KNOTTING CORKS

SELMAN AKBULUT AND KOUICHI YASUI

ABSTRACT. It is known that every exotic smooth structure on a simply connected closed 4-manifold is determined by a codimention zero compact contractible Stein submanifold and an involution on its boundary. Such a pair is called a cork. In this paper, we construct infinitely many knotted imbeddings of corks in 4-manifolds such that they induce infinitely many different exotic smooth structures. We also show that we can imbed an arbitrary finite number of corks disjointly into 4-manifolds, so that the corresponding involutions on the boundary of the contractible 4-manifolds give mutually different exotic structures. Furthermore, we construct similar examples for plugs.

1. INTRODUCTION

In [1] the first author proved that $E(2)\#\overline{\mathbf{CP}^2}$ changes its diffeomorphism type if we remove an imbedded copy of a Mazur manifold inside and reglue it by a natural involution on its bounday. This was later generalized to $E(n)\#\overline{\mathbf{CP}^2}$ $(n \geq 2)$ by Bižaca-Gompf [8]. Here $E(n)$ denotes the relatively minimal elliptic surface with no multiple fibers and with Euler characteristic $12n$. Recently, the authors [6] and the first author [4] constructed many such examples for other 4-manifolds. The following general theorem was first proved independently by Matveyev [16], Curtis-Freedman-Hsiang-Stong [9], and later on strengthened by the first author and Matveyev [5]:

Theorem 1.1 ([16], [9], [5]). *Let X be a simply connected closed smooth 4-manifold. If a smooth 4-manifold Y is homeomorphic but not diffeomorphic to X, then there exist a codimention zero contractible submanifold C of X and an involution τ on the boundary ∂C such that Y is obtained from X by removing the submanifold C and regluing it via the involution τ. Such a pair (C, τ) is called a Cork of X. Furthermore, corks and their complements can always be made Stein manifolds.*

Hence smooth structures on simply connected closed 4-manifolds are determined by corks. In this paper, subsequent to [6], we explore the behavior of corks. It is a natural question whether every smooth structure on a 4-manifold can be induced from a fixed cork (C, τ). In [6], we showed that two different exotic smooth structures on a 4-manifold can be obtained from the same cork (imbedded differently). Here, we prove that infinitely many different smooth structures on 4-manifolds can be obtained from a fixed cork:

Date: August 10, 2009.

2000 *Mathematics Subject Classification.* Primary 57R55, Secondary 57R65.

Key words and phrases. 4-manifold; cork; plug; knot surgery; rational blowdown.

The first author is partially supported by NSF, and the second author was partially supported by JSPS Research Fellowships for Young Scientists, and by GCOE, Kyoto University.

Theorem 1.2. *There exist a compact contractible Stein* 4*-manifold* C*, an involution* τ *on the boundary* ∂C*, and infinitely many simply connected closed smooth* 4*-manifolds* X_n $(n \geq 0)$ *with the following properties:*

(1) *The* 4*-manifolds* X_n $(n \geq 0)$ *are mutually homeomorphic but not diffeomorphic;*

(2) *For each* $n \geq 1$*, the* 4*-manifold* X_n *is obtained from* X_0 *by removing a copy of* C *and regluing it via* τ*. Consequently, the pair* (C, τ) *is a cork of* X_0*.*

In particular, from X_0 *we can produce infinitely many different smooth structures by the process of removing a copy of* C *and regluing it via* τ*. Consequently, these embeddings of* C *into* X_0 *are mutually non-isotopic* (*knotted copies of each other*).

It is interesting to discuss positions of corks in 4-manifolds. The next theorem says that we can put an arbitrary finite number of corks into mutually disjoint positions in 4-manifolds:

Theorem 1.3. *For each* $n \geq 1$*, there exist simply connected closed smooth* 4*-manifolds* Y_i $(0 \leq i \leq n)$*, codimension zero compact contractible Stein submanifolds* C_i $(1 \leq i \leq n)$ *of* Y_0*, and an involution* τ_i *on the each boundary* ∂C_i $(1 \leq i \leq n)$ *with the following properties:*

(1) *The submanifolds* C_i $(1 \leq i \leq n)$ *of* Y_0 *are mutually disjoint;*

(2) Y_i $(1 \leq i \leq n)$ *is obtained from* Y_0 *by removing the submanifold* C_i *and regluing it via* τ_i*;*

(3) *The* 4*-manifolds* Y_i $(0 \leq i \leq n)$ *are mutually homeomorphic but not diffeomorphic. In particular, the pairs* (C_i, τ_i) $(1 \leq i \leq n)$ *are corks of* Y_0*.*

The following theorem says that, for an embedding of a cork into a 4-manifold, we can produce finitely many different cork structures of the 4-manifold by only changing the involution of the cork without changing its embedding:

Theorem 1.4. *For each* $n \geq 1$*, there exist simply connected closed smooth* 4*-manifolds* Y_i $(0 \leq i \leq n)$*, an embedding of a compact contractible Stein* 4*-manifold* C *into* Y_0*, and involutions* τ_i $(1 \leq i \leq n)$ *on the boundary* ∂C *with the following properties:*

(1) *For each* $1 \leq i \leq n$*, the* 4*-manifold* Y_i *is obtained from* Y_0 *by removing the submanifold* C *and regluing it via* τ_i*;*

(2) *The* 4*-manifolds* Y_i $(0 \leq i \leq n)$ *are mutually homeomorphic but not diffeomorphic, hence the pairs* (C, τ_i) $(1 \leq i \leq n)$ *are mutually different corks of* Y_0*.*

In [6], we introduced new objects which we call *Plugs*. We also construct similar examples for plugs (See Section 7).

Acknowledgement . The second author would like to thank his adviser Hisaaki Endo for encouragement.

2. Corks

In this section, we recall corks. For details, see [6].

Definition 2.1. Let C be a compact contractible Stein 4-manifold with boundary and $\tau : \partial C \to \partial C$ an involution on the boundary. We call (C, τ) a *Cork* if τ extends to a self-homeomorphism of C, but cannot extend to any self-diffeomorphism of C. A cork (C, τ) is called a cork of a smooth 4-manifold X, if $C \subset X$ and X changes its diffeomorphism type when we remove C and reglue it via τ. Note that this operation does not change the homeomorphism type of X.

Remark 2.2. In this paper, we always assume that corks are contractible. (We did not assume this in the more general definition of [6].) Note that Freedman's theorem tells us that every self-diffeomorphism of the boundary of C extends to a self-homeomorphism of C when C is a compact contractible smooth 4-manifold.

Definition 2.3. Let W_n be the contractible smooth 4-manifold shown in Figure 1. Let $f_n : \partial W_n \to \partial W_n$ be the obvious involution obtained by first surgering $S^1 \times B^3$ to $B^2 \times S^2$ in the interior of W_n, then surgering the other imbedded $B^2 \times S^2$ back to $S^1 \times B^3$ (i.e. replacing the dot and "0" in Figure 1). Note that the diagram of W_n comes from a symmetric link.

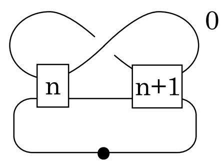

FIGURE 1. W_n

Theorem 2.4 ([6])**.** *For $n \geq 1$, the pair (W_n, f_n) is a cork.*

3. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2 by using Fintushel-Stern's knot surgery and arguments similar to the proofs of [6, Theorem 3.4 and 3.5].

First recall the following useful theorem from Gompf-Stipsicz [13, Section 9.3].

