ON THE TOPOLOGY OF COMPACT STEIN SURFACES

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Abstract. In this paper we obtain the following results: (1) Any compact Stein surface with boundary embeds naturally into a symplectic Lefschetz fibration over S^2. (2) There exists a minimal elliptic fibration over D^2, which is not Stein. (3) The circle bundle over a genus $n \geq 2$ surface with euler number $e = -1$ admits at least $n+1$ mutually non-homeomorphic simply-connected Stein fillings. (4) Any surface bundle over S^1, whose fiber is a closed surface of genus $n \geq 1$ can be embedded into a closed symplectic 4-manifold, splitting the symplectic 4-manifold into two pieces both of which have positive b_2^+. (5) Every closed, oriented connected 3-manifold has a weakly symplectically fillable double cover, branched along a 2-component link.

0. Introduction

In [AO] (see also [LP]), it was proved that every compact Stein surface admits a PALF (positive allowable Lefschetz fibration over D^2 with bounded regular fibers) and conversely every PALF is Stein. In this paper we first prove that any compact Stein surface with boundary embeds naturally into a symplectic Lefschetz fibration over S^2. In particular, this shows that we can embed a Stein surface into a minimal closed symplectic 4-manifold.

Next we show that the result in [AO] does not necessarily hold if the fiber of the Lefschetz fibration is closed, by constructing non-Stein minimal elliptic fibrations (with closed regular fibers) over D^2. Minimality of our examples is important, otherwise one can easily find non-Stein examples by blowing-up an elliptic (or in general a Lefschetz) fibration over D^2.

We also prove that for every integer $n \geq 2$, there exits an irreducible 3-manifold M_n with at least $n+1$ mutually non-homeomorphic simply-connected Stein fillings. We will identify M_n as the circle bundle over a genus n surface with euler number $e = -1$.

Moreover we show that any surface bundle over S^1, whose fiber is a closed surface of genus $n \geq 1$ can be embedded into a closed symplectic 4-manifold, splitting the symplectic 4-manifold into two pieces both of which have positive b_2^+.

Date: August 14, 2001.

Finally we prove that every closed, oriented connected 3-manifold has a weakly symplectically fillable double cover, branched along a 2-component link. This result is interesting in our point of view, since not every 3-manifold is weakly symplectically fillable as shown in [L]. We advise the reader to turn to [EH] for notions like weak/strong symplectic filling, Stein filling, etc.

1. Natural symplectic compactifications of Stein surfaces

Theorem 1. *Any compact Stein surface with boundary embeds naturally into a symplectic Lefschetz fibration over S^2. In particular, any compact Stein surface with boundary embeds into a minimal closed symplectic 4-manifold. Conversely given an irreducible symplectic Lefschetz fibration over S^2 with a section, then if we remove a neighborhood of this section union a regular fiber we get a Stein surface.*

Proof. We know that a compact Stein surface X with boundary admits a PALF. We may also assume that the regular fiber F has only one boundary component. The fibration induces an open book decomposition of ∂X with connected binding ∂F. First we enlarge X to X' by attaching a 2-handle along ∂F with 0-framing. Note that $\partial X'$ is an $\hat{F}$-bundle over S^1, where $\hat{F}$ denotes the closed surface obtained by capping off the surface F by gluing a 2-disk along its boundary. Also X' admits a positive Lefschetz fibration over D^2 with regular fiber $\hat{F}$. Let $\text{Map}(\hat{F})$ denote the mapping class group of the closed surface $\hat{F}$. The second author learned the proof of the next lemma from Ivan Smith.

Lemma 2. *Any element in $Map(\hat{F})$ can be expressed as a product of nonseparating positive Dehn twists.*

Proof. Let the curves A_i, B_i on a genus $n \geq 1$ surface $\hat{F}$ be drawn as in Figure 1 and write a_i and b_i for the positive Dehn twists about A_i and B_i, respectively.

