP}}^2$ admits a handle decomposition as in Figure 23. Note that Figure 23 contains C_5 .

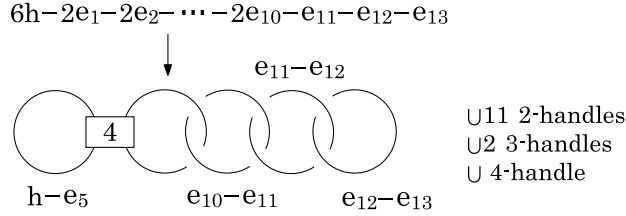


FIGURE 22. $\mathbf{CP}^2 \# 13\overline{\mathbf{CP}}^2$

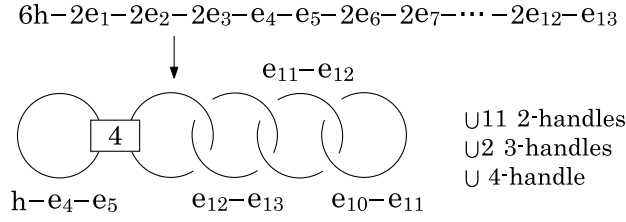


FIGURE 23. $\mathbf{CP}^2 \# 13\overline{\mathbf{CP}}^2$

Proof. (1) Handle slides, introducing 2-handle/3-handle pairs, and blowing ups as in [26, Figure 8 ~ 18], i.e. the moves similar to Figure 41 $\rightsquigarrow$ Figure 44, give Figure 54 of $\mathbf{CP}^2 \# 3\overline{\mathbf{CP}}^2$. By an isotopy, we get Figure 55. By the handle slide indicated in Figure 55, we get Figure 56. Blowing ups give Figure 57. We blow up as in Figure 19. Then we obtain Figure 22.

(2) Blowing ups in Figure 55 of $\mathbf{CP}^2 \# 3\overline{\mathbf{CP}}^2$ give Figure 58 of $\mathbf{CP}^2 \# 5\overline{\mathbf{CP}}^2$. By further blow-ups we get Figure 59. We then obtain Figure 23 of $\mathbf{CP}^2 \# 13\overline{\mathbf{CP}}^2$, by a similar construction to [26, Figure 19 ~ 25]. $\square$

Definition 6.7. Let X_5 (resp. X_3) be the smooth 4-manifold obtained from $\mathbf{CP}^2 \# 13\overline{\mathbf{CP}}^2$ by rationally blowing down the copy of C_5 in Proposotion 6.6.(1) (resp. in Proposotion 6.6.(2)).

Corollary 6.8. (1) $X_5 \# \overline{\mathbf{CP}}^2$ is obtained from $\mathbf{CP}^2 \# 14\overline{\mathbf{CP}}^2$ by rationally blowing down the copy of C_5 in Figure 24.

(2) $X_3 \# \overline{\mathbf{CP}}^2$ is obtained from $\mathbf{CP}^2 \# 14\overline{\mathbf{CP}}^2$ by rationally blowing down the copy of C_5 in Figure 25.

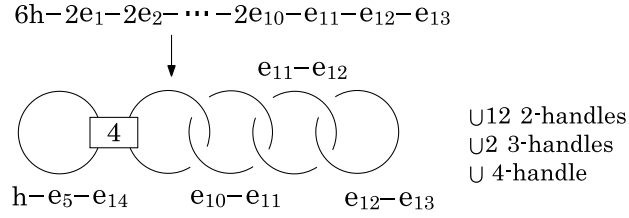


FIGURE 24. $\mathbf{CP}^2 \# 14\overline{\mathbf{CP}}^2$

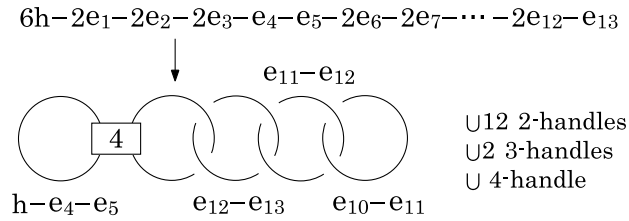


FIGURE 25. $\mathbf{CP}^2 \# 14\overline{\mathbf{CP}}^2$

Corollary 6.9. (1) $X_5 \# 5\overline{\mathbf{CP}}^2$ is obtained from $\mathbf{CP}^2 \# 14\overline{\mathbf{CP}}^2$ by removing W_3 and regluing it via f_3 .

(2) $X_3 \# 5\overline{\mathbf{CP}}^2$ is obtained from $\mathbf{CP}^2 \# 14\overline{\mathbf{CP}}^2$ by removing W_3 and regluing it via f_3 .

Proof. This obviously follows from Corollary 6.8 and Theorem 5.1. $\square$

Corollary 6.10. (1) X_3 is homeomorphic to the elliptic surface $E(1)_{2,3}$ and has the same Seiberg-Witten invariant as $E(1)_{2,3}$. In particular, X_3 is homeomorphic but not diffeomorphic to $\mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2$.

(2) X_5 is homeomorphic to the elliptic surface $E(1)_{2,5}$ and has the same Seiberg-Witten invariant as $E(1)_{2,5}$. In particular, X_5 is homeomorphic but not diffeomorphic to $\mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2$.

(3) $X_3 \# k\overline{\mathbf{CP}}^2$, $X_5 \# k\overline{\mathbf{CP}}^2$ and $\mathbf{CP}^2 \# (9+k)\overline{\mathbf{CP}}^2$ ($k \geq 0$) are mutually homeomorphic but not diffeomorphic to each other.

