KNOT SURGERY AND SCHARLEMANN MANIFOLDS

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Abstract. We discuss the relation between Fintushel-Stern knot surgery operation on 4-manifolds and "Scharlemann manifolds", and as a corollary show that they all are standard. Along the way we show the fishtail can exotically knot in S^4 infinitely many ways.

0. Introduction

Let X be a smooth 4-manifold, and $T^2 \times D^2 \subset X$ be an imbedded torus with trivial normal bundle, and $K \subset S^3$ be a knot, $N(K)$ be its tubular neighborhood. The Fintushel-Stern *knot surgery operation* is the operation of replacing $T^2 \times D^2$ with $(S^3 - N(K)) \times S^1$, so that the meridian $p \times \partial D^2$ of the torus coincides with the longitude of K [FS].

$$X \rightsquigarrow X_K = (X - T^2 \times D^2) \cup (S^3 - N(K)) \times S^1$$

The handlebody picture of this operation was given in [A1]. Let $K \subset S^3$ be a knot, and S^3_K be the 3-manifold obtained from S^3 by ± 1 surgery to K (either one). The (generalized) *Scharlemann manifold* $M(K)$ is the manifold obtained by surgering the circle $C \subset S^1 \times S^3_K$ (with even framing) which corresponds to the meridian of the knot K. It is clear that $M(K)$ is homotopy equivalent to $S^1 \times S^3 \# (S^2 \times S^2)$. in [S] Scharlemann had posed the question whether $M(K)$ is standard when K is the trefoil knot; and in [A2] this question was answered affirmatively. Here we show that $M(K)$ is also standard for any K. We decided to write this paper after seeing [T] which claims the same result. We felt that there should be a natural direct proof generalizing the steps of [A2] by using the knot-surgery description of 4-manifolds [A1]. It turns out that the stabilization theorem of [A3] provides the necessary tool to link these two. Along the way we relate Scharlemann manifolds $M(K)$ to the knot surgery operation $X \rightsquigarrow X_K$, and give a sufficient criterion when a knot surgery operation doesn't change the smooth structure of the underlying manifold. I thank M.Tange for stimulating my interest to revisit this problem.

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1. A review of the stabilization

In [A3] (and also in [Au]) it was shown that X_K is stably trivial, i.e.:

$$X_K \#(S^2 \times S^2) = X\#(S^2 \times S^2)$$

In [A3] a specific trivialization move was described in terms of handles (i.e. turning a "ribbon 1-handle" to 2-handle'). More specifically it was shown that surgering the circle $A \subset T^2 \times D^2$ (as shown in Figure 1) gives the same manifold as surgering the corresponding $A \subset (T^2 \times D^2)_K$.

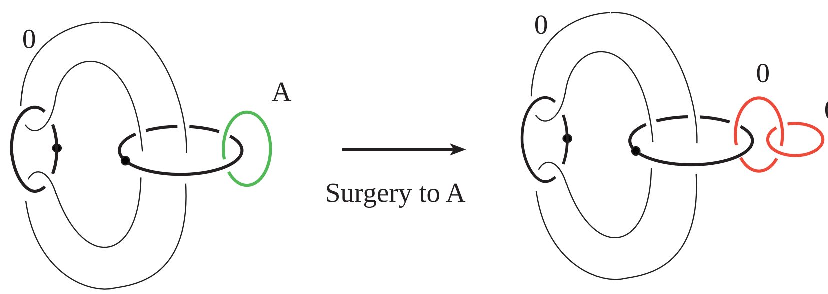

Figure 1. Surgering $T^2 \times D^2$

Notice, if we attach a 2-handle h^2 to $T^2 \times D^2$ along A with zero framing (as in Figure 2), we get $\Gamma := T^2 \times D^2 + h^2 = S^1 \times B^3 \natural (S^2 \times B^2)$, and this identification takes the loop B to the meridian of $S^2 \times B^2$.