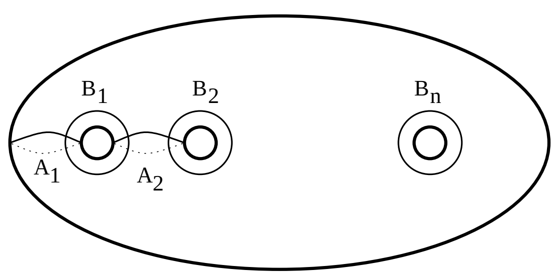

FIGURE 1. A surface of genus $n \geq 1$

The following is a standard word in the mapping class group $\mathrm{Map}(\hat{F})$. (cf. [B]).

$$1 = (a_1 b_1 a_2 b_2 ... a_n b_n)^{4n+2}.$$

Hence we conclude that a_1^{-1} is a product of nonseparating positive Dehn twists. Therefore any negative nonseparating Dehn twist is a product of nonseparating positive Dehn twists since any two negative nonseparating Dehn twists are equivalent. This finishes the proof of the lemma combined with the fact that $\mathrm{Map}(\hat{F})$ is generated by (positive and negative) nonseparating Dehn twists.

□

We use Lemma 2 to extend X' into a positive Lefschetz fibration $\tilde{X}$ over S^2 with regular fiber $\hat{F}$ as follows. Let $c_1 c_2 ... c_k$ be the global monodromy of the PALF on X, where c_i denotes the positive Dehn twist along a simple closed curve C_i on F. Then this product (after capping off the boundary component) can be viewed as a product in $\mathrm{Map}(\hat{F})$. We clearly have

$$c_1 c_2 ... c_k c_k^{-1} c_{k-1}^{-1} ... c_1^{-1} = 1.$$

By Lemma 2 we can replace every negative twist by a product of positive twists to obtain $\tilde{X}$. $\tilde{X}$ admits a symplectic structure with symplectic regular fibers by a Theorem of Gompf [GS]. Consequently, X is emdedded naturally into a closed symplectic 4-manifold $\tilde{X}$. Moreover we can choose $\tilde{X}$ to be minimal by taking fiber sums if necessary.

Conversely suppose that $\tilde{X}$ is an irreducible symplectic Lefschetz fibration over S^2 with a section. Here irreducible means that all the vanishing cycles are nonseparating. Then if we remove a neighborhood of this section union a regular fiber we get a Stein surface. This is clear from our description of Stein surfaces as PALF's.

□

Remark 1. A geometrically stronger version of Theorem 1 was proved in [LM], that is compact Stein surfaces embed into minimal complex surfaces of general type (they are known to be Lefschetz pencils). However, our proof is topologically stronger in the sense that we embed Stein surfaces into Lefschetz fibrations. Our proof should be viewed as the symplectic version of the standard procedure of embedding a compact manifold into a closed manifold by doubling process. In fact, this theorem gives a procedure of associating to any PALF a positive Lefschetz fibration over D^2 with closed fibers, which is then capped off by $\hat{F} \times D^2$ to get a symplectic Lefschetz fibration over S^2. In particular, we have a map

$$\{\text{PALF's}\} \rightarrow \{\text{Symplectic 4-manifolds}\}.$$

Since any fibered knot K in S^3 with positive allowable monodromy induces a $(PALF)_K$ on B^4, we can associate to K a closed symplectic 4-manifold. For example,

$$(PALF)_{trefoil} \rightarrow K3 \text{ surface}$$

To see this implication consider the PALF induced by the trefoil knot as indicated in Figure 2. Its monodromy ab is given by Dehn twists along the two generators of the first homology of the fiber which is a punctured torus.

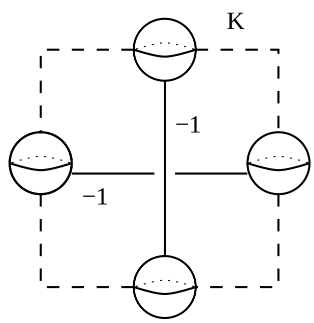

FIGURE 2. PALF induced by trefoil

Since $(ab)^6 = 1$, we have $a^{-1} = b(ab)^5$ and $b^{-1} = (ab)^5 a$, hence by our algorithm $abb^{-1}a^{-1} = (ab)^{12}$. To get the symplectic completion we need to attach 0 framed 2−handle to the binding (trefoil knot) and 22 more 2−handles with −1 framing each, as indicated in Figure 3. We then cap it off with $D^2 \times T^2$ at the end, to get a closed manifold, which corresponds to attaching the indicated −2 framed handle in Figure 3 and two 3−handles (which we don't need to indicate). This gives K3 surface (compare [HKK]) which is usually denoted by E(2).