Proof. Corollary 6.9 implies that X_3 and X_5 are simply connected. Thus Freedman's theorem and Rochlin's theorem imply that X_3 and X_5 are homeomorphic to $\mathbf{CP}^2 \# 9\overline{\mathbf{CP}}^2$. The same argument as in [26] (see also [28]) shows X_3 (resp. X_5) has the same Seiberg-Witten invariant as $E(1)_{2,3}$ (resp. $E(1)_{2,5}$). (X_3 and X_5 corresponds to E'_3 and E_5 in [26], respectively.) We thus can prove that the Seiberg-Witten invariants of $X_3 \# k\overline{\mathbf{CP}}^2$, $X_5 \# k\overline{\mathbf{CP}}^2$ and $\mathbf{CP}^2 \# (9+k)\overline{\mathbf{CP}}^2$ ($k \geq 0$) are mutually different, similar to Szabó [24, Lemmas 3.6 and 3.5]. $\square$

Proof of Theorem 6.3. Theorem 6.3 now clearly follows from Corollary 6.9 and 6.10. $\square$

Proof of Theorem 6.4. It easily follows from [26, Proposition 3.3] that $E'_3 \# 3\overline{\mathbf{CP}}^2$ is obtained from $\mathbf{CP}^2 \# 16\overline{\mathbf{CP}}^2$ by rationally blowing down C_5 inside $V_{5,5}$. (In this proof, E'_3 denotes the manifold defined in [26], not the manifold defined in Theorem 6.1 in this paper.) Theorem 5.1 thus shows that $E'_3 \# 7\overline{\mathbf{CP}}^2$ in [26] is obtained from $\mathbf{CP}^2 \# 16\overline{\mathbf{CP}}^2$ by removing $W_{4,5}$ and regluing it via $f_{4,5}$. Similarly, [26, Proposition 3.2.(2)] implies that $E_5 \# 7\overline{\mathbf{CP}}^2$ is obtained from $\mathbf{CP}^2 \# 16\overline{\mathbf{CP}}^2$ by removing $W_{4,5}$ and regluing it via $f_{4,5}$. Since E_5 has the same Seiberg-Witten invariant as the elliptic surface $E(1)_{2,5}$, we can prove that $E'_3 \# 7\overline{\mathbf{CP}}^2$, $E_5 \# 7\overline{\mathbf{CP}}^2$ and $\mathbf{CP}^2 \# 16\overline{\mathbf{CP}}^2$ are mutually homeomorphic but not diffeomorphic, similar to Corollary 6.10. $\square$

We finally point out an interesting property of E_3 and E'_3 .

Proposition 6.11. *The minimal 4-manifold in Figure 21 is obtained from $\mathbf{CP}^2 \# 14\overline{\mathbf{CP}}^2$ by rationally blowing down C_6 .*

Proof. By blowing up in Figure 47, we get Figure 60. We can easily find a copy of C_6 in Figure 60. Draw a diagram of the rational blow-down of $\mathbf{CP}^2 \# 14\overline{\mathbf{CP}}^2$ along this C_6 . Then we get Figure 61. By handle slides, we can change the diagram into Figure 21. $\square$

Remark 6.12. We can change the construction of E'_3 so that, for any $5 \leq p \leq 11$, E'_3 can be obtained from $\mathbf{CP}^2 \# (8+p)\overline{\mathbf{CP}}^2$ by rationally blowing down C_p . The smooth 4-manifold E_3 can also be obtained from $\mathbf{CP}^2 \# (8+p)\overline{\mathbf{CP}}^2$ ($3 \leq p \leq 11$) by rationally

blowing down C_p . Note that E_3 and E'_3 are homeomorphic to $E(1)_{2,3}$ and have the same Seiberg-Witten invariants as $E(1)_{2,3}$. However, it is not known whether or not $E(1)_{2,3}$ itself can be obtained from $\mathbf{CP}^2 \# (8+p)\overline{\mathbf{CP}}^2$ by rationally blowing down C_p , for $p \geq 4$. Moreover, there is no known other minimal 4-manifolds for which more than three different rational blow-down constructions exist.

For various handle diagrams of $E(1)_{2,3}$, see Harer-Kas-Kirby [18], Gompf [16], the first author [5], and the second author [29]. The handlebody of $E(1)_{2,3}$ in [5] has no 1- and 3-handles. For more diagrams of exotic rational surfaces, see [5], [27], [29].

7. Further remarks

We conclude this paper by making some remarks.

7.1. General remarks

Remark 7.1. (1) Theorem 2.5 and 3.4 show that W_n and $\overline{W}_n$ are corks and that $W_{m,n}$ are plugs. By a similar technique, we can easily give more examples of corks (which consist of one 1-handle and one 2-handle) and plugs.

(2) In Figure 1, we defined the smooth 4-manifold $\overline{W}_n$. Note that this symbol does not denote the reverse orientation of W_n .

Remark 7.2. Theorem 5.1. (1) and (2) describe relations between rational blow-down operations and cork operations, under some conditions. We can similarly prove the following theorem. Notice the slight difference from Theorem 5.1.

Theorem 7.3. (1) Suppose that a smooth 4-manifold X contains $T_{p,p-1}$ (resp. $T_{p,p+1}$). Let $X_{(p)}$ be the rational blow-down of X along C_p . Then X contains W_{p-1} (resp. W_p) such that $X_{(p)} \# (p-1)\overline{\mathbf{CP}}^2$ is obtained from X by removing W_{p-1} (resp. W_p) and regluing it via f_{p-1} (resp. f_p).

(2) Suppose that a smooth 4-manifold X contains $U_{p,p-1}$ (resp. $U_{p,p+1}$). Let $X_{(p)}$ be the rational blow-down of X along C_p . Then X contains W_{p-1} (resp. W_p) such that $X_{(p)} \# (p-1)\overline{\mathbf{CP}}^2$ is obtained from X by removing W_{p-1} (resp. W_p) and regluing it via f_{p-1} (resp. f_p).

Remark 7.4. Let Z be a simply connected closed smooth 4-manifold, and $\{Z_n\}_{n \geq 1}$ mutually different smooth structures on Z . Then, by Theorem 1.1, there exist corks (C_i, τ_i) of Z such that Z_i is obtained from Z by removing C_i and regluing it via τ_i . It seems natural to ask whether or not (C_i, τ_i) and (C_j, τ_j) ($i \neq j$) can be the same. Theorem 6.3 shows that (C_1, τ_1) and (C_2, τ_2) can be the same, namely, the embeddings of corks are different. In a forthcoming paper, we further discuss knotting corks and plugs.