Theorem 3.1 (Gompf-Stipsicz [13])**.** *For $n \geq 1$, the elliptic surface $E(n)$ has the handle decomposition in Figure 2. The obvious cusp neighborhood (i.e. the dotted circle, one of the -1 framed meridians of the dotted circle, and the left most 0-framed unknot) is isotopic to the regular neighborhood of a cusp fiber of $E(n)$.*

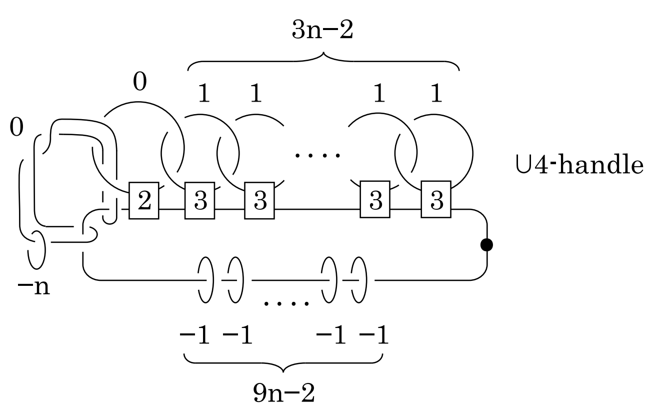

FIGURE 2. $E(n)$

Corollary 3.2. *For $n \geq 1$, the elliptic surface $E(n)$ has a handle decomposition as in Figure 3. The obvious cusp neighborhood (i.e. 0-framed trefoil knot) is isotopic to the regular neighborhood of a cusp fiber of $E(n)$.*

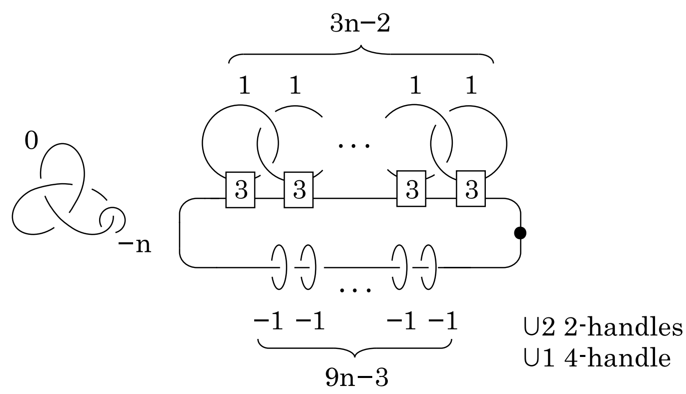

FIGURE 3. $E(n)$

Proof. In Figure 2, pull off the leftmost 0-framed unknot from the dotted circle by sliding over −1-framed knot. □

Corollary 3.3. *For each $p_1, p_2, \ldots, p_n \geq 2$, the elliptic surface $E(p_1 + p_2 + \cdots + p_n)$ has a handle decomposition as in Figure 4. The obvious cusp neighborhood is isotopic to the regular neighborhood of a cusp fiber of $E(p_1 + p_2 + \cdots + p_n)$. Here $k = 9(p_1 + p_2 + \cdots + p_n) - 5n - 4$.*

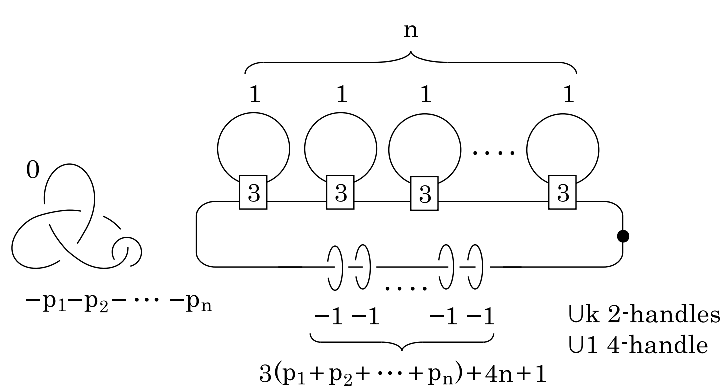

FIGURE 4. $E(p_1 + p_2 + \cdots + p_n)$ $(p_1, p_2, \ldots, p_n \geq 2)$

Remark 3.4. In this section, we do not use Corollary 3.3, it will be used in Section 5.

We can now easily get the following proposition.

Proposition 3.5. *The 4-manifold $E(n)\#\overline{\mathbf{CP}^2}$ $(n \geq 2)$ has a handle decomposition as in Figure 5. The obvious cusp neighborhood in the figure is isotopic to the regular neighborhood of a cusp fiber of $E(n)$. Note that the figure contains W_1.*

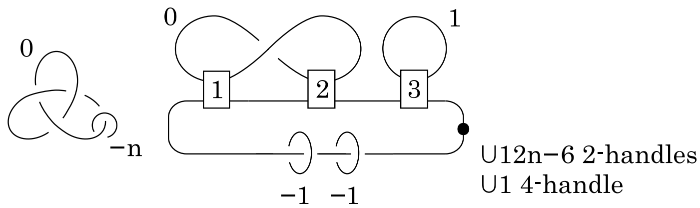

FIGURE 5. $E(n)\#\overline{\mathbf{CP}^2}$ $(n \geq 2)$

Proof. It follows from Corollary 3.2 that the submanifold of $E(n)$ in the first diagram of Figure 13 is disjoint from the cusp neighborhood of $E(n)$. Slide handles as in Figure 13. Then we clearly get Figure 5 by blowing up. □

Definition 3.6. (1) Let X be the smooth 4-manifold obtained from $E(n)\#\overline{\mathbf{CP}^2}$ $(n \geq 2)$ by removing the copy of W_1 in Figure 5 and regluing it via f_1. Note that X contains a cusp neighborhood because the copy of W_1 in $E(n)\#\overline{\mathbf{CP}^2}$ is disjoint from the cusp neighborhood in Figure 5.
(2) Let K be a knot in S^3 and X_K the Fintushel-Stern's knot surgery ([11]) with K in the cusp neighborhood of X.
(3) Let $E(n)_K$ be the knot surgery with K in the cusp neighborhood of $E(n)$.

The following corollary clearly follows from Proposition 3.5 and the definitions above.

Corollary 3.7. (1) *The* 4*-manifold* X *splits off* $\mathbf{CP}^2\#2\overline{\mathbf{CP}^2}$ *as a connected summand. Furthermore, the cusp neighborhood of* X *is disjoint from* $\mathbf{CP}^2\#2\overline{\mathbf{CP}^2}$ *in this connected sum decomposition.*
(2) *The* 4*-manifold* X_K *contains a copy of* W_1 *such that* $E(n)_K\#\overline{\mathbf{CP}^2}$ *is obtained from* X_K *by removing the copy of* W_1 *and regluing it via* f_1*.*

Corollary 3.8. (1) *The* 4*-manifold* X *splits off* $S^2 \times S^2$ *as a connected summand. Furthermore, the cusp neighborhood of* X *is disjoint from* $S^2 \times S^2$ *in this connected sum decomposition. Consequently, the Seiberg-Witten invariant of* X *vanishes.*
(2) *The* 4*-manifold* X_K *is diffeomorphic to* X*. In particular, the Seiberg-Witten invariant of* X_K *vanishes.*
(3) *For each knot* K *in* S^3*, there exists a copy of* W_1 *in* X *such that* $E(n)_K\#\overline{\mathbf{CP}^2}$ *is obtained from* X *by removing the copy of* W_1 *and regluing it via* f_1*.*
(4) *If a knot* K *in* S^3 *has the non-trivial Alexander polynomial, then* $E(n)_K\#\overline{\mathbf{CP}^2}$ *is homeomorphic but not diffeomorphic to* X*, in particular* (W_1, f_1) *is a cork of* X*.*

Proof. The claim (1) easily follows from Corollary 3.7.(1) and the fact that, for every non-spin 4-manifold Y, the 4-manifold $Y\#\mathbf{CP}^2\#\overline{\mathbf{CP}^2}$ is diffeomorphic to $Y\#(S^2 \times S^2)$. The claim (1) and the definition of X_K together with the stabilization theorem of knot surgery by the first author [3] and Auckly [7] show the claim (2). The claim (3) thus follows from Corollary 3.7.(2). Since the Seiberg-Witten invariant of $E(n)_K\#\overline{\mathbf{CP}^2}$ does not vanish (Fintushel-Stern [11]), the claim (4) follows from the claim (1). □

Now we can easily prove Theorem 1.2.