Notice, that since $(ab)^{-1} = (ab)^5$, rather then using the algorithm we can write in a shorter way $abb^{-1}a^{-1} = (ab)^6$, and end up constructing a smaller symplectic completion by attaching ten 2-handles to Figure 2 instead, as indicated in Figure 4. This time the last 2-handle corresponding to $D^2 \times T^2$ has to be attached by −1 framing (see discussion in [HKK] about determining framing of this handle). This gives us the half Kummer surface $E(1)$.

Remark 2. The last statement in Theorem 1 is implicit in Donaldson's work ([D]). Our proof, however, is purely topological. Combining with the existence of irreducible Lefschetz pencils (cf. [D], [S]) on symplectic 4-manifolds one concludes that every closed symplectic 4-manifold, possibly after blowing up at some points, contains a "big" Stein piece.

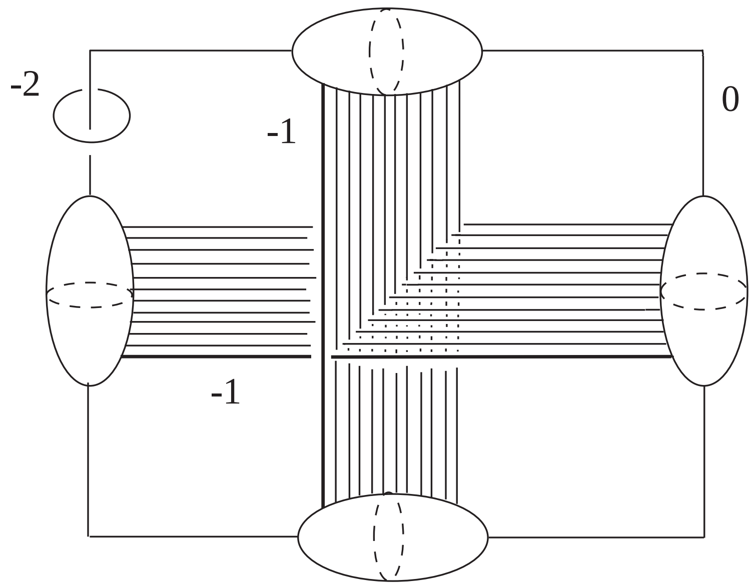

FIGURE 3. E(2)

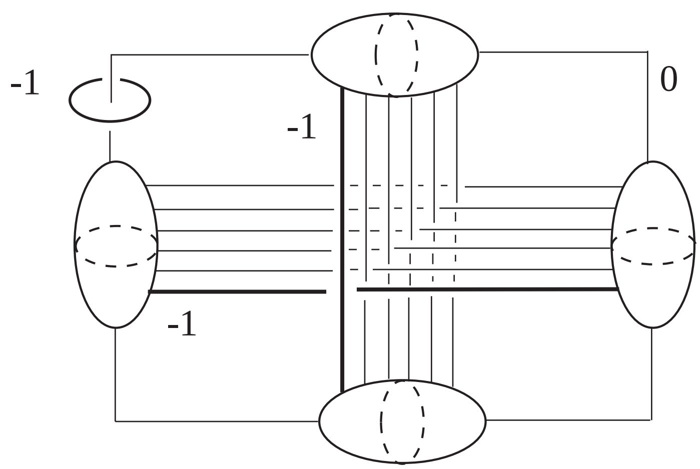

FIGURE 4. E(1)

Remark 3. In Lemma 2 we proved that any element in Map($\hat{F}$) can be expressed as a product of nonseparating positive Dehn twists. This is not true for the mapping class group of a surface with boundary. Otherwise any 3-manifold would be Stein fillable (cf. [AO]), which is shown to be false, for example, in Theorem 7.

2. Non-Stein minimal elliptic fibrations

Let $Y(e,n)$ denote the circle bundle over a genus $n \geq 1$ surface with euler number e. First we observe that $Y(1,1)$ is also a torus bundle over S^1 with monodromy a single positive Dehn twist along a simple closed curve on the torus T^2. (A proof of this is given in the appendix). Next we show that $Y(1,1)$ is Stein fillable. In fact a regular neighborhood of so called fishtail fiber in an elliptic fibration is a Stein filling of $Y(1,1)$ by [G] as shown in Figure 5.