7.2. 1- and 3-handles of 4-manifolds

It is not known if the minimal numbers of 1- and 3-handles of smooth 4-manifolds (which are invariants of smooth structures) are non-trivial for simply connected closed

4-manifolds. However, the following example shows that this invariant is non-trivial for simply connected 4-manifolds with boundary.

Example 7.5. There exist two homeomorphic simply connected smooth 4-manifolds with boundary with the following property: one admits a handle decomposition without 1- and 3-handles; the other has at least a 1- or 3-handle in each of its handle decompositions. In particular, these smooth structures can be detected by minimal numbers of its 1- and 3-handles.

Proof. The first author [4] constructed smooth 4-manifolds which are homeomorphic but not diffeomorphic to the cusp neighborhood, by knot surgery in the cusp neighborhood. Ozsváth-Szabó [23] proved that if Dehn surgery along a knot K in S^3 is orientation-preserving diffeomorphic to 0-framed right trefoil surgery, then K is isotopic to right trefoil. These two theorems imply the claim. $\square$

As far as the authors know, no other such examples are known.

Question 7.6. Does W_n^2 (n : large) in Figure 4 admit a handle decomposition without 1- and 3-handles? Note that W_n^1 ($n \geq 1$) is homeomorphic but not diffeomorphic to W_n^2 and has a handle decomposition without 1- and 3-handles.

Remark 7.7. W_1^2 admits a handle decomposition without 1- and 3-handles.

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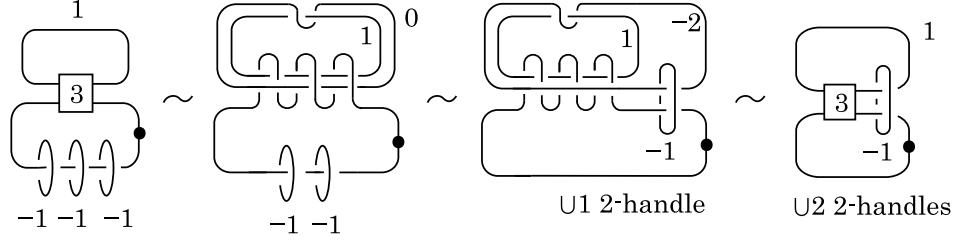
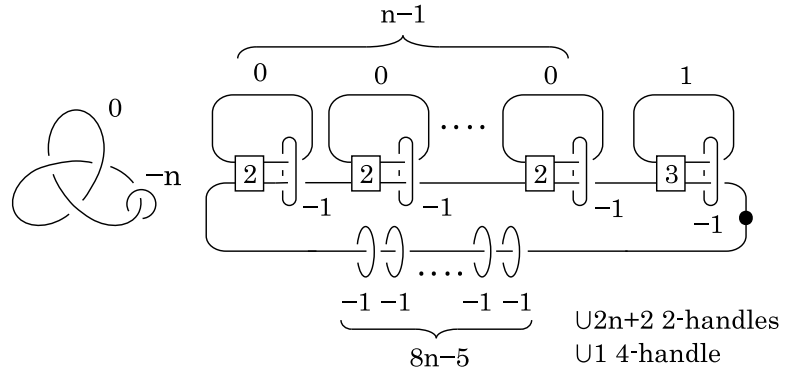
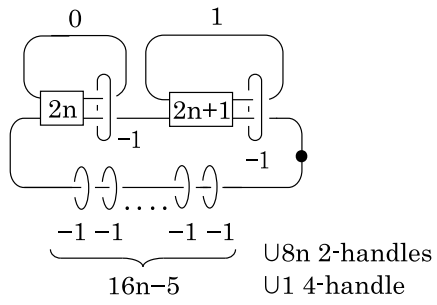


FIGURE 26. Handle slides


 FIGURE 27. $E(n)$

 FIGURE 28. $E(2n)$

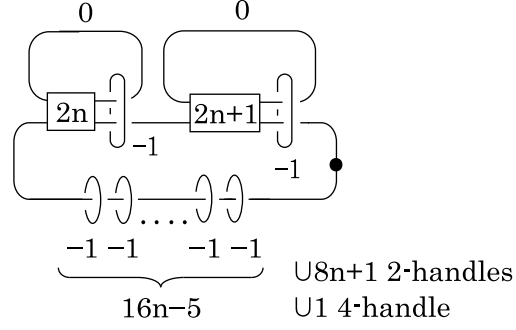


FIGURE 29. $E(2n) \# \overline{\mathbf{CP}}^2$

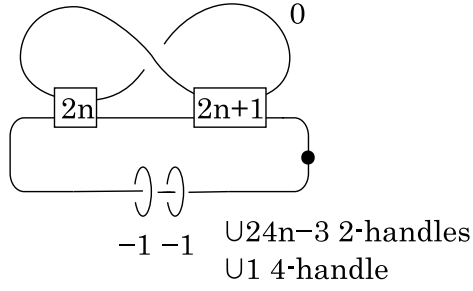


FIGURE 30. $E(2n) \# \overline{\mathbf{CP}}^2$

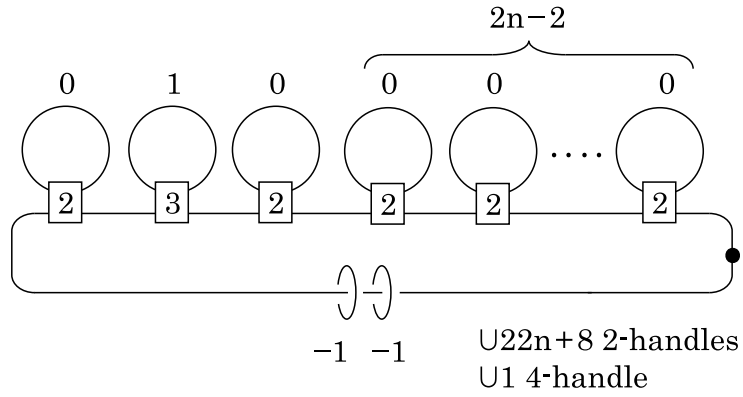


FIGURE 31. $E(2n+1)$ ($n \geq 1$)

