

HARVEY LAWSON MANIFOLDS AND DUALITIES

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ABSTRACT. The purpose of this paper is to introduce Harvey-Lawson manifolds and review the construction of certain “mirror dual” Calabi-Yau submanifolds inside a G_2 manifold. More specifically, given a Harvey-Lawson manifold HL , we explain how to assign a pair of tangent bundle valued 2 and 3-forms to a G_2 manifold $(M, HL, \varphi, \Lambda)$, with the calibration 3-form φ and an oriented 2-plane field Λ . As in [AS2] these forms can then be used to define different complex and symplectic structures on certain 6-dimensional subbundles of $T(M)$. When these bundles are integrated they give mirror CY manifolds (related thru HL manifolds).

1. INTRODUCTION

Let (M^7, φ) be a G_2 manifold with the calibration 3-form φ . If φ restricts to be the volume form of an oriented 3-dimensional submanifold Y^3 , then Y is called an associative submanifold of M . In [AS2] the authors introduced a notion of mirror duality in any G_2 manifold (M^7, φ) based on the associative/coassociative splitting of its tangent bundle $TM = \mathbb{E} \oplus \mathbb{V}$ by the non-vanishing 2-fame fields provided by [T]. This duality initially depends on the choice of two non-vanishing vector fields, one in \mathbb{E} and the other in \mathbb{V} . In this article we give a natural form of this duality where the choice of these vector fields are made more canonical, in the expense of possibly localizing this process to the tubular neighborhood of the 3-skeleton of (M, φ) .

2. BASIC DEFINITIONS

Let us recall some basic facts about G_2 manifolds (e.g. [B1], [HL], [AS1]). Octonions give an 8 dimensional division algebra $\mathbb{O} = \mathbb{H} \oplus l\mathbb{H} = \mathbb{R}^8$ generated by $\langle 1, i, j, k, l, li, lj, lk \rangle$. The imaginary octonions $im\mathbb{O} = \mathbb{R}^7$ is equipped with the cross product operation $\times : \mathbb{R}^7 \times \mathbb{R}^7 \rightarrow \mathbb{R}^7$ defined by $u \times v = im(\bar{v}.u)$. The exceptional Lie group G_2 is the linear automorphisms of $im\mathbb{O}$ preserving this cross product. Alternatively:

$$(1) \quad G_2 = \{(u_1, u_2, u_3) \in (\mathbb{R}^7)^3 \mid \langle u_i, u_j \rangle = \delta_{ij}, \langle u_1 \times u_2, u_3 \rangle = 0\}.$$

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$$(2) \quad G_2 = \{A \in GL(7, \mathbb{R}) \mid A^* \varphi_0 = \varphi_0\}.$$

where $\varphi_0 = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$ with $e^{ijk} = dx^i \wedge dx^j \wedge dx^k$. We say a 7-manifold M^7 has a G_2 structure if there is a 3-form $\varphi \in \Omega^3(M)$ such that at each $p \in M$ the pair $(T_p(M), \varphi(p))$ is (pointwise) isomorphic to $(T_0(\mathbb{R}^7), \varphi_0)$. This condition is equivalent to reducing the tangent frame bundle of M from $GL(7, \mathbb{R})$ to G_2 . A manifold with G_2 structure (M, φ) is called a G_2 manifold (integrable G_2 structure) if at each point $p \in M$ there is a chart $(U, p) \rightarrow (\mathbb{R}^7, 0)$ on which φ equals to φ_0 up to second order term, i.e. on the image of the open set U we can write $\varphi(x) = \varphi_0 + O(|x|^2)$.

One important class of G_2 manifolds are the ones obtained from Calabi-Yau manifolds. Let (X, ω, Ω) be a complex 3-dimensional Calabi-Yau manifold with Kähler form ω and a nowhere vanishing holomorphic 3-form Ω , then $X^6 \times S^1$ has holonomy group $SU(3) \subset G_2$, hence is a G_2 manifold. In this case $\varphi = \text{Re } \Omega + \omega \wedge dt$. Similarly, $X^6 \times \mathbb{R}$ gives a noncompact G_2 manifold.