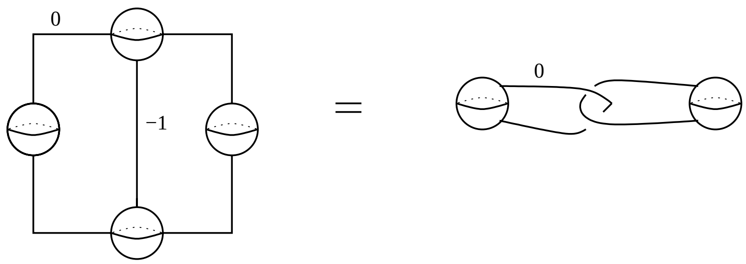

Figure 5. Regular neigborhood of a fishtail fiber

Now consider the elliptic surface X fibered over D^2 with global monodromy

$$(ab)^6 a$$

where a and b denote positive Dehn twists along simple closed curves A and B, respectively as shown in Figure 6. Note that $\partial X = Y(1,1)$.

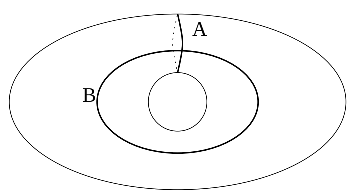

Figure 6. Torus

Theorem 3. *X is a minimal, symplectic and simply-connected elliptic fibration over D^2, which does not admit any Stein structure.*

Proof. X is minimal since X embeds in, say $E(2)$, which is minimal. X is simply-connected since $X = E(1)\sharp Z_1$ and $E(1)$ is simply-connected. X admits a symplectic structure by a theorem of Gompf [GS].
Suppose that X is Stein. We can embed X into a complex surface S of general type by a theorem of Lisca and Matic [LM]. We will show that we can assume $b_2^+(S-X) > 0$.

Suppose that $b_2^+(S-X)=0$. First we extend X into X' by attaching a 2-handle with framing $tb(K)-1$ along a Legendrian knot K satisfying $tb(K)>1$ and contained in a standard 3-ball D^3 in ∂X. (Here tb denotes the Thurston-Bennequin invariant). Because of the framing of the 2-handle, X' is also a Stein surface. So we can embed X' into a complex surface S' of general type. (cf. Figure 7). Thus $b_2^+(S'-X)>0$, since we created a second homology class with positive self-intersection in $S'-X$. Also note that $b_2^+(X)>0$.

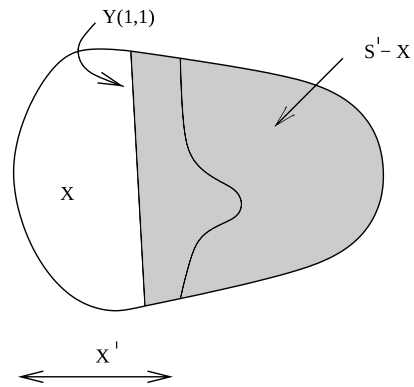

FIGURE 7

This gives a contradiction to the following result of Ozsvath and Szabo, hence showing that X is not Stein.

Theorem 4. [OzS] *If S is a complex surface of general type, and $|e|\geq 2n-1$, then S admits no splitting along an embedded copy of $Y=Y(e,n)$ of the form $S=X_1\#_Y X_2$ with $b_2^+(X_1), b_2^+(X_2)>0$.*

□

Remark 4. We can generalize Theorem 3 as follows. Define $X_k=E(k)\sharp Z_1$ for $k\geq 1$. Then for each $k\geq 1$, X_k is a minimal, symplectic and simply-connected elliptic fibration over D^2, which does not admit any Stein structure. We can also generalize Theorem 3 in a different direction. One can easily find non-Stein *achiral* minimal Lefschetz fibrations over D^2.

Theorem 5. *Suppose that $\tilde{X}$ is a Stein filling of some $Y(e,n)$ with $|e|\geq 2n-1$, then $b_2^+(\tilde{X})=0$.*

Proof. We use the same trick as in the proof of Theorem 3. □

The following result was proved in [GS]. We give a different proof as an application of Theorem 5.

Corollary 6. *Let $X(e,n)$ denote the disk bundle over a genus n surface with euler number e. If $e \geq 2n-1$ then $X(e,n)$ is not Stein.*

Proof. First note that $\partial X(e,n) = Y(e,n)$. Suppose that $e \geq 2n-1$. Then we can apply Theorem 5 to conclude that $X(e,n)$ is not Stein since $b_2^+(X(e,n)) = 1$.

□

Remark 5. [GS] The result in Corollary 6 is optimal, i.e., $X(e,n)$ is realized as a Stein surface iff $e < 2n-1$. $Y(e,n) = \partial X(e,n)$, however, is Stein fillable for all e, n. (cf [G])

It is known that the Poincare homology sphere with reversed orientation is not weakly symplectically semi-fillable [L]. We prove the following weaker result by our technique.