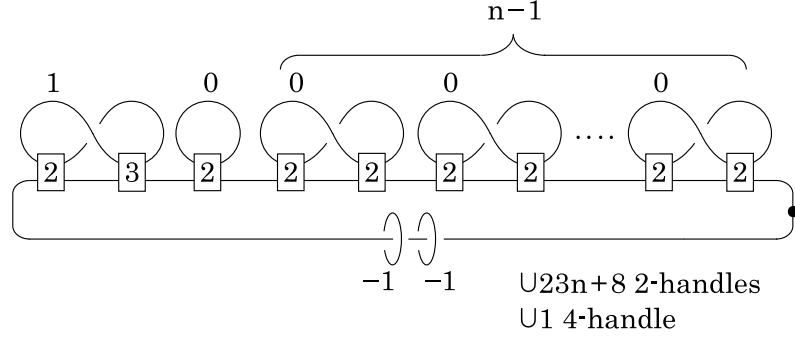


FIGURE 32. $E(2n+1)$ ($n \geq 1$)

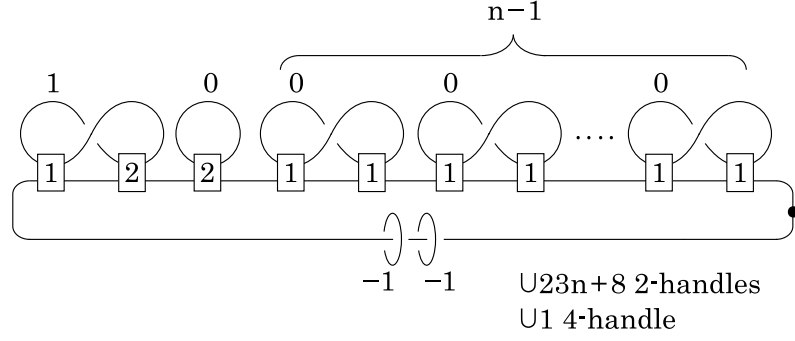


FIGURE 33. $E(2n+1)$ ($n \geq 1$)

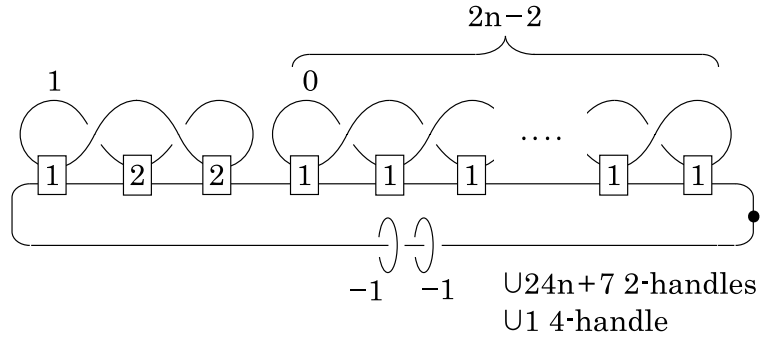


FIGURE 34. $E(2n+1)$ ($n \geq 1$)

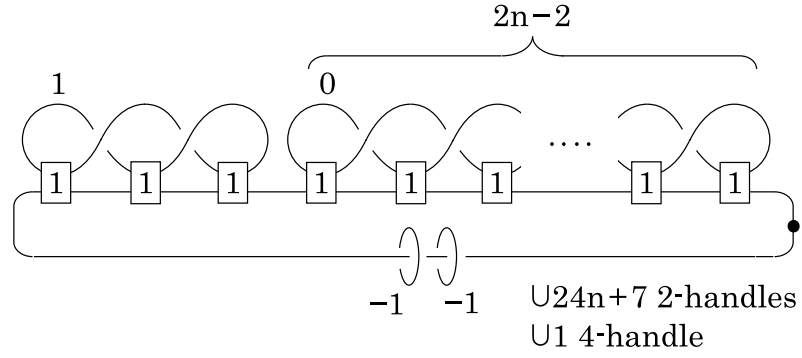


FIGURE 35. $E(2n+1)$ ($n \geq 1$)

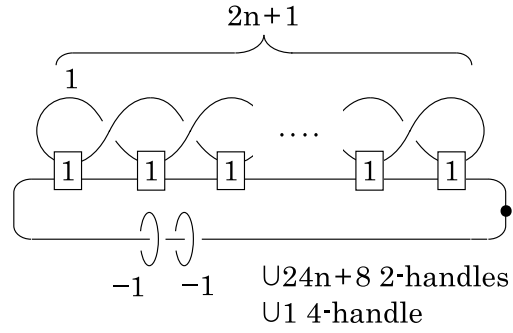


FIGURE 36. $E(2n+1)$ ($n \geq 1$)

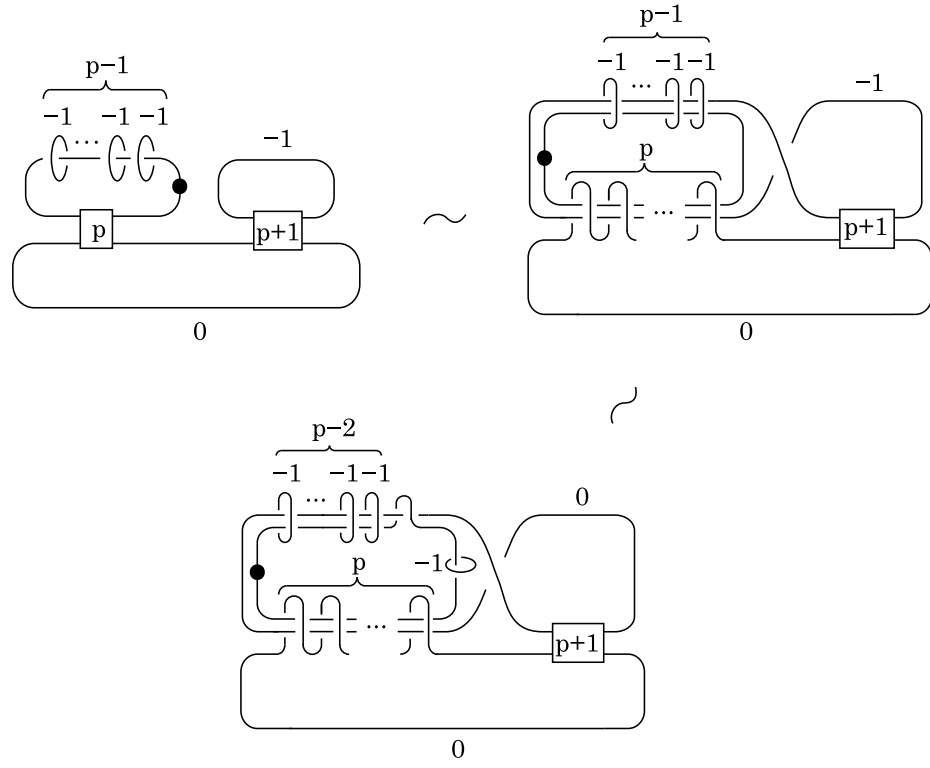


FIGURE 37. Handle slides of $T_{p,p+1}$

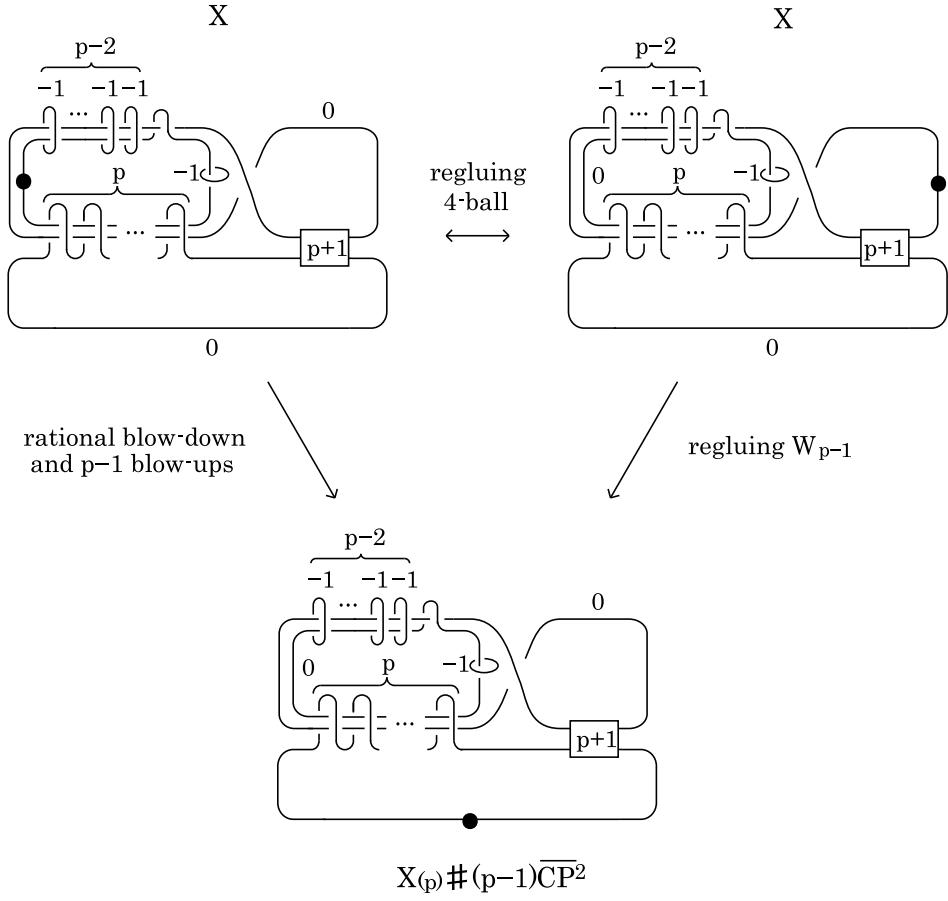


FIGURE 38

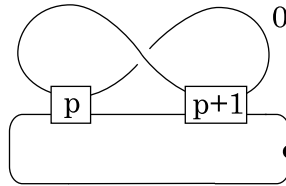


FIGURE 39

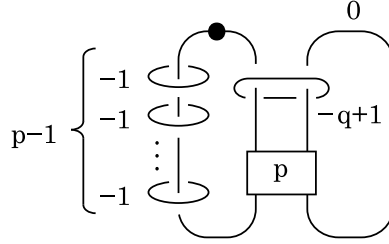


FIGURE 40

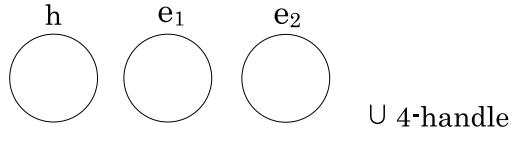


FIGURE 41. $\mathbf{CP}^2 \# 2\overline{\mathbf{CP}}^2$

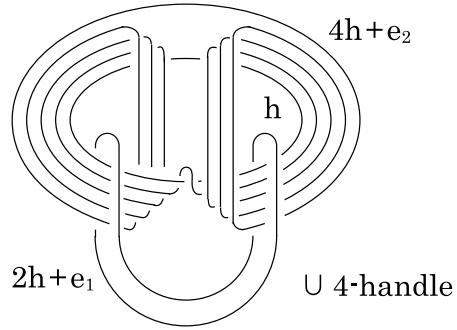


FIGURE 42. $\mathbf{CP}^2 \# 2\overline{\mathbf{CP}}^2$

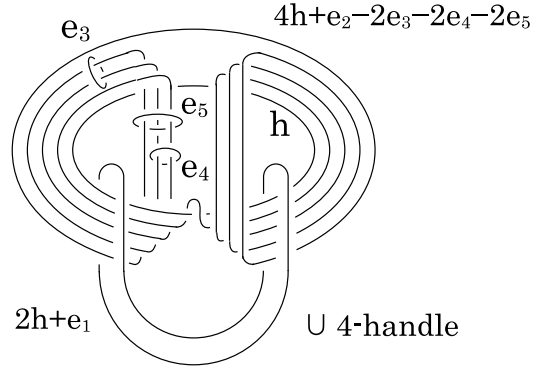


FIGURE 43. $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$

