

ISOTOPING 2-SPHERES IN 4-MANIFOLDS

SELMAN AKBULUT

ABSTRACT. Here we discuss an example of topologically isotopic but smoothly non-isotopic pair of 2-spheres in a simply connected 4-manifold, which become smoothly isotopic after stabilizing by connected summing with $S^2 \times S^2$.

0. THE EXAMPLE

in [AKMR] examples of topologically isotopic but smoothly non-isotopic spheres in simply connected 4-manifolds, which become smoothly isotopic after connected summing with $S^2 \times S^2$, were discussed. In this note we show that such an example already follows from [A2].

First review [A2]: Let $f : \partial W \rightarrow \partial W$ be the cork twisting involution of the Mazur cork (W, f) . Since f fixes the boundary ∂D of a properly imbedded disk $D \subset W$ up to isotopy, $f(\partial D)$ bounds a disk in W as well (isotopy in the collar union D), hence we can extend f across the tubular neighborhood of $N(D)$ of D by the carving process of [A1]. This results a manifold $Q = W - N(D)$ of Figure 1, homotopy equivalent to $B^3 \times S^1$, and an involution on its boundary $\tau : \partial Q \rightarrow \partial Q$.

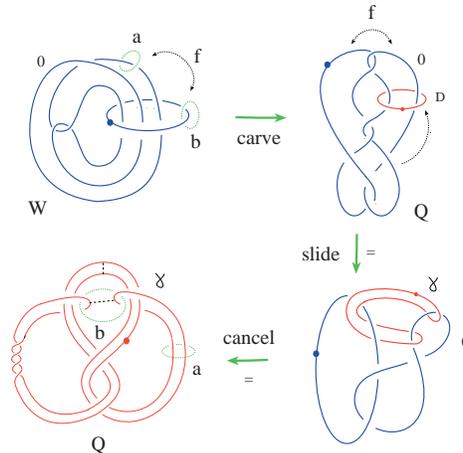


FIGURE 1

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τ does not extend to Q as a diffeomorphism (otherwise f would extend to a self diffeomorphism of W). So τ gives an exotic structure to Q relative to its boundary (just as in the cork case). In [A4] such (Q, τ) 's are called anticorks because they live inside of corks (W, f) , and twisting Q by the involution τ undoes the effect of twisting W by f . Notice that the loop $\gamma = \partial D$ bounds two different disks in B^4 with the same complement Q (where the identity map between their boundaries can not extend to a diffeomorphism inside), they are described by the two different ribbon moves indicated in the last picture of Figure 1. The two disks are the obvious disks which γ bounds in the third picture of Figure 1, and the same disk after zero and dot exchanges of the figure.

Now let M be the 4-manifold obtained by attaching a 2-handle to B^4 along the ribbon knot γ of Figure 1, with $+1$ framing. Clearly M has two imbedded 2-spheres $S_i, i = 1, 2$ of self intersection $+1$ representing $H_2(M) \cong \mathbb{Z}$, corresponding to the two different 2-disks which γ bounds in B^4 . Blowing down either S_1 or S_2 turns M into the *positron* cork \bar{W}_1 of Figure 2 ([AM]), and the two different blowings down process turns the identity map $\partial M \rightarrow \partial M$ to the cork involution $f : \partial \bar{W}_1 \rightarrow \partial \bar{W}_1$, i.e. the maps in Figure 2 commute (this can be seen by blowing down γ of Figure 1 by using the two different disks). Hence S_1 and S_2 are not smoothly isotopic in M by any isotopy keeping ∂M fixed, though they are topologically isotopic (by Freedman's theorem); but they are isotopic In $M \# S^2 \times S^2$ rel boundary (since surgery corresponds turning the dotted circle to 0-framed circle, in the third picture of Figure 1).

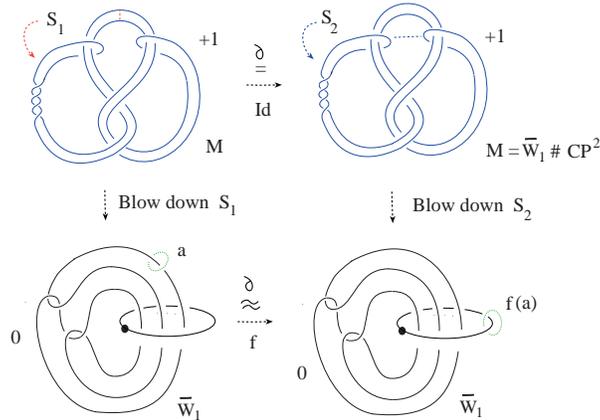


FIGURE 2

Since $M = \bar{W}_1 \# \mathbb{C}\mathbb{P}^2$, this example shows that the blowing up $\mathbb{C}\mathbb{P}^2$ operation undoes the cork twisting operation of (\bar{W}_1, f) . Also since the Dolgachev surface $E(1)_{2,3}$ differs from its standard copy $\mathbb{C}\mathbb{P}^2 \# 9\bar{\mathbb{C}}\mathbb{P}^2$ by twisting the positron cork (\bar{W}_1, f) inside (Theorem 1 of [A3]), the manifold M in this example can be made to be closed (without boundary).

Remark 1. *Reader can check that the two ribbon disks of the ribbon knot in Figure 1 are actually the same ribbon disks (isotoping the last picture of Figure 1, by forcing the two stands going through b stay parallel, results the same picture except the positions of a and b are exchanged) but f induces nontrivial identifications on the boundaries of the ribbon complements.*

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DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, MI, 48824
E-mail address: akbulut@math.msu.edu