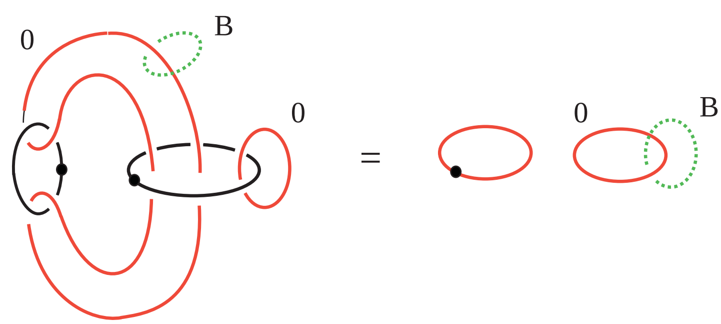

Figure 2. $S^1 \times B^3 \natural (S^2 \times B^2)$

The proof of [A3] shows that the knot surgery of $S^1 \times B^3 \natural (S^2 \times B^2)$ along this $T^2 \times D^2 \subset S^1 \times B^3 \natural (S^2 \times B^2)$ keeps it standard:

Theorem 1 ([A3])**.** $[\, S^1 \times B^3 \natural (S^2 \times B^2)\,]_K = S^1 \times B^3 \natural (S^2 \times B^2)$

Proof. (Sketch) Figure 4 gives the handlebody of the knot surgery (where K is drawn as the trefoil knot). The zero framed linking circle to of the "ribbon 1-handle" cancels this ribbon 1-handle, and in the process the rest of the handlebody becomes standard (cf. [A3]). □

This theorem gives a sufficient condition for showing that a knot surgery operation does not change the underlying smooth manifold. More specifically, If a torus $T^2 \subset X$ has a $\Gamma = S^1 \times B^3 \#(S^2 \times B^2)$ neighborhood in X (put another way, if the loop $A \subset \partial(T^2 \times D^2)$ bounds a disk in the complement $X - T^2 \times D^2$, whose normal framing induces the zero framing on A), then $X_K = X$. For example S^4 can be decomposed as a union of two fishtails glued along boundaries as in Figure 3, and clearly the torus inside has a Γ neighborhood.

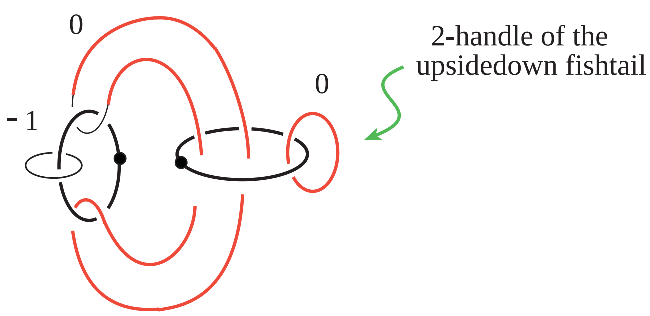

FIGURE 3. S^4 as a union of two fishtails

Similarly Figure 5 describes of $S^2 \times S^2$ as the double of the cusp, and Figure 6 describes $S^1 \times S^3 \#(S^2 \times S^2)$ as the double of the fishtail. Note that in these figures we give some alternative pictures of these handlebodies by using the diffeomorphism $\varphi : S^2 \times T^2 \to S^2 \times T^2$ of Figure 7, which carries the loop A to itself by twisting its tubular neighborhood. Clearly (from the pictures) in all of these cases the sub-torus lies in a Γ neighborhood. Therefore we have:

Corollary 2.

(a) $S^4_K = S^4$

(b) $(S^2 \times S^2)_K = S^2 \times S^2$

(c) $[\, S^1 \times S^3 \#(S^2 \times S^2)\,]_K = S^1 \times S^3 \#(S^2 \times S^2)$

By taking $K \subset S^3$ to be knots with different Alexander polynomials and using [A1] we can state Corollary 2 (a) in the following useful form:

Theorem 3. *The fishtail F (the 2-sphere with one self intersection) can imbed into S^4 infinitely many different ways $f_K : F \hookrightarrow S^4$, so that each $S^4 - f_K(F) = F_K$ is a different exotic copy of F, where K are knots with different Alexander polynomials.*