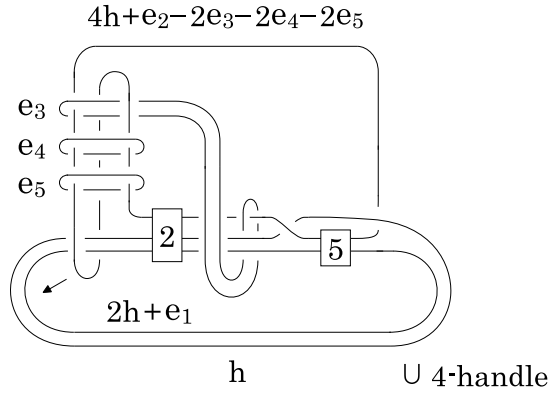


FIGURE 44. $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$

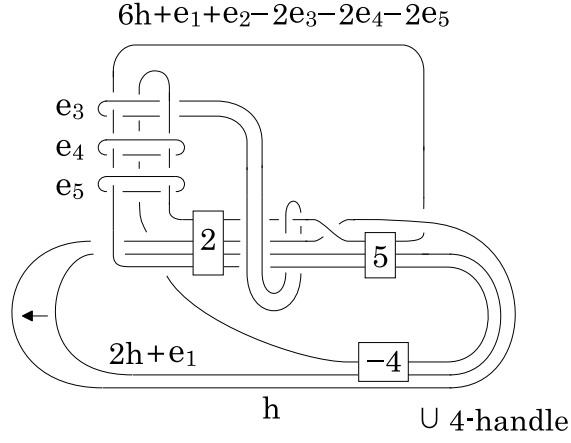


FIGURE 45. $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$

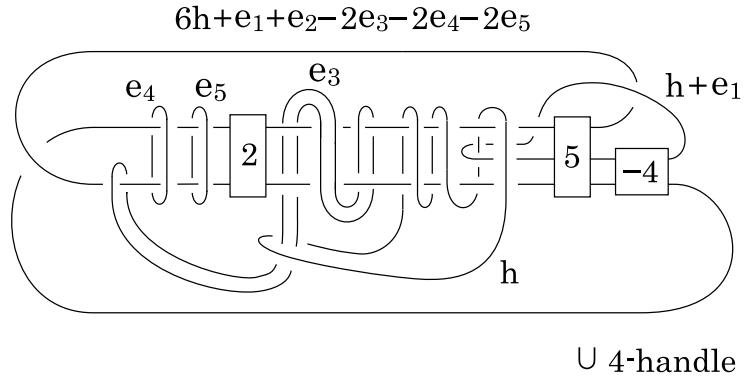


FIGURE 46. $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}}^2$

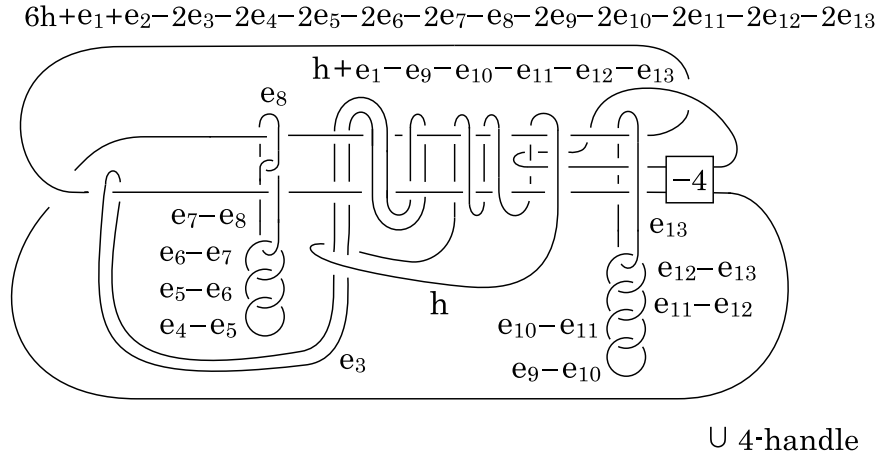


FIGURE 47. $\mathbf{CP}^2 \# 13\overline{\mathbf{CP}}^2$

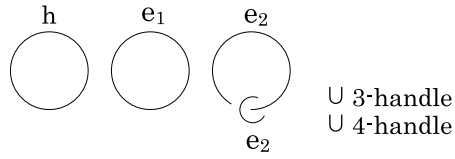
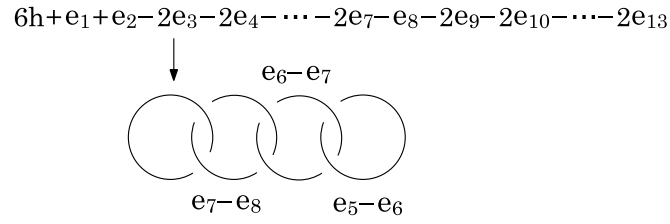


FIGURE 49. $\mathbf{CP}^2 \# 2\overline{\mathbf{CP}}^2$

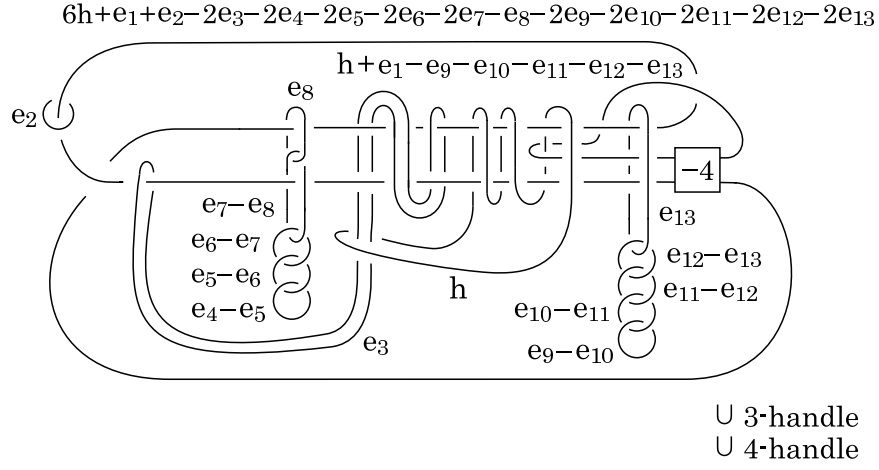


FIGURE 50. $\mathbf{CP}^2 \# 13\overline{\mathbf{CP}}^2$

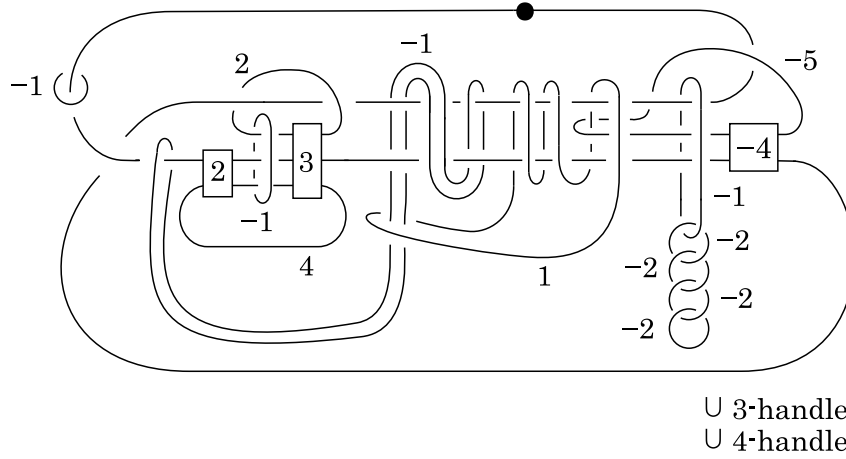


FIGURE 51. A different diagram of the 4-manifold in Figure 21

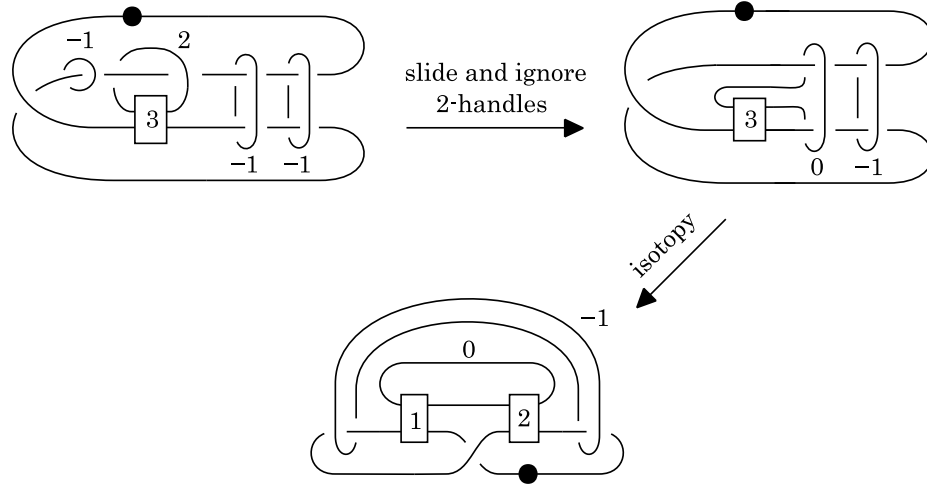


FIGURE 52. Construction of a cork in E'_3

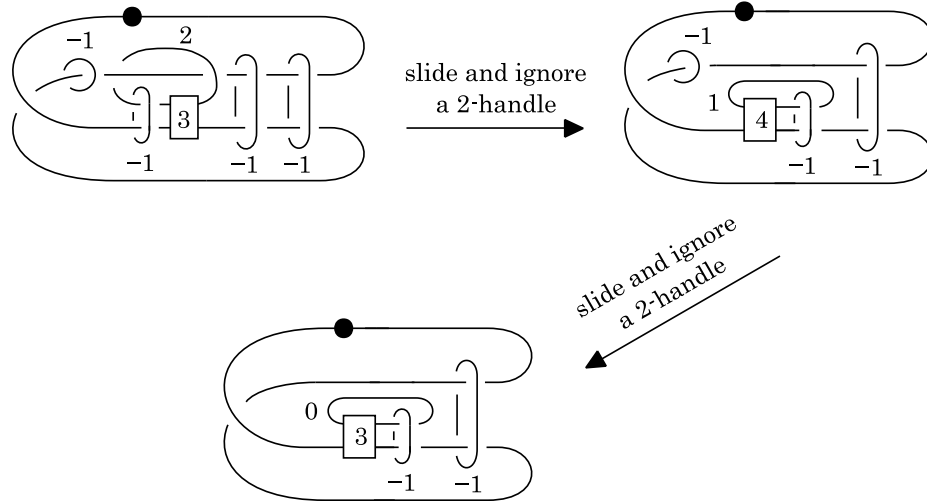


FIGURE 53. Construction of a plug in E'_3

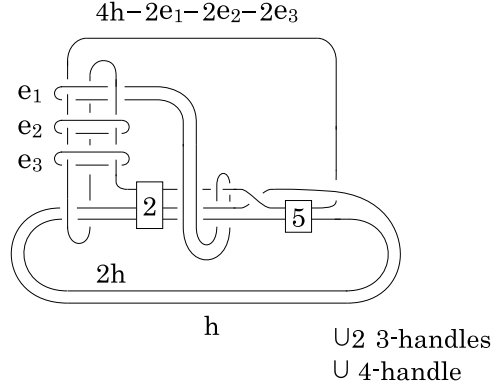


FIGURE 54. $\mathbf{CP}^2 \# 3\overline{\mathbf{CP}}^2$

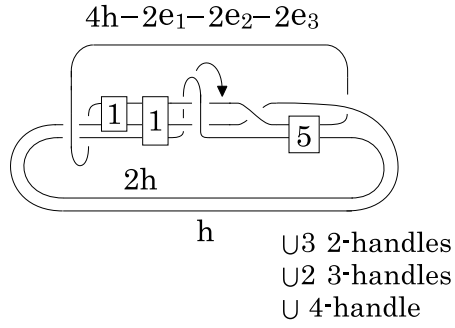


FIGURE 55. $\mathbf{CP}^2 \# 3\overline{\mathbf{CP}}^2$