Theorem 7. *Poincare homology sphere with reversed orientation is not Stein fillable.*

Proof. Let Y denote the Poincare homology sphere oriented as the boundary of the $-E_8$ plumbing as shown in Figure 8.

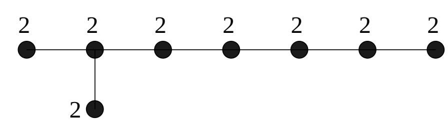

FIGURE 8. $-E_8$ plumbing

Note that Y has a metric with positive scalar curvature. Suppose X is a Stein filling of Y. Embed X into a complex surface S of general type with $b_2^+(S - X) > 0$ as in the proof of Theorem 3. Since S has non-vanishing Seiberg-Witten invariants and Y has a metric with positive scalar curvature, we have $b_2^+(X) = 0$. (see [OhO] for a proof). Hence $X \cup (E_8)$ is a closed, smooth and negative definite 4-manifold with a nonstandard intersection form, which can not exist by a theorem of Donaldson.

□

3. Stein fillings of some circle bundles

In this section we show that for every integer $n \geq 2$, there exits an irreducible 3-manifold M_n with at least $n+1$ mutually non-homeomorphic simply-connected Stein fillings. We will also identify M_n as the circle bundle $Y(-1,n)$ over a genus n surface with euler number $e = -1$.

Let t_α denote a positive Dehn twist about a simple closed curve α on an oriented surface F. The following result is standard. (cf. [B]).

Lemma 8. *For any two simple closed curves α and β on F we have $t_\beta t_\alpha = t_\alpha t_{t_\beta(\alpha)}$.*

Let the curves A_i, B_i, D_2, E_2 on a genus $n \geq 2$ surface F with one boundary component be drawn as in Figure 9 and write lower-case letters a_i etc. for the positive Dehn twist about A_i etc. Let δ denote a positive Dehn twist about a curve parallel to the boundary.

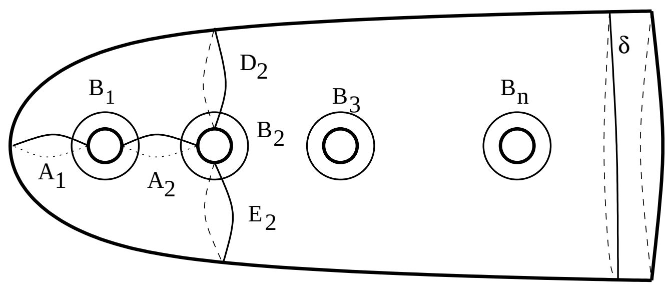

FIGURE 9. Genus $n \geq 2$ surface F with boundary

Proposition 9.

$$(a_1 b_1 a_2 b_2 ... a_n b_n)^4 = (a_1 b_1 a_2)^4 \psi,$$

where ψ is a product of $8n - 12$ nonseparating positive Dehn twists.

Proof. We apply Lemma 8 as many times as needed to obtain $(a_1 b_1 a_2)^4$ at the beginning of the given product.

□

Lemma 10. [LP] *We have $\delta = (a_1 b_1 a_2 b_2 ... a_n b_n)^{4n+2}$ in Map$(F, \partial F)$.*

Theorem 11. *For every integer $n \geq 2$, there exits an irreducible 3-manifold M_n with at least $n + 1$ mutually non-homeomorphic simply-connected Stein fillings.*

Proof. Fix an integer $n \geq 2$. Define

$$F_\delta = (F \times [0,1]) / \ (\delta(x), 0) \sim (x, 1)$$

where $\delta \in \text{Map}(F, \partial F)$ is the map in Lemma 10. Consider the closed oriented 3-manifold

$$M_n = (S^1 \times D^2) \cup F_\delta.$$

In other words, M_n has a positive open book decomposition with binding ∂F, page F and monodromy δ which is expressed as a product of nonseparating positive Dehn twists. By the results in [AO], M_n admits a Stein filling X_0 which is a PALF (positive allowable Lefschetz fibration over D^2 with bounded fibers) of genus n with $2n(4n+2)$ singular fibers. We can calculate the Euler characteristic $\chi(X_0)$ as follows:

$$\chi(X_0) = 2 - 2n - 1 + 2n(4n+2) = 8n^2 + 2n + 1.$$

The following is a standard relation in the mapping class group. (cf. [B]).