Definition 1. Let (M, φ) be a G_2 manifold. A 4-dimensional submanifold $X \subset M$ is called coassociative if $\varphi|_X = 0$. A 3-dimensional submanifold $Y \subset M$ is called associative if $\varphi|_Y \equiv \text{vol}(Y)$; this condition is equivalent to the condition $\chi|_Y \equiv 0$, where $\chi \in \Omega^3(M, TM)$ is the tangent bundle valued 3-form defined by the identity:

$$(3) \quad \langle \chi(u, v, w), z \rangle = * \varphi(u, v, w, z)$$

The equivalence of these conditions follows from the ‘associator equality’ of [HL]

$$\varphi(u, v, w)^2 + |\chi(u, v, w)|^2/4 = |u \wedge v \wedge w|^2$$

Similar to the definition of χ one can define a tangent bundle 2-form, which is just the cross product of M (nevertheless viewing it as a 2-form has its advantages).

Definition 2. Let (M, φ) be a G_2 manifold. Then $\psi \in \Omega^2(M, TM)$ is the tangent bundle valued 2-form defined by the identity:

$$(4) \quad \langle \psi(u, v), w \rangle = \varphi(u, v, w) = \langle u \times v, w \rangle$$

On a local chart of a G_2 manifold (M, φ) , the form φ coincides with the form $\varphi_0 \in \Omega^3(\mathbb{R}^7)$ up to quadratic terms, we can express the tangent valued forms χ and ψ in terms of φ_0 in local coordinates. More generally, if e_1, \dots, e_7 is any local orthonormal frame and e^1, \dots, e^7 is the dual frame, from definitions we get:

$$\begin{aligned}
\chi = & (e^{256} + e^{247} + e^{346} - e^{357})e_1 \\
& + (-e^{156} - e^{147} - e^{345} - e^{367})e_2 \\
& + (e^{157} - e^{146} + e^{245} + e^{267})e_3 \\
& + (e^{127} + e^{136} - e^{235} - e^{567})e_4 \\
& + (e^{126} - e^{137} + e^{234} + e^{467})e_5 \\
& + (-e^{125} - e^{134} - e^{237} - e^{457})e_6 \\
& + (-e^{124} + e^{135} + e^{236} + e^{456})e_7.
\end{aligned}$$

$$\begin{aligned}
\psi = & (e^{23} + e^{45} + e^{67})e_1 \\
& + (e^{46} - e^{57} - e^{13})e_2 \\
& + (e^{12} - e^{47} - e^{56})e_3 \\
& + (e^{37} - e^{15} - e^{26})e_4 \\
& + (e^{14} + e^{27} + e^{36})e_5 \\
& + (e^{24} - e^{17} - e^{35})e_6 \\
& + (e^{16} - e^{25} - e^{34})e_7.
\end{aligned}$$

Here are some useful facts :

Lemma 1. ([AS1]) *To any 3-dimensional submanifold $Y^3 \subset (M, \varphi)$, χ assigns a normal vector field, which vanishes when Y is associative.*

Lemma 2. ([AS1]) *To any associative manifold $Y^3 \subset (M, \varphi)$ with a non-vanishing oriented 2-plane field, χ defines a complex structure on its normal bundle (notice in particular that any coassociative submanifold $X \subset M$ has an almost complex structure if its normal bundle has a non-vanishing section).*

Proof. Let $L \subset \mathbb{R}^7$ be an associative 3-plane, that is $\varphi_0|_L = \text{vol}(L)$. Then for every pair of orthonormal vectors $\{u, v\} \subset L$, the form χ defines a complex structure on the orthogonal 4-plane L^\perp , as follows: Define $j : L^\perp \rightarrow L^\perp$ by

$$(5) \quad j(X) = \chi(u, v, X)$$

This is well defined i.e. $j(X) \in L^\perp$, because when $w \in L$ we have:

$$\langle \chi(u, v, X), w \rangle = * \varphi_0(u, v, X, w) = - * \varphi_0(u, v, w, X) = \langle \chi(u, v, w), X \rangle = 0$$