2. Proving $M(K)$ is standard

Theorem 4. $M(K) = S^1 \times S^3 \# (S^2 \times S^2)$

Proof. The first picture of Figure 8 is the handlebody of $S^1 \times S^3_K$ surgered along the linking loop C (in the figure K is drawn as the trefoil knot), as discussed in [A2]. Here the pair of small red linking handles denotes the surgering the loop C in $S^1 \times S^3_K$. By sliding the zero framed circle over the $+1$ framed circle we obtain the second picture of Figure 8. Then by sliding the small -1 framed red circle over one of the long zero framed circles (the ones going through the 1-handle), and then sliding the large -1-framed circle over this small -1 framed circle we obtain the first picture of Figure 9 (now the large -1 framed circle becomes 0 framed). Note that this last move is from [A2] (e.g. going from Figure 30 to Figure 31 of [A2]). Then by sliding $+1$ framed circle over the -1 framed circle we obtain the second picture of Figure 9 (i.e. the reverse of the first move of Figure 8). Now by [A1], this is just the handlebody of the knot surgered manifold of Figure 6, which is $[\, S^1 \times S^3 \# (S^2 \times S^2)\,]_K$, so by the Corollary 2 it is $S^1 \times S^3 \# (S^2 \times S^2)$. $\square$.

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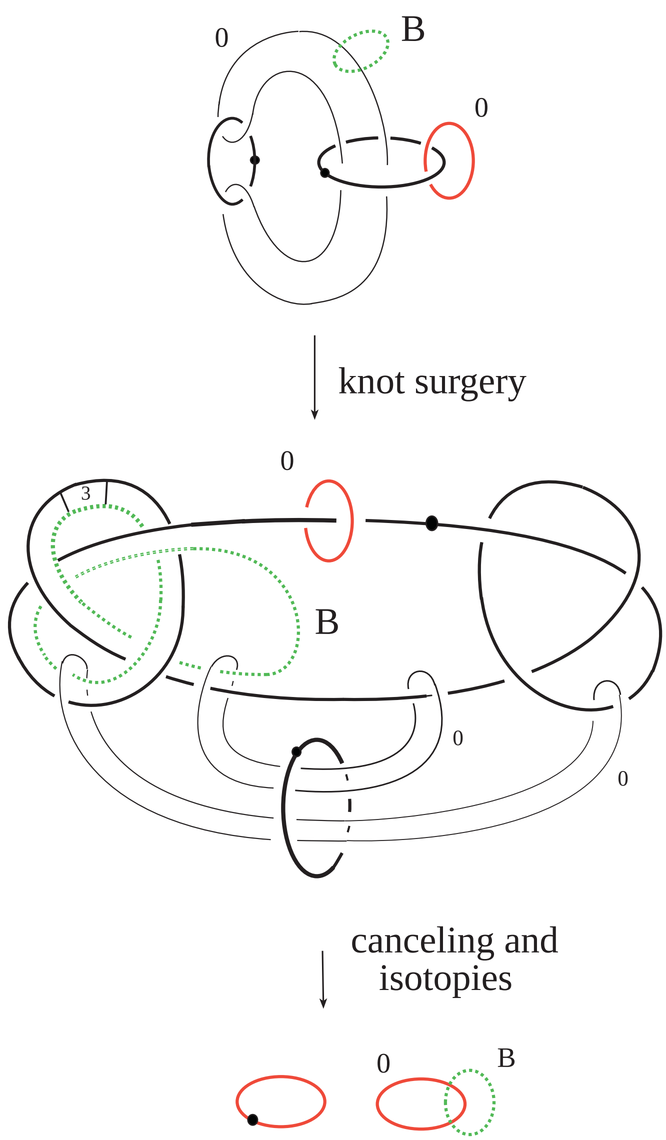