$$(a_1 b_1 a_2)^4 = d_2 e_2.$$

Using this relation and Proposition 9, we can write the map δ as follows:

$$\begin{aligned}
\delta &= (a_1 b_1 a_2 b_2 ... a_n b_n)^{4n+2} \\
&= (a_1 b_1 a_2 b_2 ... a_n b_n)^4 (a_1 b_1 a_2 b_2 ... a_n b_n)^{4(n-1)+2} \\
&= (a_1 b_1 a_2)^4 \psi (a_1 b_1 a_2 b_2 ... a_n b_n)^{4(n-1)+2} \\
&= d_2 e_2 \psi (a_1 b_1 a_2 b_2 ... a_n b_n)^{4(n-1)+2}
\end{aligned}$$

This last product gives another Stein filling X_1 of M_n as a PALF with 10 less singular fibers. Thus

$$\chi(X_1) = \chi(X_0) - 10.$$

We can iterate the substitution above n-times to get $n+1$ mutually non-homeomorphic Stein fillings $X_0, X_1, ..., X_n$ of M_n such that

$$\chi(X_i) = \chi(X_{i-1}) - 10.$$

It is easy to see that all the Stein fillings are simply-connected. Next we show that $M_n = Y(-1, n)$, which implies in particular, that M_n is irreducible.

□

Lemma 12. [St] *We can identify M_n as the circle bundle $Y(-1, n)$ over a genus n surface with euler number $e = -1$.*

Proof. Fix an integer $n \geq 2$. Let F denote a genus n surface with one boundary component and let $\hat{F}$ denote the closed surface obtained by capping off the surface F by gluing a 2-disk along its boundary. Then there is a natural map

$$Map(F, \partial F) \to Map(\hat{F}).$$

So the relation in Lemma 10 induces a word in $Map(\hat{F})$. Consequently this gives a positive Lefschetz fibration over S^2 with a section a sphere of square -1 and regular fiber $\hat{F}$. Now consider a neighborhood U_n of a regular fiber union this section. First we observe that $\partial U_n = \overline{M_n}$. Moreover U_n is obtained by plumbing a disk bundle over

$\hat{F}$ and a disk bundle over S^2. We can blow down the -1 sphere to get a disk bundle over $\hat{F}$ with euler number $+1$. We prove our result by reversing the orientations.

□

Remark 6. In [G], Gompf proves that $Y(-1,n)$ has at least n Stein fillable contact structures. The Stein fillings are mutually diffeomorphic but they admit different complex structures distinguished by their first Chern classes.

4. Splittings of symplectic 4-manifolds along surface bundles over S^1

Proposition 13. *Any closed surface bundle over S^1 can be embedded into a closed symplectic 4-manifold, splitting the symplectic 4-manifold into two pieces both of which have positive b_2^+.*

Proof. Let Σ denote a closed, connected and oriented surface of genus n. Let B_ϕ denote the Σ-bundle over S^1 with monodromy $\phi \in \mathrm{Map}(\Sigma)$. We can express ϕ as a product $c_1c_2...c_k$ of (nonseparating) positive Dehn twists by Lemma 2. Using this product we can fill in the Σ-bundle over S^1 with a positive Lefschetz fibration X_1 over D^2 with regular fiber Σ. We trivially have

$$c_1c_2...c_kc_k^{-1}c_{k-1}^{-1}...c_1^{-1} = 1.$$

Now replace each negative twist in this word by a product of (nonseparating) positive Dehn twists again by Lemma 2. Hence we get a positive Lefschetz fibration X over S^2 which is a union of X_1 and $X_2 = X - X_1$. Suppose that $b_2^+(X_i) = 0$ for some $i = 1, 2$. (Otherwise the theorem is proved). Let $G(n)$ denote the nontrivial genus n Lefschetz fibration over S^2 given by the word

$$1 = (a_1b_1a_2b_2...a_nb_n)^{4n+2}.$$

These higher genus Lefschetz fibrations can be considered as the generalization of the elliptic fibration on $E(1)$. We have $b_2^+(G(n)) > 0$ since $G(n)$ has a symplectic structure by a theorem of Gompf [GS]. Define

$$X_i' = X_i \sharp G(n)$$

where $\sharp$ denotes the fiber sum along a regular fiber. It is easy to see that $b_2^+(X_i') > 0$ for $i = 1, 2$. Hence $G(n)\sharp X \sharp G(n)$ is a closed symplectic 4-manifold which is a union of X_1' and X_2' (with $b_2^+(X_i') > 0$), glued along the Σ-bundle B_ϕ.