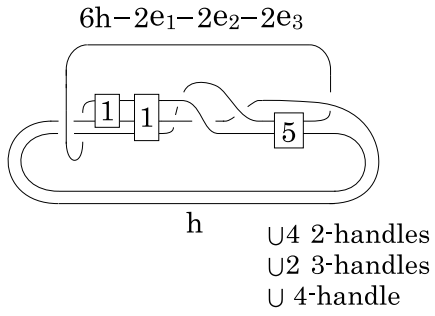


FIGURE 56. $\mathbf{CP}^2 \# 3\overline{\mathbf{CP}}^2$

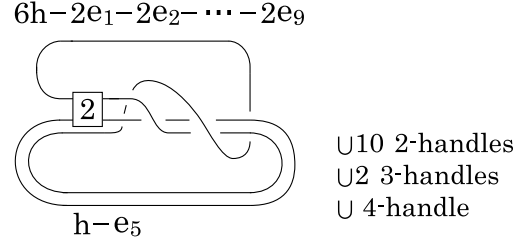


FIGURE 57. $\mathbb{CP}^2 \# 9 \overline{\mathbb{CP}}^2$

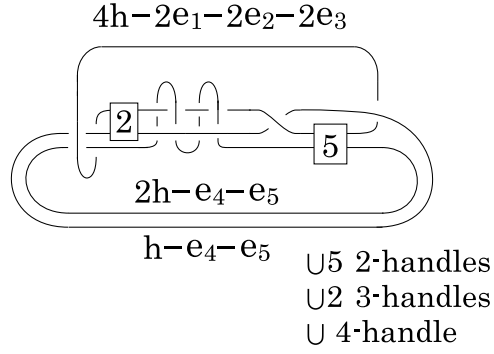


FIGURE 58. $\mathbb{CP}^2 \# 5 \overline{\mathbb{CP}}^2$

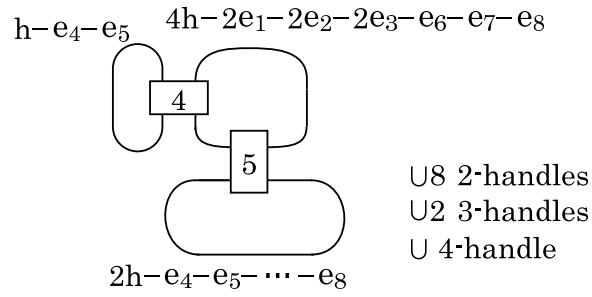


FIGURE 59. $\mathbb{CP}^2 \# 8 \overline{\mathbb{CP}}^2$

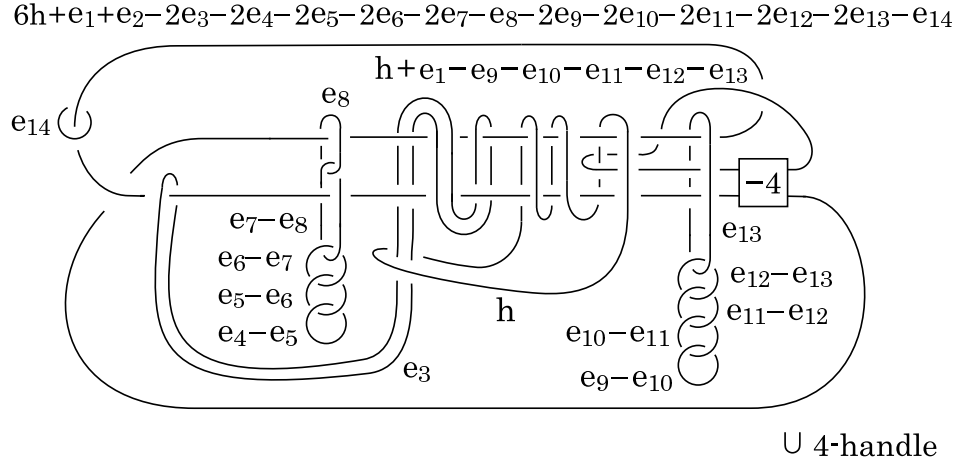
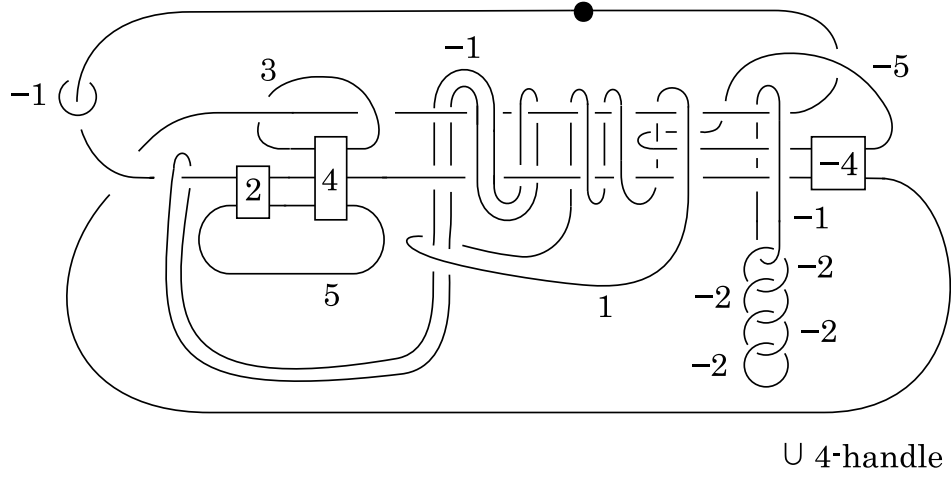

 FIGURE 60. $\mathbf{CP}^2 \# 14 \overline{\mathbf{CP}}^2$


FIGURE 61. A different diagram of the 4-manifold in Figure 21