Proof of Theorem 1.2. Let $X_0 := X$, and K_i ($i \geq 1$) be knots in S^3 with mutually different non-trivial Alexander polynomials. Define $X_i = E(n)_{K_i} \# \overline{\mathbf{CP}^2}$. Then the claim easily follows from Cororally 3.8 and the Fintushel-Stern's formula ([11]) of the Seiberg-Witten invariant of $E(n)_{K_i}$. □

Remark 3.9. (1) Freedman's theorem shows that X_0 is homeomorphic to $(2n-1)\mathbf{CP}^2 \# 10n\overline{\mathbf{CP}^2}$. Since X_0 splits off $\mathbf{CP}^2 \# \overline{\mathbf{CP}^2}$, it is likely that X_0 is diffeomorphic to $(2n-1)\mathbf{CP}^2 \# 10n\overline{\mathbf{CP}^2}$.
(2) The complement of the each copy of W_1 in $X_0 (\cong X_K)$ given in Corollary 3.8.(3) is simply connected. This claim easily follows from Figure 5 and the definition of X_K. Note that the knot surgered Gompf nuclei N_{nK} is simply connected.
(3) We proved Theorem 1.2 for the cork (W_1, f_1). We can similarly prove Theorem 1.2 for many other corks, including (W_n, f_n) and $(\overline{W}_n, \overline{f}_n)$. (For the definition of $(\overline{W}_n, \overline{f}_n)$, see [6].) See also the proofs of [6, Proposition 3.3.(1) and (3)].

4. Rational blowdown

In this section we review the rational blowdown introduced by Fintushel-Stern [10]. We also introduce a new relation between rational blowdowns and corks.

Let C_p and B_p be the smooth 4-manifolds defined by handlebody diagrams in Figure 6, and $u_1, \dots, u_{p-1}$ elements of $H_2(C_p; \mathbf{Z})$ given by corresponding 2-handles in the figure such that $u_i \cdot u_{i+1} = +1$ ($1 \leq i \leq p-2$). The boundary ∂C_p of C_p is diffeomorphic to the lens space $L(p^2, p-1)$, and also diffeomorphic to the boundary ∂B_p of B_p.

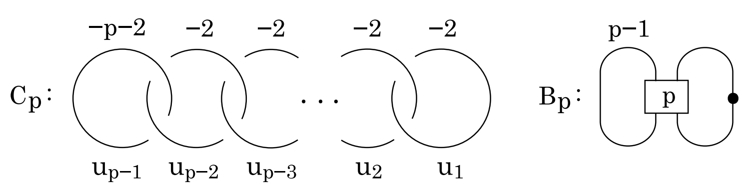

Figure 6.

Suppose that C_p embeds in a smooth 4-manifold Z. Let $Z_{(p)}$ be the smooth 4-manifold obtained from Z by removing C_p and gluing B_p along the boundary. The smooth 4-manifold $Z_{(p)}$ is called the rational blowdown of Z along C_p. Note that $Z_{(p)}$ is uniquely determined up to diffeomorphism by a fixed pair (Z, C_p) (see Fintushel-Stern [10]). This operation preserves b_2^+, decreases b_2^-, may create torsion in the first homology group.

Rational blowdown has some relations with corks ([6]). In this paper, we give the relation below, similarly to [6]. This relation is a key of our proof of Theorem 1.3 and 1.4.

Theorem 4.1. *Let D_p be the smooth 4-manifold in Figure 7 (notice D_p is C_p with two 2-handles attached). Suppose that a smooth 4-manifold Z contains D_p. Let $Z_{(p)}$ be the rational blowdown of Z along the copy of C_p contained in D_p. Then the submanifold D_p of Z contains W_{p-1} such that $Z_{(p)} \# (p-1)\overline{\mathbf{CP}^2}$ is obtained from Z by removing W_{p-1} and regluing via f_{p-1}.*

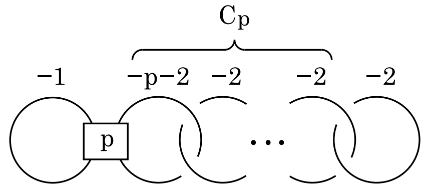

Figure 7. D_p

Proof. We can easily get a handle decomposition of D_p in the first diagram of Figure 14, following the procedure in [13, Section 8.5]. (See also [6, Figure 14 and 15].) Slide handles as in Figure 14. In the third diagram of Figure 14, we can find a 0-framed unknot which links the dotted circle geometrically once. Replace this dot and 0 as in the first row of Figure 15. This operation corresponds to removing B^4 in the submanifold D_p of Z and regluing B^4. Since every self-diffeomorphism of S^3 extends to a self-diffeomorphism of B^4, the operation above keeps the diffeomorphism type of D_p and the embedding of D_p into X up to isotopy (In our situation, we can easily show this claim by checking that the operation above corresponds to canceling the 1-handle/2-handle pair and introducing a 1-handle/2-handle pair differently). As a consequence, we can easily find W_{p-1} in D_p. Note that Figure 16 is isotopic to the standard diagram of W_{p-1}. By removing W_{p-1} in the submanifold D_p of Z and regluing it via f_{p-1}, we get the lower diagram of Figure 15.

The rational blowdown procedure in [13, Section 8.5] shows that $Z_{(p)}\#(p-1)\overline{\mathbf{CP}^2}$ is obtained by replacing the dot and 0 as in the left side of Figure 15. Hence, we obtain $Z_{(p)}\#(p-1)\overline{\mathbf{CP}^2}$ from Z by removing W_{p-1} and regluing it via f_{p-1}. □

5. Construction

In this section, we construct the examples of Theorem 1.3 and 1.4, by imitating rational blowdown constructions in [20] and [21].

Let T be the class of a regular fiber of $E(p_1+p_2+\cdots+p_n)$ in $H_2(E(p_1+p_2+\cdots+p_n);\mathbf{Z})$. Let $e_1, e_2, \dots, e_n$ be the standard basis of $H_2(n\overline{\mathbf{CP}^2};\mathbf{Z})$ such that $e_i^2=-1$ $(1\le i\le n)$ and $e_i\cdot e_j=0$ $(i\neq j)$.

Proposition 5.1. *For each $n\ge 1$ and each $p_1, p_2, \dots, p_n\ge 2$, the 4-manifold $E(p_1+p_2+\cdots+p_n)\#n\overline{\mathbf{CP}^2}$ admits a handle decomposition as in Figure 8. The obvious cusp neighborhood in the figure is isotopic to the regular neighborhood of a cusp fiber of $E(p_1+p_2+\cdots+p_n)$. The homology classes in the figure represent the homology classes given by corresponding 2-handles. Here $k=11(p_1+p_2+\cdots+p_n)+n-4$.*

Proof. We begin with the diagram of $E(p_1+p_2+\cdots+p_n)$ in Figure 4. Introduce a canceling 2-handle/3-handle pair and slide handles as in Figure 17. An isotopy gives the first diagram of Figure 18. Slide handles and blow up as in Figure 18. Handle slides give the first diagram of Figure 19. Slide handles as in Figure 19. We now have the diagram of D_p. By repeating this process and canceling the 1-handle with a −1-framed 2-handle, we get Figure 8 of $E(p_1+p_2+\cdots+p_n)\#n\overline{\mathbf{CP}^2}$. □