□

Remark 7. We were pointed out by I. Smith that the first part of our proof (not including the b_2^+ arguement) had appeared in [Sm]. He actually shows that the Lefschetz fibration X_1 is a weak symplectic filling for the contact structure arising from a C^0 perturbation of the obvious foliation on ∂X_1.

Next we use branched coverings to obtain the following result.

Theorem 14. *Every closed, oriented connected 3-manifold has a weakly symplectically fillable double cover, branched along a 2-component link.*

Proof. We use the following result of Montesinos [M] (see also [S]).

Lemma 15. [M] *Every closed, oriented connected 3-manifold M contains a two component link L, such that there is a 2-fold covering of M branched over L which is a surface bundle over S^1.*

Proof. It is well known that every closed, oriented connected 3-manifold M has an open book decomposition with connected binding. Let F denote the page and ϕ denote the monodromy of an open book on M. We use $2F$ to denote the double of F, where two copies of F are glued along their boundary. Define

$$N = 2F \times [-1, 1]/\ (\phi \# \phi^{-1}(x), -1) \sim (x, 1)$$

So, N is a $2F$-bundle over S^1 with monodromy $\phi \# \phi^{-1}$, where ϕ acts on one copy of F and ϕ^{-1} acts on the other copy. Let u be the involution on N given by

$$u(x, t) = (ix, -t),$$

where i denotes the involution on $2F$ interchanging the two copies of F. Then it is easy to see that $N/u = M$. Moreover the fixed point set of the action of u on N is a disjoint union of two circles. This finishes the proof of the lemma.

□

Since every 3-manifold M is double branched covered by a surface bundle N, and every surface bundle over S^1 is weakly symplectically fillable as in the proof of Proposition 13, we conclude that every closed oriented connected 3-manifold has a weakly symplectically fillable double cover, branched along a 2-component link.

□

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5. Appendix

We prove that the circle bundle over torus with euler number $e = 1$ is diffeomorphic to a torus bundle over circle, whose monodromy is a single positive Dehn twist.

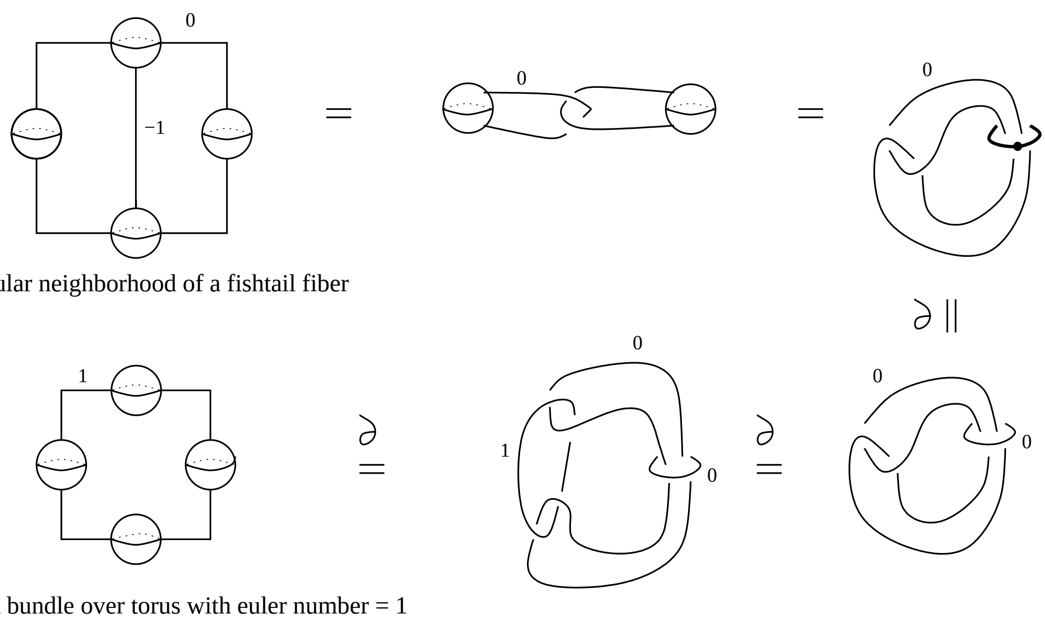

Figure 10

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