Also $j^2(X) = j(\chi(u, v, X)) = \chi(u, v, \chi(u, v, X)) = -X$. We can check the last equality by taking an orthonormal basis $\{X_j\} \subset L^\perp$ and calculating

$$\begin{aligned}
\langle \chi(u, v, \chi(u, v, X_i)), X_j \rangle &= * \varphi_0(u, v, \chi(u, v, X_i), X_j) = - * \varphi_0(u, v, X_j, \chi(u, v, X_i)) \\
&= - \langle \chi(u, v, X_j), \chi(u, v, X_i) \rangle = -\delta_{ij}
\end{aligned}$$

The last equality holds since the map j is orthogonal, and the orthogonality can be seen by polarizing the associator equality, and by noticing $\varphi_0(u, v, X_i) = 0$. Observe that the map j only depends on the oriented 2-plane $\Lambda = \langle u, v \rangle$ generated by $\{u, v\}$ (i.e. it only depends on the complex structure on Λ). \square

3. CALABI-YAU'S HYPERSURFACES IN G_2 MANIFOLDS

In [AS2] authors proposed a notion of *mirror duality* for Calabi-Yau submanifold pairs lying inside of a G_2 manifold (M, φ) . This is done first by assigning almost Calabi-Yau structures to hypersurfaces induced by hyperplane distributions. The construction goes as follows. Suppose ξ be a nonvanishing vector field $\xi \in \Omega^0(M, TM)$, which gives a codimension one integrable distribution $V_\xi := \xi^\perp$ on M . If X_ξ is a leaf of this distribution, then the forms χ and ψ induce a non-degenerate 2-form ω_ξ and an almost complex structure J_ξ on X_ξ as follows:

$$(6) \quad \omega_\xi = \langle \psi, \xi \rangle \quad \text{and} \quad J_\xi(u) = u \times \xi.$$

$$(7) \quad \text{Re } \Omega_\xi = \varphi|_{V_\xi} \quad \text{and} \quad \text{Im } \Omega_\xi = \langle \chi, \xi \rangle.$$

where the inner products, of the vector valued differential forms ψ and χ with vector field ξ , are performed by using their vector part. So $\omega_\xi = \xi \lrcorner \varphi$, and $\text{Im } \Omega_\xi = \xi \lrcorner * \varphi$. Call $\Omega_\xi = \text{Re } \Omega_\xi + i \text{Im } \Omega_\xi$. These induce almost Calabi-Yau structure on X_ξ , analogous to Example 1.

Theorem 3. ([AS2]) *Let (M, φ) be a G_2 manifold, and ξ be a unit vector field such that ξ^\perp comes from a codimension one foliation on M , then $(X_\xi, \omega_\xi, \Omega_\xi, J_\xi)$ is an almost Calabi-Yau manifold such that $\varphi|_{X_\xi} = \text{Re } \Omega_\xi$ and $*\varphi|_{X_\xi} = *_3 \omega_\xi$. Furthermore, if $\mathcal{L}_\xi(\varphi)|_{X_\xi} = 0$ then $d\omega_\xi = 0$, and if $\mathcal{L}_\xi(*\varphi)|_{X_\xi} = 0$ then J_ξ is integrable; when both conditions are satisfied $(X_\xi, \omega_\xi, \Omega_\xi, J_\xi)$ is a Calabi-Yau manifold.*

Here is a brief discussion of [AS2] with explanation of its terms: Let $\xi^\#$ be the dual 1-form of ξ , and $e_{\xi^\#}$ and $i_\xi = \xi \lrcorner$ denote the exterior and interior product operations on differential forms, then

$$\varphi = e_{\xi^\#} \circ i_\xi(\varphi) + i_\xi \circ e_{\xi^\#}(\varphi) = \omega_\xi \wedge \xi^\# + \text{Re } \Omega_\xi.$$

This is the decomposition of the form φ with respect to $\xi \oplus \xi^\perp$. The condition that the distribution V_ξ to be integrable is $d\xi^\# \wedge \xi^\# = 0$. Also it is clear from definitions that J_ξ is an almost complex structure on X_ξ , and the 2-form ω_ξ is non-degenerate on X_ξ , because

$$\omega_\xi^3 = (\xi \lrcorner \varphi)^3 = \xi \lrcorner [(\xi \lrcorner \varphi) \wedge (\xi \lrcorner \varphi) \wedge \varphi] = \xi \lrcorner (6|\xi|^2 \mu) = 6\mu_\xi$$

where $\mu_\xi = \mu|_{V_\xi}$ is the induced orientation form on V_ξ . For $u, v \in V_\xi$.

$$\begin{aligned}\omega_\xi(J_\xi(u), v) &= \omega_\xi(u \times \xi, v) = \langle \psi(u \times \xi, v), \xi \rangle = \varphi(u \times \xi, v, \xi) \\ &= -\varphi(\xi, \xi \times u, v) = -\langle \xi \times (\xi \times u), v \rangle \\ &= -\langle -|\xi|^2 u + \langle \xi, u \rangle \xi, v \rangle = |\xi|^2 \langle u, v \rangle - \langle \xi, u \rangle \langle \xi, v \rangle \\ &= \langle u, v \rangle.\end{aligned}$$

implies $\langle J_\xi(u), J_\xi(v) \rangle = -\omega_\xi(u, J_\xi(v)) = \langle u, v \rangle$. By a calculation of J_ξ , one checks that the 3-form Ω_ξ is a (3,0) form, furthermore it is non-vanishing because

$$\begin{aligned}\frac{1}{2i} \Omega_\xi \wedge \bar{\Omega}_\xi = \text{Im } \Omega_\xi \wedge \text{Re } \Omega_\xi &= (\xi \lrcorner * \varphi) \wedge [\xi \lrcorner (\xi^\# \wedge \varphi)] \\ &= -\xi \lrcorner [(\xi \lrcorner * \varphi) \wedge (\xi^\# \wedge \varphi)] \\ &= \xi \lrcorner [* (\xi^\# \wedge \varphi) \wedge (\xi^\# \wedge \varphi)] \\ &= |\xi^\# \wedge \varphi|^2 \xi \lrcorner \text{vol}(M) \\ &= 4|\xi^\#|^2 (*\xi^\#) = 4 \text{vol}(X_\xi).\end{aligned}$$

It is easy to see $*\text{Re } \Omega_\xi = -\text{Im } \Omega_\xi \wedge \xi^\#$ and $*\text{Im } \Omega_\xi = \text{Re } \Omega_\xi \wedge \xi^\#$.

$$*_3 \text{Re } \Omega_\xi = \text{Im } \Omega_\xi.$$

Notice that ω_ξ is a symplectic structure on X_ξ when $d\varphi = 0$ and $\mathcal{L}_\xi(\varphi)|_{V_\xi} = 0$, (\mathcal{L}_ξ is the Lie derivative along ξ), since $\omega_\xi = \xi \lrcorner \varphi$ and:

$$d\omega_\xi = \mathcal{L}_\xi(\varphi) - \xi \lrcorner d\varphi = \mathcal{L}_\xi(\varphi)$$

J_ξ is integrable complex structure if $d^*\varphi = 0$ and $\mathcal{L}_\xi(*\varphi)|_{V_\xi} = 0$ since

$$d(\text{Im } \Omega_\xi) = d(\xi \lrcorner * \varphi) = \mathcal{L}_\xi(*\varphi) - \xi \lrcorner d(*\varphi) = 0$$

Also notice that $d\varphi = 0 \implies d(\text{Re } \Omega_\xi) = d(\varphi|_{X_\xi}) = 0$.