Definition 5.2. (1) Define $Y_0 := E(p_1+p_2+\cdots+p_n)\#n\overline{\mathbf{CP}^2}$. Let Y_i' $(1\le i\le n)$ be the rational blowdown of Y_0 along the copy of C_{p_i} in Figure 8. Put

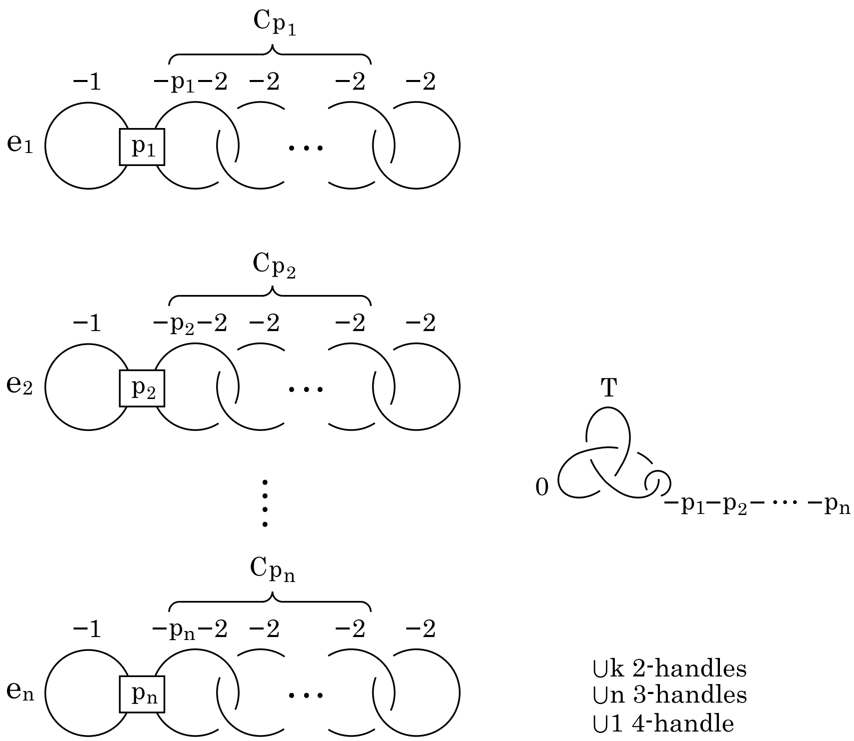

FIGURE 8. $E(p_1 + p_2 + \cdots + p_n)\#n\overline{\mathbf{CP}^2}$ $(p_1, p_2, \ldots, p_n \geq 2)$

$Y_i := Y_i' \# (p_i - 1)\overline{\mathbf{CP}^2}$.

(2) For $k_1, k_2, \ldots, k_n \geq 1$, let $W(k_1, k_2, \ldots, k_n)$ be the boundary sum $W_{k_1} \natural W_{k_2} \natural \cdots \natural W_{k_n}$. Figure 9 is a diagram of $W(k_1, k_2, \ldots, k_n)$. Let $f^i(k_1, k_2, \ldots, k_n)$ be the involution on the boundary $\partial W(k_1, k_2, \ldots, k_n)$ obtained by replacing the dot and zero of the component of W_{k_i}.

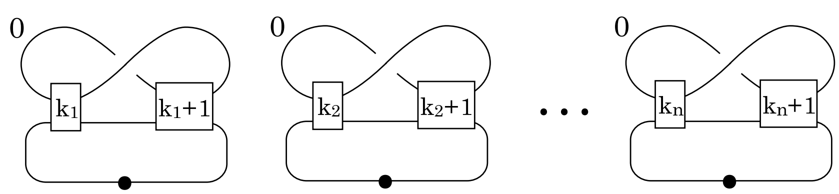

FIGURE 9. $W(k_1, k_2, \ldots, k_n)$

Lemma 5.3. *For each $k_1, k_2, \ldots, k_n \geq 1$, the manifold $W(k_1, k_2, \ldots, k_n)$ is a compact contractible Stein 4-manifold.*

Proof. We can check this by changing the 1-handle notations of $W(k_1, k_2, \ldots, k_n)$, and putting the 2-handles in Legendrian positions. (For such a diagram of W_n, see Figure 10.) Now all we have to check is the Eliashberg criterion: the framings on the 2-handles are less than Thurston-Bennequin number. $\square$

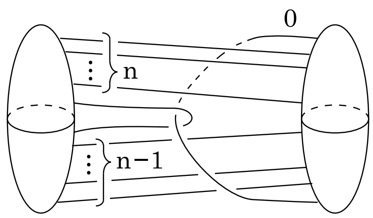

FIGURE 10. W_n

Proposition 5.4. (1) *The 4-manifold Y_0 contains mutually disjoint copies of $W_{p_1}, W_{p_2}, \ldots, W_{p_n}$ such that, for each i, the 4-manifold Y_i is obtained from Y_0 by removing the copy of W_{p_i} and regluing it via the involution f_{p_i}.*

(2) *The 4-manifold Y_0 contains a fixed copy of $W(p_1-1, p_2-1, \ldots, p_n-1)$ such that, for each i, the 4-manifold Y_i is obtained from Y_0 by removing the copy of $W(p_1 - 1, p_2 - 1, \ldots, p_n - 1)$ and regluing it via the involution $f^i(p_1 - 1, p_2 - 1, \ldots, p_n - 1)$.*

Proof. Proposition 5.1 and Theorem 4.1 clearly show the claims (1) and (2). □

6. PROOF OF THEOREM 1.3 AND 1.4

6.1. **Seiberg-Witten invariants.** In this subsection, we briefly review facts about the Seiberg-Witten invariants. For details and examples of computations, see, for example, Fintushel-Stern [12].

Suppose that Z is a simply connected closed smooth 4-manifold with $b_2^+(Z) > 1$ and odd. Let $\mathcal{C}(Z)$ be the set of characteristic elements of $H^2(Z; \mathbf{Z})$. Then the Seiberg-Witten invariant $SW_Z : \mathcal{C}(Z) \to \mathbf{Z}$ is defined. Let $e(Z)$ and $\sigma(Z)$ be the Euler characteristic and the signature of Z, respectively, and $d_Z(K)$ the even integer defined by $d_Z(K) = \frac{1}{4}(K^2 - 2e(Z) - 3\sigma(Z))$ for $K \in \mathcal{C}(Z)$. If $SW_Z(K) \neq 0$, then K is called a Seiberg-Witten basic class of Z. We denote $\beta(Z)$ as the set of the Seiberg-Witten basic classes of Z. The following theorem is well-known.

Theorem 6.1 (Witten [19], cf. Gompf-Stipsicz [13]). *For $n \geq 2$,*
(1) $\beta(E(n)) = \{k \cdot PD(T) \mid k \equiv 0 \ (\mathrm{mod}\ 2), |k| \leq n - 2\}$;
(2) $\beta(E(n) \# m\overline{\mathbf{CP}^2}) = \{k \cdot PD(T) \pm E_1 \pm E_2 \pm \cdots \pm E_m \mid k \equiv 0 \ (\mathrm{mod}\ 2), |k| \leq n - 2\}$. *Here T denotes the class of a regular fiber of $E(n)$ in $H_2(E(n); \mathbf{Z})$, and $E_1, E_2, \ldots, E_m$ denotes the standard basis of $H^2(m\overline{\mathbf{CP}^2}; \mathbf{Z})$.*

We here recall the change of the Seiberg-Witten invariants by rationally blowing down. Assume further that Z contains a copy of C_p. Let $Z_{(p)}$ be the rational blowdown of Z along the copy of C_p. Suppose that $Z_{(p)}$ is simply connected. The following theorems are obtained by Fintushel-Stern [10].

