Corks, plugs and Stein manifolds

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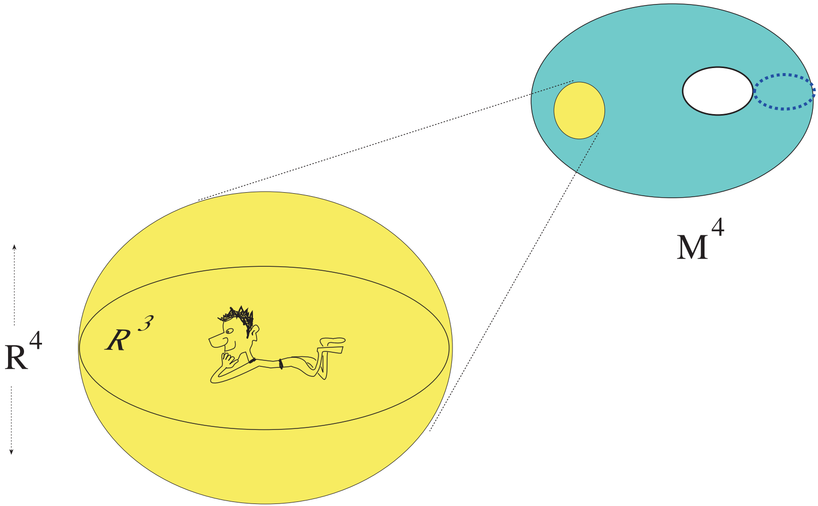
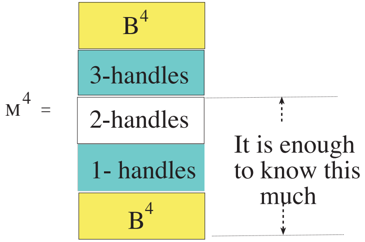
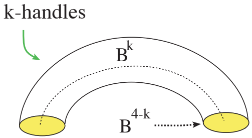
4-manifolds are spaces which are locally R^4

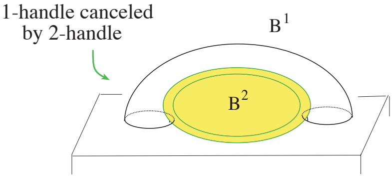


We visualize 4-manifolds by handles

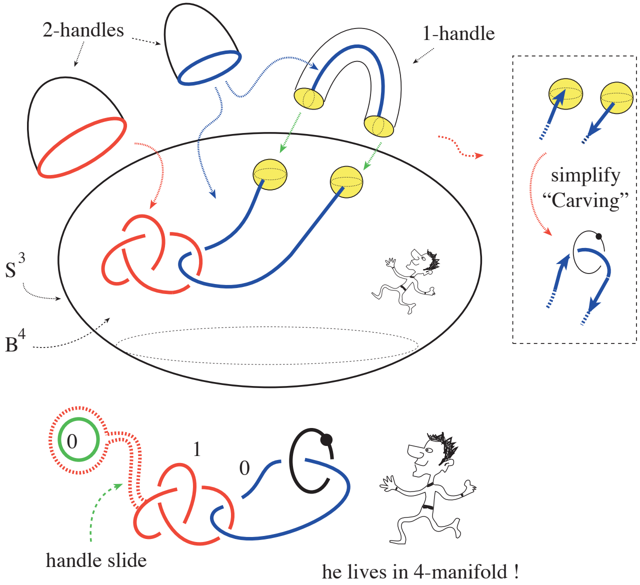


A k-handle is just a ball $B^4 = \mathbf{B}^k \times B^{4-k}$ $(k = 0, 1, 2, 3, 4)$ attached along $\partial \mathbf{B}^k \times B^{4-k} = \mathbf{S}^{k-1} \times B^{4-k}$.





All the tools you need to study 4-manifolds



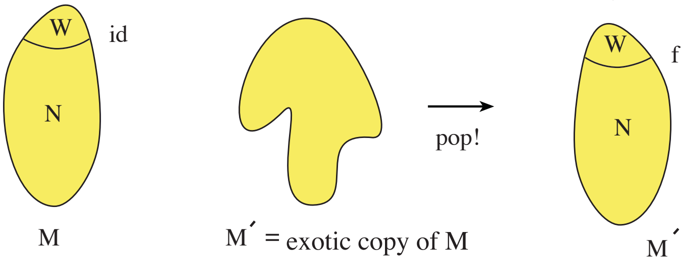
Popping Corks

Let M be a smooth closed 1-connected 4-manifold, and M' be its exotic copy of M. Then $\exists$ a compact contractible codim zero $W \subset M$ with

$$M = N \cup_{id} W \ , \quad M' = N \cup_f W$$

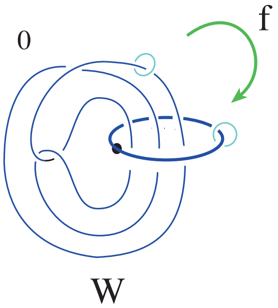
Furthermore, we can make the each piece W and N Stein manifolds!