4. HL MANIFOLDS AND MIRROR DUALITY IN G_2 MANIFOLDS

By [T] any 7-dimensional Riemannian manifold admits a non-vanishing orthonormal 2-frame field $\Lambda = \langle u, v \rangle$, in particular (M, φ) admits such a field. Λ gives a section of the bundle of oriented 2-frames $V_2(M) \rightarrow M$, and hence gives an associative/coassociative splitting of the tangent bundle $TM = \mathbf{E} \oplus \mathbf{V}$, where $\mathbf{E} = \mathbf{E}_\Lambda = \langle u, v, u \times v \rangle$ and $\mathbf{V} = \mathbf{V}_\Lambda = \mathbf{E}^\perp$. When there is no danger of confusion we will denote the 2-frame fields and the 2-planes fields which they induce by the same symbol Λ . Also, any unit section ξ of $\mathbf{E} \rightarrow M$ induces a complex structure J_ξ on the bundle $\mathbf{V} \rightarrow M$ by the cross product $J_\xi(u) = u \times \xi$.

In [AS2] any two almost Calabi-Yau's X_ξ and $X_{\xi'}$ inside (M, φ) were called *dual* if the defining vector fields ξ and ξ' are chosen from \mathbf{V} and \mathbf{E} , respectively. Here we

make this correspondence more precise, in particular showing how to choose ξ and ξ' in a more canonical way.

Definition 3. A 3-dimensional submanifold $Y^3 \subset (M, \varphi)$ is called *Harvey-Lawson manifold (HL in short)* if $\varphi|_Y = 0$.

Definition 4. ([AS1]) Call any orthonormal 3-frame field $\Gamma = \langle u, v, w \rangle$ on (M, φ) , a G_2 -frame field if $\varphi(u, v, w) = \langle u \times v, w \rangle = 0$, equivalently w is a unit section of $\mathbf{V}_\Lambda \rightarrow X$, with $\Lambda = \langle u, v \rangle$ (see (1)).

Now pick a nonvanishing 2-frame field $\Lambda = \langle u, v \rangle$ on M and let $TM = \mathbf{E} \oplus \mathbf{V}$ be the induced splitting with $\mathbf{E} = \langle u, v, u \times v \rangle$. Let w be a unit section of the bundle $\mathbf{V} \rightarrow M$. Such a section w may not exist on whole M , but by obstruction theory it exists on a tubular neighborhood U of the 3-skeleton $M^{(3)}$ of M (which is a complement of some 3-complex $Z \subset M$). So $\varphi(u, v, w) = 0$, and hence $\Gamma = \langle u, v, w \rangle$ is a G_2 frame field. Next consider the non-vanishing vector fields:

- $R = \chi(u, v, w) = -u \times (v \times w)$
- $R' = \frac{1}{\sqrt{3}}(u \times v + v \times w + w \times u)$
- $R'' = \frac{1}{\sqrt{3}}(u + v + w)$

If the 6-plane fields R^\perp , R'^\perp , and R''^\perp , are integrable we get almost Calabi-Yau manifolds $(X_R, w_R, \Omega_R, J_R)$, $(X_{R'}, w_{R'}, \Omega_{R'}, J_{R'})$, and $(X_{R''}, w_{R''}, \Omega_{R''}, J_{R''})$. Let us use the convention that a, b, c are real numbers, and $[u_1, \dots, u_n]$ is the distribution generated by the vectors u_1, \dots, u_n .