Theorem 6.2 (Fintushel-Stern [10]). *For every element K of $\mathcal{C}(Z_{(p)})$, there exists an element $\tilde{K}$ of $\mathcal{C}(Z)$ such that $K|_{Z_{(p)} - \mathrm{int}\, B_p} = \tilde{K}|_{Z - \mathrm{int}\, C_p}$ and $d_{Z_{(p)}}(K) = d_Z(\tilde{K})$. Such an element $\tilde{K}$ of $\mathcal{C}(Z)$ is called a lift of K.*

Theorem 6.3 (Fintushel-Stern [10]). *If an element $\tilde{K}$ of $\mathcal{C}(Z)$ is a lift of some element K of $\mathcal{C}(Z_{(p)})$, then $SW_{Z_{(p)}}(K) = SW_Z(\tilde{K})$.*

Theorem 6.4 (Fintushel-Stern [10], cf. Park [18]). *If an element $\tilde{K}$ of $\mathcal{C}(Z)$ satisfies that $(\tilde{K}|_{C_p})^2 = 1 - p$ and $\tilde{K}|_{\partial C_p} = mp \in \mathbf{Z}_{p^2} \cong H^2(\partial C_p; \mathbf{Z})$ with $m \equiv p - 1 \pmod 2$, then there exists an element K of $\mathcal{C}(Z_{(p)})$ such that $\tilde{K}$ is a lift of K.*

Corollary 6.5. *If an element $\tilde{K}$ of $\mathcal{C}(Z)$ satisfies $\tilde{K}(u_1) = \tilde{K}(u_2) = \cdots = \tilde{K}(u_{p-2}) = 0$ and $\tilde{K}(u_{p-1}) = \pm p$, then $\tilde{K}$ is a lift of some element K of $\mathcal{C}(Z_{(p)})$.*

6.2. **Computation of SW invariants.** In this subsection, we prove Theorem 1.3 and 1.4 by computing the Seiberg-Witten invariants of the 4-manifolds Y_i $(0 \leq i \leq n)$ in Definition 5.2.

For a smooth 4-manifold Z we denote $N(Z)$ as the number of elements of $\beta(Z)$.

Lemma 6.6. $N(Y_i) = 2^{p_i - 1} N(Y_0)$ $(1 \leq i \leq n)$

Proof. Proposition 5.1, Theorem 6.1 and Corollary 6.5 show that every Seiberg-Witten basic class of Y_0 is a lift of some element of $\mathcal{C}(Y_i')$, and that these basic classes of Y_0 have mutually different restrictions to $Y_0 - \operatorname{int} C_{p_i} (= Y_i' - \operatorname{int} B_{p_i})$. Note that every element of $H^2(Y_i'; \mathbf{Z})$ is uniquely determined by its restriction to $Y_i' - \operatorname{int} B_{p_i}$. (We can easily check this from the cohomology exact sequence for the pair $(Y_i', Y_i' - \operatorname{int} B_{p_i})$.) Hence Theorems 6.2 and 6.3 give $N(Y_i') = N(Y_0)$. Now the required claim follows from the blow-up formula. □

Corollary 6.7. *If $p_1, p_2, \ldots, p_n \geq 2$ are mutually different, then Y_i $(0 \leq i \leq n)$ are mutually homeomorphic but not diffeomorphic.*

Proof of Theorem 1.3 and 1.4. The theorems clearly follow from the corollary above and Proposition 5.4. □

7. Further remarks

7.1. **Plugs.** In [6], we introduced new objects which we call *Plugs*.

Definition 7.1. Let P be a compact Stein 4-manifold with boundary and $\tau : \partial P \to \partial P$ an involution on the boundary, which cannot extend to any self-homeomorphism of P. We call (P, τ) a *Plug* of X, if $P \subset X$ and X keeps its homeomorphism type and changes its diffeomorphism type when removing P and gluing it via τ. We call (P, τ) a *Plug* if there exists a smooth 4-manifold X such that (P, τ) is a plug of X.

Definition 7.2. Let $W_{m,n}$ be the smooth 4-manifold in Figure 11, and let $f_{m,n} : \partial W_{m,n} \to \partial W_{m,n}$ be the obvious involution obtained by first surgering $S^1 \times B^3$ to $B^2 \times S^2$ in the interior of $W_{m,n}$, then surgering the other imbedded $B^2 \times S^2$ back to $S^1 \times B^3$ (i.e. replacing the dots in Figure 1). Note that the diagram of $W_{m,n}$ is a symmetric link.

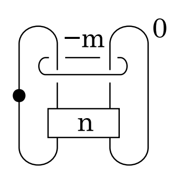

Figure 11.

Theorem 7.3 ([6]). *For $m \geq 1$ and $n \geq 2$, the pair $(W_{m,n}, f_{m,n})$ is a plug.*

As we pointed out in [6, Remark 5.3], removing and regluing plugs have naturally the same effect as cork operations, under some conditions. We can easily check that our constructions in this paper are such cases. Therefore we can similarly prove the theorems below.

The following theorem shows that infinitely many different smooth structures on 4-manifolds can be obtained from a fixed plug:

Theorem 7.4. *There exist a simply connected compact Stein 4-manifold P, an involution τ on the boundary ∂P, and infinitely many simply connected closed smooth 4-manifolds X_n $(n \geq 0)$ with the following properties:*
(1) *The pair (P, τ) is a plug;*
(2) *The 4-manifolds X_n $(n \geq 0)$ are mutually homeomorphic but not diffeomorphic;*
(3) *For each $n \geq 1$, the 4-manifold X_n is obtained from X_0 by removing a copy of P and regluing it via τ. Consequently, the pair (P, τ) is a plug of X_0.*

In particular, from X_0 we can produce infinitely many different smooth structures by the process of removing a copy of P and regluing it via τ. Consequently, these embeddings of P into X_0 are mutually non-isotopic (knotted copies of each other).

Proof. We begin with the proposition below.

Proposition 7.5. *The 4-manifold $E(n)\#\overline{\mathbf{CP}^2}$ $(n \geq 2)$ has a handle decomposition as in Figure 12. The obvious cusp neighborhood in the figure is isotopic to the regular neighborhood of a cusp fiber of $E(n)$. Note that the figure contains W_1 and $W_{1,2}$. Furthermore, this copy of W_1 is isotopic to the copy of W_1 obtained in Proposition 3.5.*

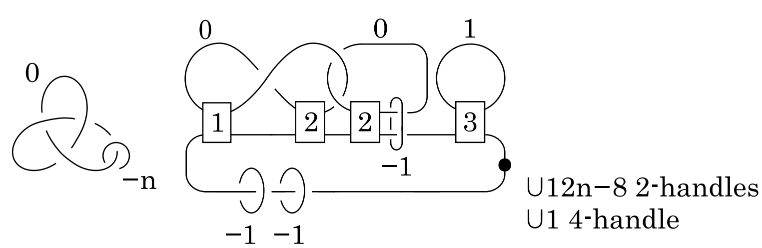

FIGURE 12. $E(n)\#\overline{\mathbf{CP}^2}$ $(n \geq 2)$

Proof of Proposition 7.5. It follows from Corollary 3.2 that the submanifold of $E(n)$ in the first diagram of Figure 13 is disjoint from the cusp neighborhood of $E(n)$. Slide handles as in Figure 20. Now we can easily check the required claim. See also the proof of Proposition 3.5. □

Let X be the smooth 4-manifold obtained from $E(n)\#\overline{\mathbf{CP}^2}$ $(n \geq 2)$ by removing the copy of $W_{1,2}$ in Figure 12 and regluing it via $f_{1,2}$. Note that two 0-framed 2-handles of the copies of W_1 and $W_{1,2}$ in Figure 12 link geometrically once. Therefore, similarly to the proof of Theorem 4.1, we can easily show that X is obtained from $E(n)\#\overline{\mathbf{CP}^2}$ by removing the copy of W_1 in Figure 12 and regluing it via f_1. The rest of the proof of Theorem 7.4 proceeds as in the proof of Theorem 1.2. □

Remark 7.6. (1) As we pointed out in the proof above, the cork operation and the plug operation are the same, for our example of Theorem 1.2 and 7.4.