(*This was first observed on an example by A, then the general result was proven by Matveyev and independently by Curtis-Freedman-Hsiang-Stong. The Stein part is due to A and Matveyev.)*



Corks:

- A **Cork** is a pair (W, f), where W is a compact contractible Stein manifold, and $f : \partial W \to \partial W$ is an involution, which extends to a self-homemorphism of W, but not a self-diffeomorphism of W. We say (W, f) is a cork of M, if $W \subset M$, and the transformation $M \rightsquigarrow M \smile_f W$ turns M into an exotic copy of itself.

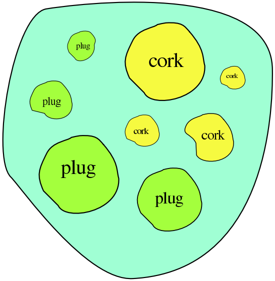


Cork! smallest exotic manifold
relative to its boundary

Plugs:

Recently Yasui and I begin systematic study of Corks. We identified some specific corks of many exotic manifolds. Along the way, we found another type of basic codimension zero submanifolds which are responsible for exotic structres on 4-manifolds, we named them "plugs"

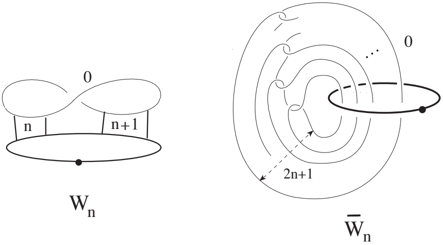
- A **Plug** is a pair (P, f), where P is a compact Stein manifold, and $f : \partial P \to \partial P$ is an involution, which does **not** extend to a self homeomorphism of P. We say (P, f) is a plug of M if $P \subset M$ and transformation $M \rightsquigarrow M \smile_f P$ turns M into an exotic copy of itself.



M

Some basic corks and plugs appear in nature:

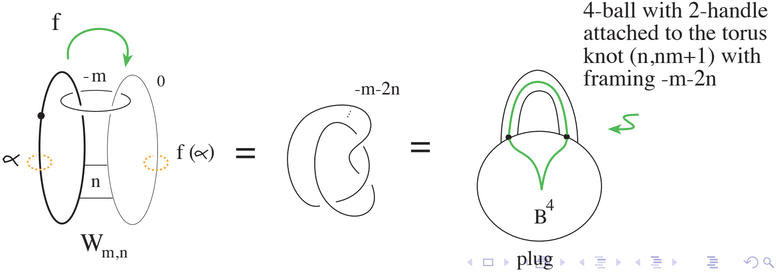
Corks:



n,m=1,2..

Corks are (usually) homotopy points, and plugs are homotopy S^2's.

Plugs:



Recent work with K.Yasui (rest of the slides):

- **(A) Knotting:** A cork can imbed $(W, f) \hookrightarrow X$ infinitely many ways.
- **(B) Realization:** Given a 2-handlebody Y (i.e. 4-manifold consisting of 0- 1- and 2-handles) with $b_2 \geq 1$. There are arbitrarily many exotic Stein manifolds $Y_0, Y_1, ..., Y_n$, with all $Y_i \subset Y$, and their topological invariants (fundamental groups, homology groups, boundary homology groups, and intersection forms) coincide with that of Y.
- **(C) Exotic imbedding:** Given a pair (X, Y) with $X - Y$ 2-handlebody $\Rightarrow \exists$ subsets $Y_i \subset X$, $i = 1, 2, .., n$ with (X, Y_i) are exotic pairs, and $Y, Y_1, .. Y_n$ have the same topological invariants.
- **(D) Packing:** It is possible to imbed arbitrarily many copies of a cork (W, f) disjointly, into a closed smooth manifold X ("cork twisting" X along each of them gives a different smooth structure).
- **(E) Infinite packing:** It is possible to imbed infinitely many copies of (W, f) disjointly, into (noncompact) smooth manifolds.