Lemma 4. *By definitions, the following hold*

- (a) $Y := [u, v, w] = [au + bv + cw \mid a + b + c = 0] \oplus [R'']$
- (b) $\mathbb{V} = [u, v, w, R]$, is a coassociative 4-plane field.
- (c) $\mathbb{E} := [u \times v, v \times w, w \times u]$ is an associative 3-plane field.
- (d) $\mathbb{E} \perp \mathbb{V}$

Theorem 5. *For $(a, b, c) \in \mathbb{R}^3$ with $a + b + c = 0$, then*

- (a) $TX_R = [au + bv + cw, R'', R', a(v \times w) + b(w \times u) + c(u \times v)]$
 $J_R(au + bv + cw) = -a(v \times w) - b(w \times u) - c(u \times v)$
 $J_R(R'') = -R'$
- (b) $TX_{R'} = [au + bv + cw, R'', R, a(v \times w) + b(w \times u) + c(u \times v)]$
 $J_{R'}(au + bv + cw) = -((b - c)u + (c - a)v + (a - b)w)/\sqrt{3}$
 $J_{R'}(a(v \times w) + b(w \times u) + c(u \times v)) =$
 $((b - c)(v \times w) + (c - a)(w \times u) + (a - b)(u \times v))/\sqrt{3}$
 $J_{R'}(R'') = R$
- (c) $TX_{R''} = [au + bv + cw, R, R', a(v \times w) + b(w \times u) + c(u \times v)]$

$$\begin{aligned}
J_{R''}(au + bv + cw) &= \\
&((b-a)(u \times v) + (c-b)(v \times w) + (a-c)(w \times u))/\sqrt{3} \\
J_{R''}(R) &= R'
\end{aligned}$$

(d) $\{u, v, w, R, u \times v, v \times w, w \times u\}$ is an orthonormal frame field.

Proof. To show (a) by using (4) we calculate:

$$\begin{aligned}
R \times u &= \chi(u, v, w) \times u = -[u \times (v \times w)] \times u = u \times [u \times (v \times w)] \\
&= -\chi(u, u, v \times w) - \langle u, u \rangle (v \times w) + \langle u, v \times w \rangle u \\
&= -(v \times w) + \varphi(u, v, w)u
\end{aligned}$$

$$(8) \quad \text{Therefore } R \times u = -(v \times w)$$

Similarly, $R \times v = -(w \times u)$ and $R \times w = -(u \times v)$. Therefore we have $J_R(au + bv + cw) = -a(v \times w) - b(w \times u) - c(u \times v)$, and $J_R(R'') = -R'$.

$$\begin{aligned}
\sqrt{3} R' \times u &= (u \times v + v \times w + w \times u) \times u \\
&= -u \times (u \times v) - u \times (v \times w) - u \times (w \times u) \\
&= \langle u, u \rangle v - \langle u, v \rangle u \\
&\quad + \chi(u, v, w) + \langle u, v \rangle w - \langle u, w \rangle v \\
&\quad + \langle u, w \rangle u - \langle u, u \rangle w
\end{aligned}$$

$$(9) \quad \text{Therefore } \sqrt{3} R' \times u = R + (v - w)$$

Similarly $\sqrt{3} R' \times v = R + (w - u)$, and $\sqrt{3} R' \times w = R + (u - v)$, which implies the first part of (b), and $J_{R'}(R'') = R$.

For the second part of (b) we need to compute the following:

$$\begin{aligned}
(10) \quad \sqrt{3} R' \times [a(v \times w) + b(w \times u) + c(u \times v)] &= \\
(u \times v + v \times w + w \times u) \times [a(v \times w) + b(w \times u) + c(u \times v)] &
\end{aligned}$$

For this first by repeatedly using (4) and $\varphi(u, v, w) = 0$ we calculate:

$$\begin{aligned}
(v \times u) \times (w \times v) &= -\chi(v \times u, w, v) - \langle v \times u, w \rangle v + \langle v \times u, v \rangle w \\
&= -\chi(v \times u, w, v) = -\chi(w, v, v \times u) \\
&= w \times (v \times (v \times u)) + \langle w, v \rangle (v \times u) - \langle w, v \times u \rangle v \\
&= w \times (v \times (v \times u)) \\
&= w \times (-\chi(v, v, u) - \langle v, v \rangle u + \langle v, u \rangle v) = -(w \times u)
\end{aligned}$$