(2) In Theorem 7.4, we obtained infinitely many knotted embeddings of $W_{1,2}$ into X. Each embedding gives the same subspace of $H_2(X;\mathbf{Z})$ corresponding to $H_2(W_{1,2};\mathbf{Z})(\cong \mathbf{Z})$. Thus the construction in the theorem might give useful applications to find homologous but non-isotopic surfaces in X.

(3) We proved Theorem 7.4 for the plug $(W_{1,2}, f_{1,2})$. We can similarly prove Theorem 7.4 for many other plugs, including $(W_{m,n}, f_{m,n})$ $(m \geq 1,\, n \geq 2)$.

The next theorem says that we can put an arbitrary finite number of plugs into mutually disjoint positions in 4-manifolds:

Theorem 7.7. *For each $n \geq 1$, there exist simply connected closed smooth 4-manifolds Y_i $(0 \leq i \leq n)$, codimension zero simply connected compact Stein submanifolds P_i $(1 \leq i \leq n)$ of Y_0, and an involution τ_i on the each boundary ∂P_i $(1 \leq i \leq n)$ with the following properties:*

(1) *The pairs (P_i, τ_i) $(1 \leq i \leq n)$ are plugs;*

(2) *The submanifolds P_i $(1 \leq i \leq n)$ of Y_0 are mutually disjoint;*

(3) *Each Y_i $(1 \leq i \leq n)$ is obtained from Y_0 by removing the submanifold P_i and regluing it via τ_i;*

(4) *The 4-manifolds Y_i $(0 \leq i \leq n)$ are mutually homeomorphic but not diffeomorphic. In particular, the pairs (P_i, τ_i) $(1 \leq i \leq n)$ are plugs of Y_0.*

The following theorem says that, for an embedding of a plug into a 4-manifold, we can produce finitely many different plug structures of the 4-manifold by only changing the involution of the plug without changing its embedding:

Theorem 7.8. *For each $n \geq 1$, there exist simply connected closed smooth 4-manifolds Y_i $(0 \leq i \leq n)$, an embedding of a simply connected compact Stein 4-manifold P into Y_0, and involutions τ_i $(1 \leq i \leq n)$ on the boundary ∂P with the following properties:*

(1) *The pairs (P, τ_i) $(1 \leq i \leq n)$ are plugs.*

(2) *For each $1 \leq i \leq n$, the 4-manifold Y_i is obtained from Y_0 by removing the submanifold P and regluing it via τ_i;*

(3) *The 4-manifolds Y_i $(0 \leq i \leq n)$ are mutually homeomorphic but not diffeomorphic, hence the pairs (P, τ_i) $(1 \leq i \leq n)$ are mutually different plugs of Y_0.*

Proof of Theorem 7.7 and 7.8. According to the theorem below, we can show the required claims similarly to the proof of Theorem 1.3 and 1.4. (As for Theorem 7.8, we also use the argument similar to [6, Lemma 2.7.(3)].)

Theorem 7.9 ([6, Theorem 5.1.(3)])**.** *Suppose that a smooth 4-manifold Z contains the 4-manifold D_p in Figure 7. Let $Z_{(p)}$ be the rational blowdown of Z along the copy of C_p contained in D_p. Then the submanifold D_p of Z contains $W_{1,p}$ such that $Z_{(p)}\#(p-1)\overline{\mathbf{CP}^2}$ is obtained from Z by removing $W_{1,p}$ and regluing it via $f_{1,p}$.*

□

7.2. **Knotted contractible 4-manifolds.** Lickorish [14] constructed large families of contractible 4-manifolds that have two non-isotopic embeddings into S^4. Livingston [15] later gave large families of contractible 4-manifolds that have infinitely many mutually non-isotopic embeddings into S^4. These embeddings are detected by the fundamental group of their complements. The corks W_n can be knotted in S^4 with simply connected complements (even PL knotted), this is because doubling W_n both by the identity and by the involution $f_n : \partial W_n \to \partial W_n$ give S^4, and f_n takes a slice knot to a non-slice knot (cf. [2]). Obviously these knotted imbeddings $W_n \subset S^4$ are not corks of S^4. It is not known whether S^4 admits cork imbeddings (i.e. it is not known whether S^4 admits an exotic smooth structure). Theorem 1.2 of this paper gives infinitely many mutually non-isotopic embeddings of W_1 (and also other contractible 4-manifolds [See Remark 3.9.(3)]) into the 4-manifold X_0 with simply connected complements, so that the imbeddings give mutually different cork structures of X_0.

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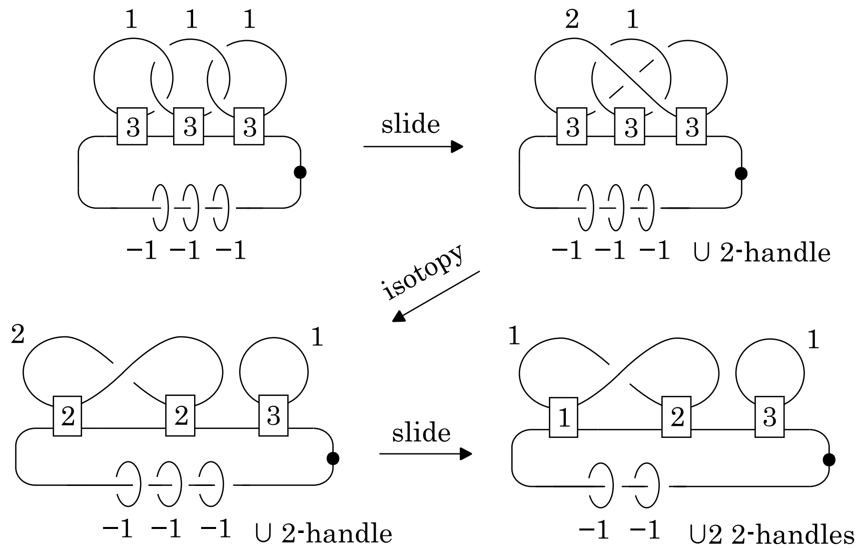

FIGURE 13. handle slides

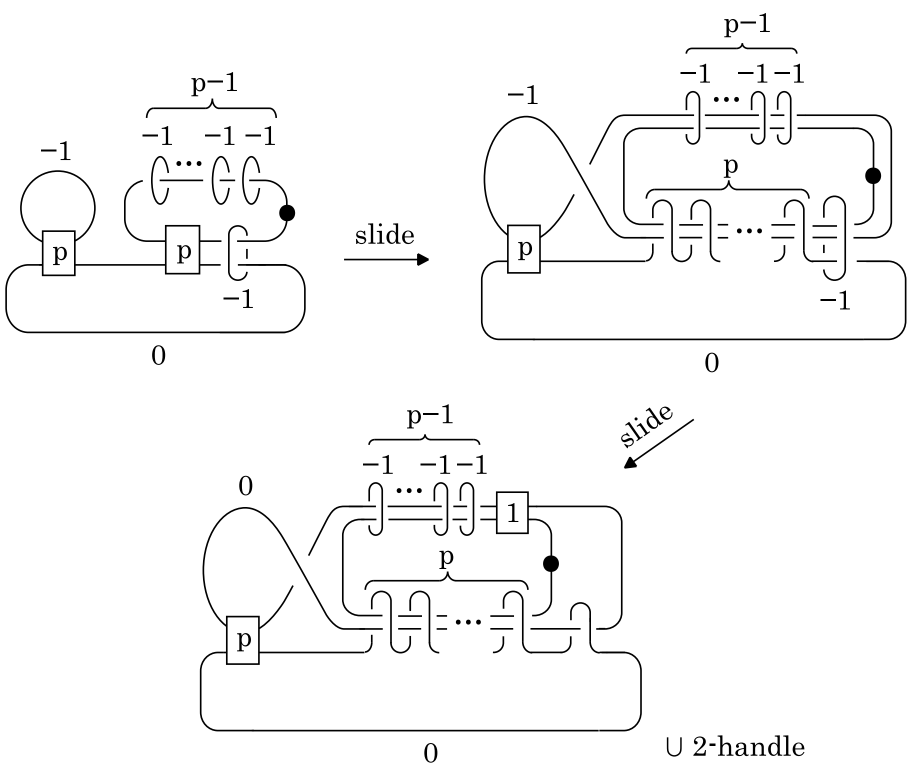

FIGURE 14. handle slides of D_p

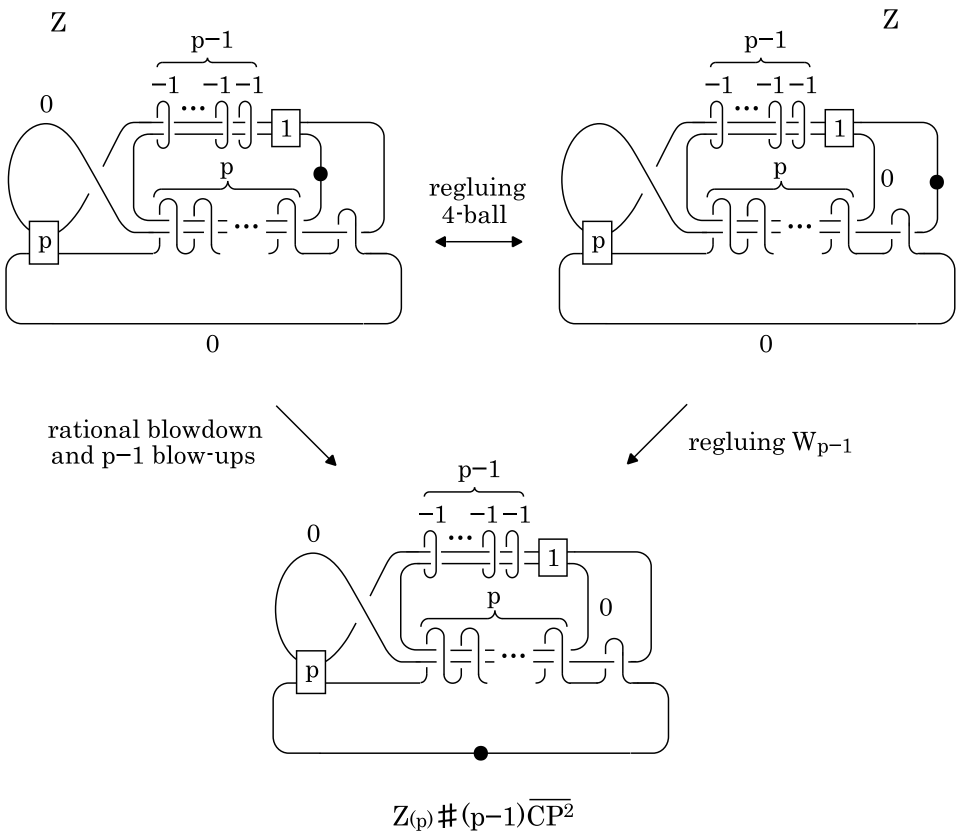

FIGURE 15.

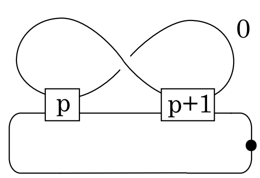

FIGURE 16. W_{p-1}

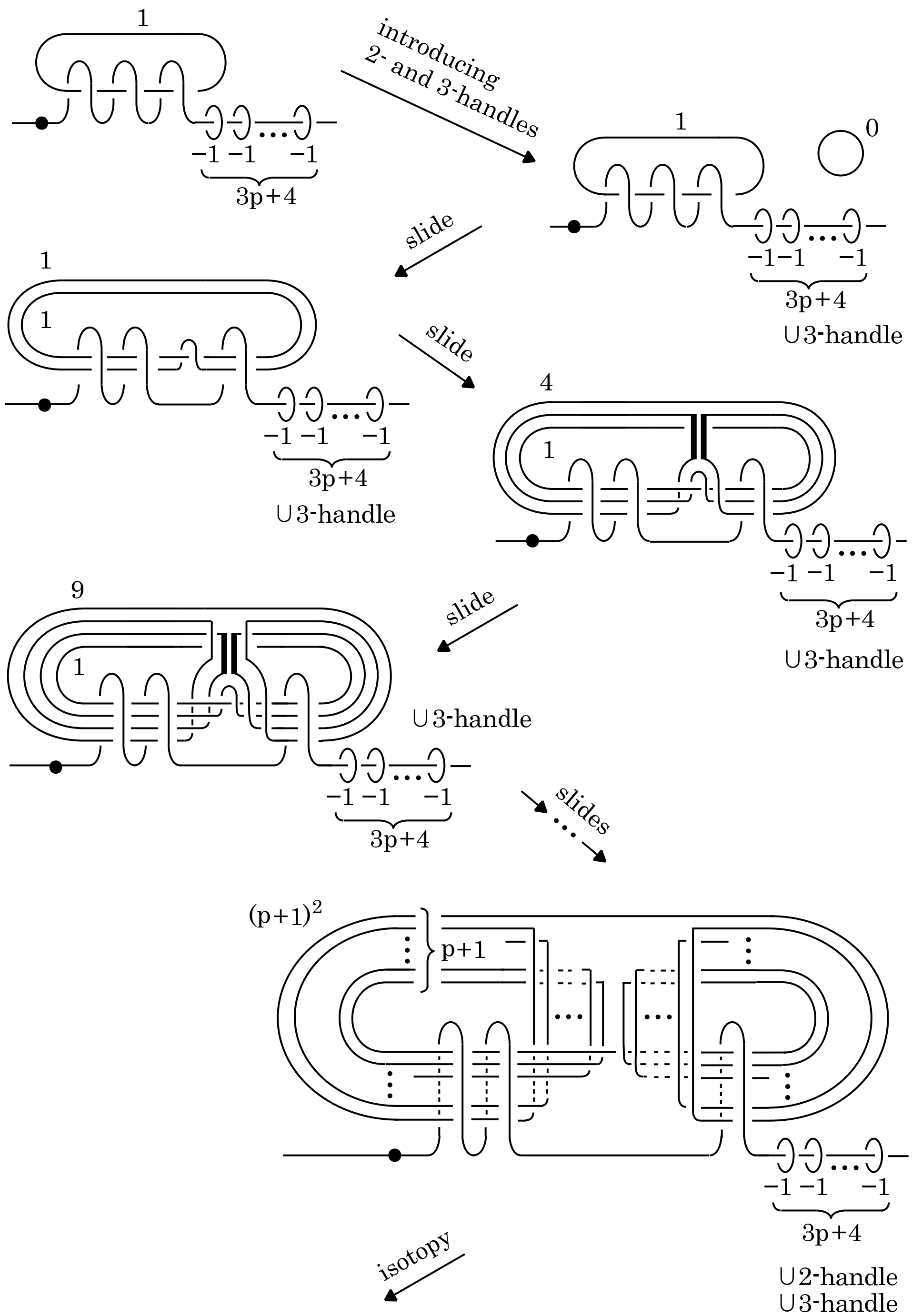

FIGURE 17. introducing a 2-handle/3-handle pair and sliding handles

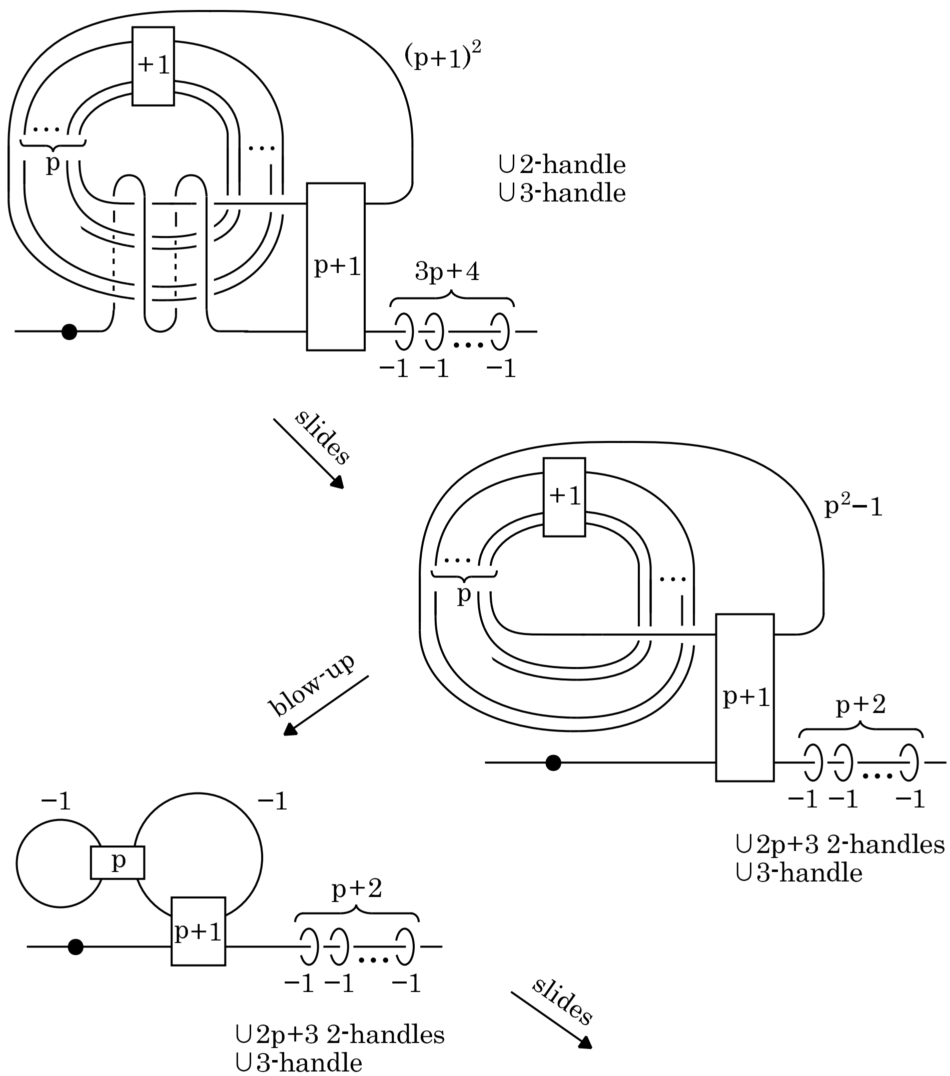

FIGURE 18. handle slides and blow-up

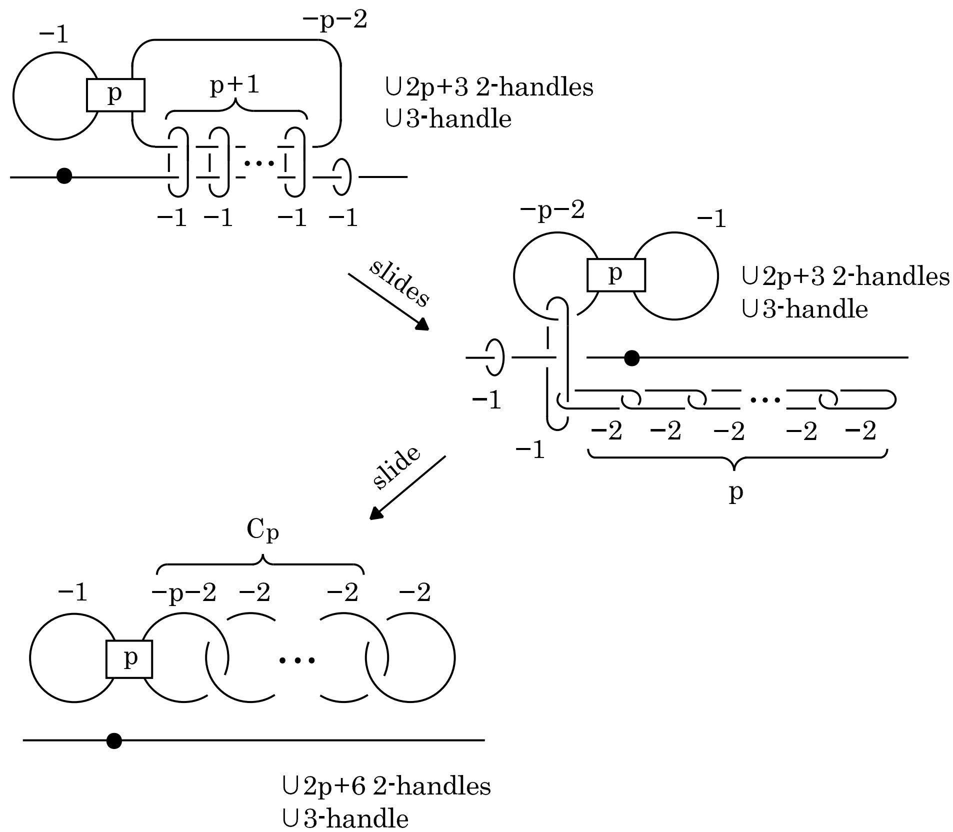

FIGURE 19. handle slides

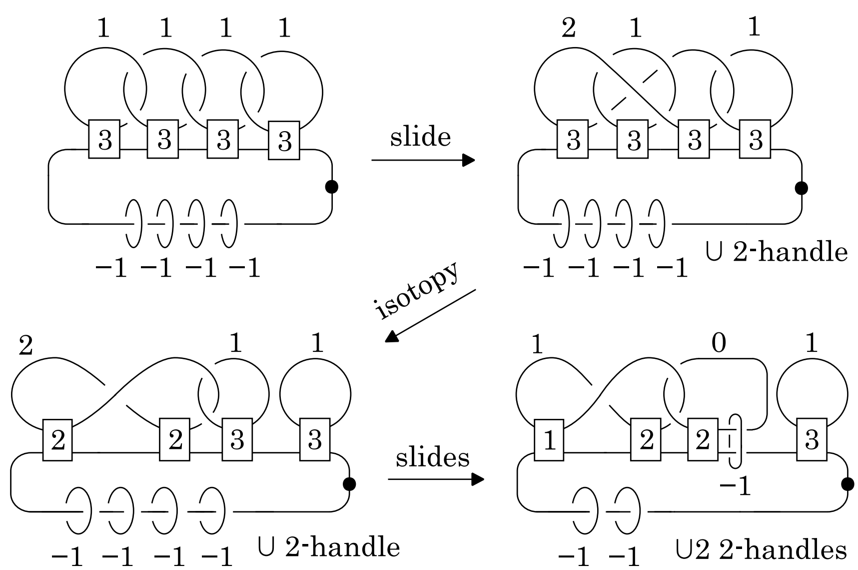

FIGURE 20. handle slides

Department of Mathematics, Michigan State University, E.Lansing, MI, 48824, USA
E-mail address: akbulut@math.msu.edu

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan
E-mail address: kyasui@kurims.kyoto-u.ac.jp