Two useful smooth 4-manifold invariants:

(1) $\mathbf{X} \rightsquigarrow \mathbf{n}(\mathbf{X})$: Assume $H_1(X)$ has no 2-torsion, then the set of $Spin^c$ structures is: $Spin^c(X) = \{ \alpha \in H_2(X) \mid \alpha = w_2(X) \bmod 2 \}$, and the Seiberg-Witten invariant is a function $SW_X : Spin^c(X) \to \mathbf{Z}$. It is known that this function is nonzero on a finite subset $\beta(X) = \{\pm\alpha_1, \pm\alpha_2, .., \pm\alpha_k\}$. We define $n(X) = |\beta(X)| = k$

(2) $\mathbf{X} \rightsquigarrow \mathbf{m}_\alpha(\mathbf{X})$: Given (X, α) with $\alpha \in H_2(X)$. Let $m_\alpha(X)$ be the minimal genus of imbedded surface representing α. This is an invariant of diffeomorphisms $f : X \to X$ with $f_*(\alpha) = \alpha$

- Note: By setting $\alpha_0 = 0$ and $t_j = exp(\alpha_j)$, the function SW_X is usually written as a single polynomial $SW(X) = \sum SW_X(\alpha_j)t_j$.

Recall “Knot surgery” operation:

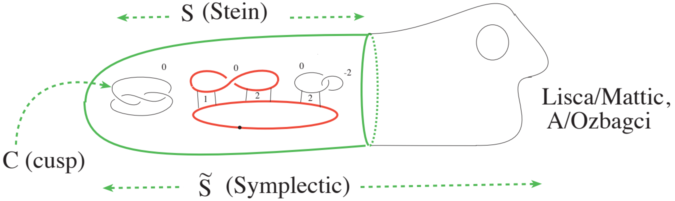
Let X be a smooth 4-manifold, and $T^2 \times D^2 \subset X$ be an imbedded torus with trivial normal bundle, and $K \subset S^3$ be a knot, $N(K)$ be its tubular neighborhood. The Fintushel-Stern **knot surgery operation** is the operation of replacing $T^2 \times D^2$ with $(S^3 - N(K)) \times S^1$, so that the meridian $p \times \partial D^2$ of the torus coincides with the longtitude of K:

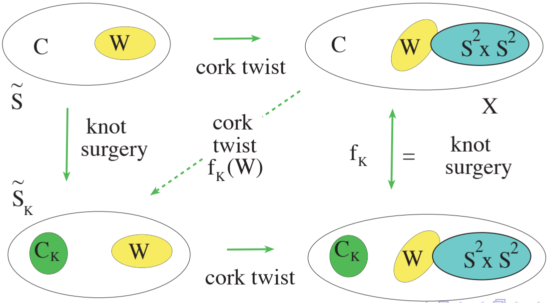
$$X \rightsquigarrow X_K = (X - T^2 \times D^2) \cup (S^3 - N(K)) \times S^1$$

$SW(X_K) = SW(X)\Delta_K(t)$, where $\Delta_K(t)$ is Alexander polynomial of K, and $t = exp(2[T])$ (Fintushel-Stern). Hence when $SW(X) \neq 0$, all X_K are different exotic copies of X, for all K with different $\Delta_K(t)$.

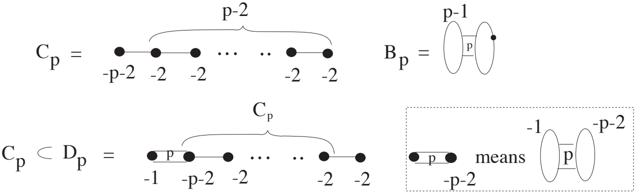
Knotting a cork infinitely many different ways:

Compactify $S \subset \tilde{S}$, with $b_2^+(\tilde{S}) > 1$ and $C \neq 0$ in homology of $\tilde{S}$.





Relating cork twist to rational blow-down:



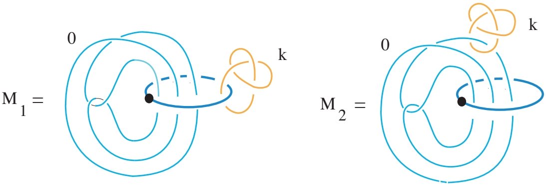
If $C_p \subset X$, the rational blowing down (defined by Fintushel-Stern) is:

$$X \rightsquigarrow X_{(p)} = (X - C_p) \cup B_p$$

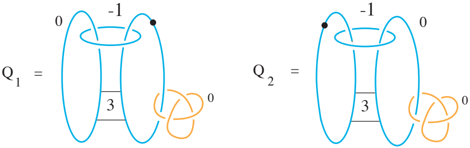
Theorem (A-Yasui): If $D_p \subset X$, then $\exists$ the cork $W_{p-1} \subset D_p$ such that

$$\text{Cork twist } X \text{ along } W_{p-1} \Longleftrightarrow X_{(p)} \# (p-1)\overline{\mathbf{CP}}^2$$