Then by plugging in (9) gives (b). Proof of (c) is similar to (a) \square

In particular from the above calculations we get can express φ as:

Corollary 6.

$$\begin{aligned} \varphi &= u^\# \wedge v^\# \wedge (u^\# \times v^\#) + v^\# \wedge w^\# \wedge (v^\# \times w^\#) + w^\# \wedge u^\# \wedge (w^\# \times u^\#) \\ &+ u^\# \wedge R^\# \wedge (v^\# \times w^\#) + (v^\# \wedge R^\#) \wedge (w^\# \times u^\#) + w^\# \wedge R^\# \wedge (u^\# \times v^\#) \\ &\quad - (u^\# \times v^\#) \wedge (v^\# \times w^\#) \wedge (w^\# \times u^\#) \end{aligned}$$

Recall that in an earlier paper we proved the following facts:

Proposition 7. [AS2] *Let $\{\alpha, \beta\}$ be orthonormal vector fields on (M, φ) . Then on X_α the following hold*

- (i) $Re \Omega_\alpha = \omega_\beta \wedge \beta^\# + Re \Omega_\beta$
- (ii) $Im \Omega_\alpha = \alpha \lrcorner (\star \omega_\beta) - (\alpha \lrcorner Im \Omega_\beta) \wedge \beta^\#$
- (iii) $\omega_\alpha = \alpha \lrcorner Re \Omega_\beta + (\alpha \lrcorner \omega_\beta) \wedge \beta^\#$

Proof. Since $Re \Omega_\alpha = \varphi|_{X_\alpha}$ (i) follows. Since $Im \Omega_\alpha = \alpha \lrcorner \star \varphi$ following gives (ii)

$$\begin{aligned} \alpha \lrcorner (\star \omega_\beta) &= \alpha \lrcorner [\beta \lrcorner \star (\beta \lrcorner \varphi)] \\ &= \alpha \lrcorner \beta \lrcorner (\beta^\# \wedge \star \varphi) \\ &= \alpha \lrcorner \star \varphi + \beta^\# \wedge (\alpha \lrcorner \beta \lrcorner \star \varphi) \\ &= \alpha \lrcorner \star \varphi + (\alpha \lrcorner Im \Omega_\beta) \wedge \beta^\# \end{aligned}$$

(iii) follows from the following computation

$$\alpha \lrcorner Re \Omega_\beta = \alpha \lrcorner \beta \lrcorner (\beta^\# \wedge \varphi) = \alpha \lrcorner \varphi + \beta^\# \wedge (\alpha \lrcorner \beta \lrcorner \varphi) = \alpha \lrcorner \varphi - (\alpha \lrcorner \omega_\beta) \wedge \beta^\#$$

□

Note that even though the identities of this proposition hold only after restricting the right hand side to X_α , all the individual terms are defined everywhere on (M, φ) . Also, from the construction, X_α and X_β inherit vector fields β and α , respectively.

Corollary 8. [AS2] *Let $\{\alpha, \beta\}$ be orthonormal vector fields on (M, φ) . Then there are $A_{\alpha\beta} \in \Omega^3(M)$, and $W_{\alpha\beta} \in \Omega^2(M)$ satisfying*

- (a) $\varphi|_{X_\alpha} = Re \Omega_\alpha$ and $\varphi|_{X_\beta} = Re \Omega_\beta$
- (b) $A_{\alpha\beta}|_{X_\alpha} = Im \Omega_\alpha$ and $A_{\alpha\beta}|_{X_\beta} = \alpha \lrcorner (\star \omega_\beta)$
- (c) $W_{\alpha\beta}|_{X_\alpha} = \omega_\alpha$ and $W_{\alpha\beta}|_{X_\beta} = \alpha \lrcorner Re \Omega_\beta$

Now we can choose α as R and β as R' of the given HL manifold. That concludes that given a HL submanifold of a G_2 manifold, it will determine a “canonical” mirror pair of Calabi-Yau manifolds (related thru the HL manifold) with the complex and symplectic structures given above.

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