FIGURE 4

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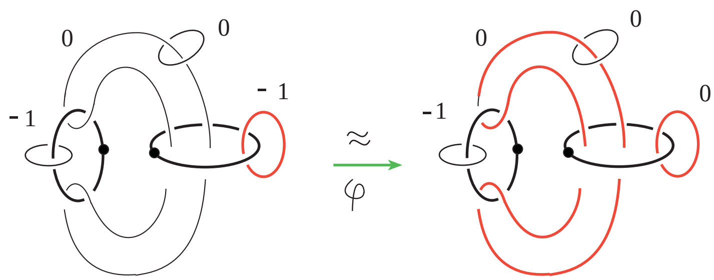

FIGURE 5. $S^2 \times S^2$ as double of two cusps

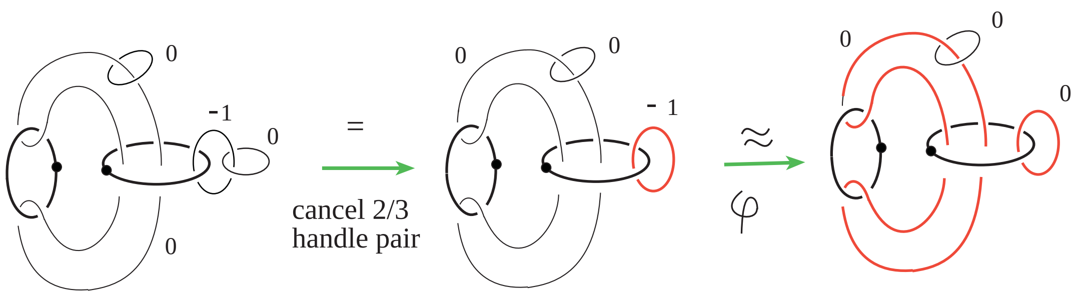

FIGURE 6. $S^1 \times S^3 \#(S^2 \times S^2)$ as double of two fishtails

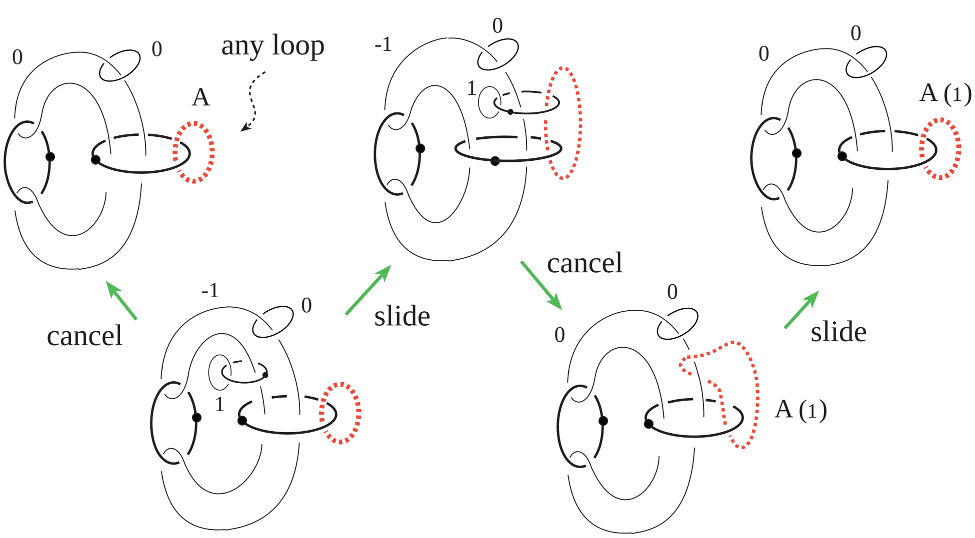

FIGURE 7. Diffeomorphism $\varphi : S^2 \times T^2 \rightarrow S^2 \times T^2$

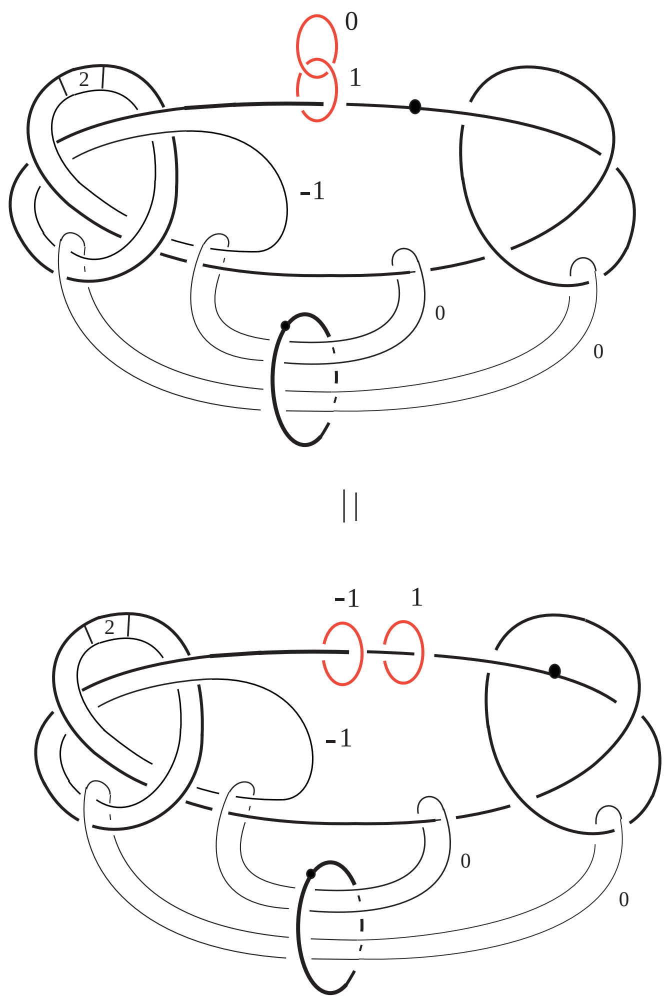

Figure 8

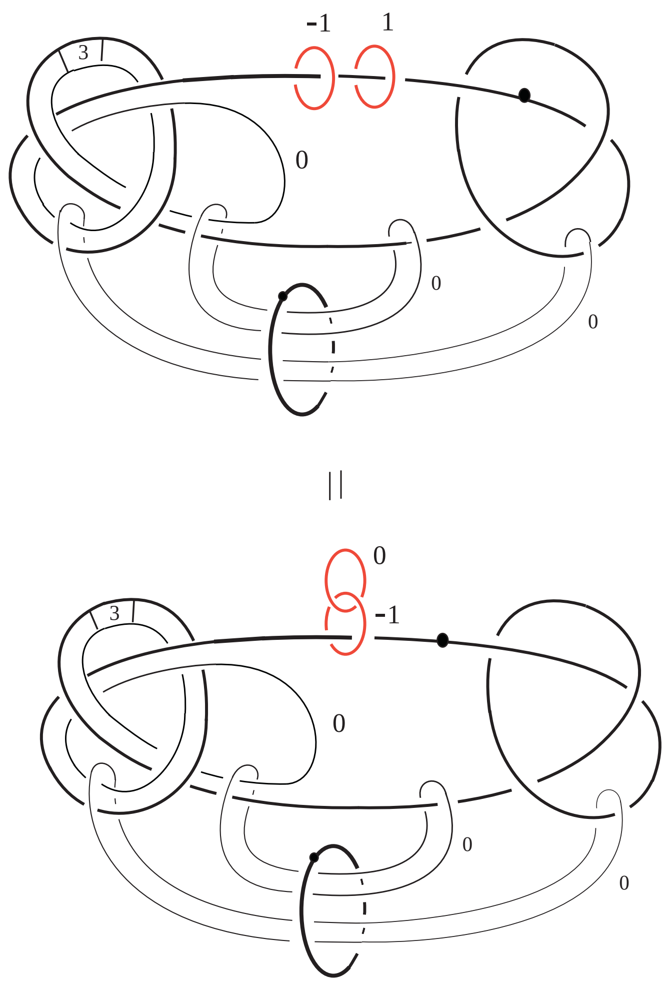

Figure 9