Constructing exotic Stein pairs by cork twisting:

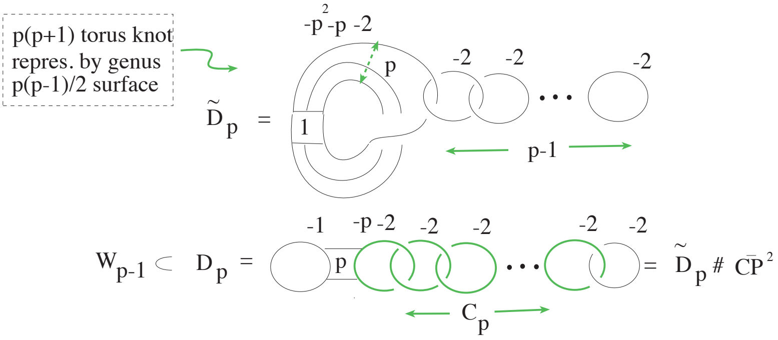


Exotic Stein manifold pairs
obtained by inflating a cork



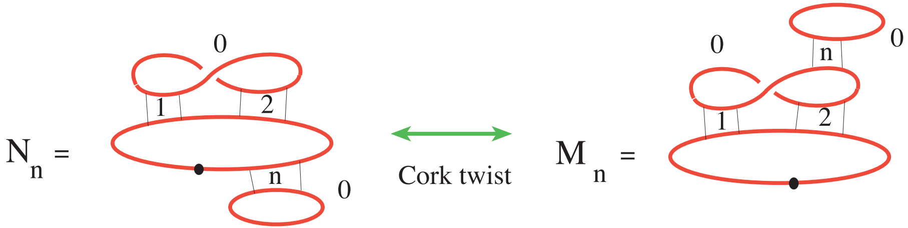
Exotic Stein manifold pairs
obtained by inflating a plug

Imbedding disjoint corks:



- Imbed $\tilde{D} = \natural_{i=1}^{n} \tilde{D}_{p_i} \subset S$ (closed symplectic) with $b_2^+(S) > 1$.
  Adjunction inequality $\Rightarrow$ all basic classes of S vanishes on $\tilde{D}$
- $\natural W_{p_i-1} \subset \natural_{i=1}^{n} D_{p_i} \subset S \# n\,\mathbf{CP}^2 := X$
- $X_i := X$ cork twisted along $W_{p_i} \Rightarrow X_i = X_{(p_i)} \# (p_i - 1)\mathbf{CP}^2$
  $\Rightarrow n(X_i) = 2^{p_i-1} n(X) \Rightarrow$ Therefore all X_i are different !.

Key Lemma:

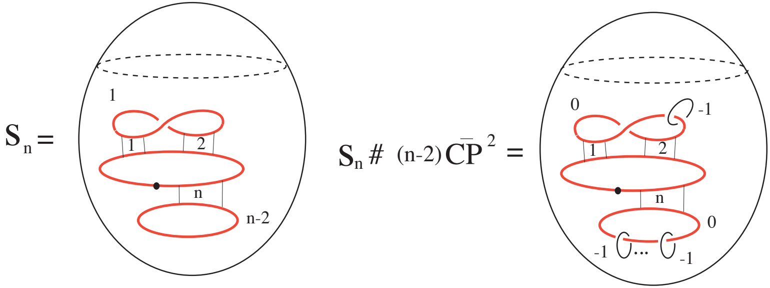


(a) The generator $\alpha \in H_2(N_n; \mathbf{Z})$ is represented by genus n surface Σ_n

(b) If $k\alpha$ is represented by Σ_g with $g < n$ then $k = 0$

Proof of (a) is by inspection. To prove (b) we imbed the Stein manifold N_n into a closed symplectic manifold $N_n \subset S_n$

Proof of Key lemma (b):



$$K \in \beta(S_n) \Rightarrow L := K \pm e_1 \pm e_2 ... \pm e_{n-1} \in \beta(S_n \# (n-1)\mathbf{CP}^2)$$

$$\Rightarrow | < L, k\alpha > | \geq k \; |(n-2) + n| = k(2n-2)$$

$$2g - 2 \geq (k\alpha)^2 + k(2n-2) \Leftarrow \text{ Adjunction inequality}$$

$$n - 1 > g - 1 \geq k(n-1) \Rightarrow k < 1 \Rightarrow k = 0$$

Infinite packing: