Topology of multiple log transforms of 4-manifolds

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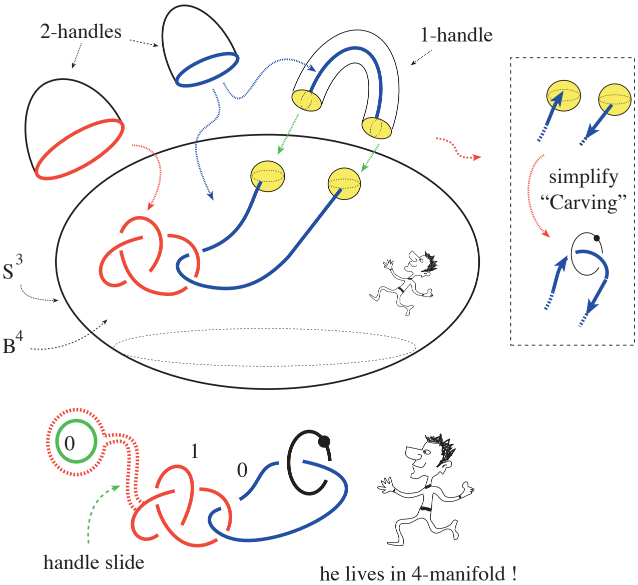
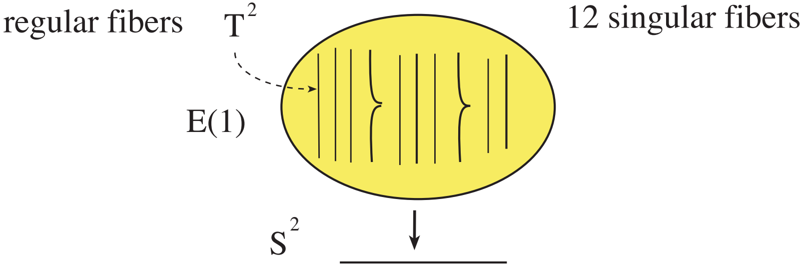
4-dim handlebodies



The elliptic surface $E(1)$

- $E(1)$: Let $P_0(z)$ and $P_1(z)$ be a generic pair of homogenious cubic polynomials in $\mathbf{C}^3$. For each $t = [t_0, t_1] \in \mathbf{CP}^1$ the following sets $Z_t = \{z \in \mathbf{CP}^2 \mid t_0 P_0(z) + t_1 P_1(z) = 0\}$ fill $\mathbf{CP}^2$ (generically each Z_t is a torus).



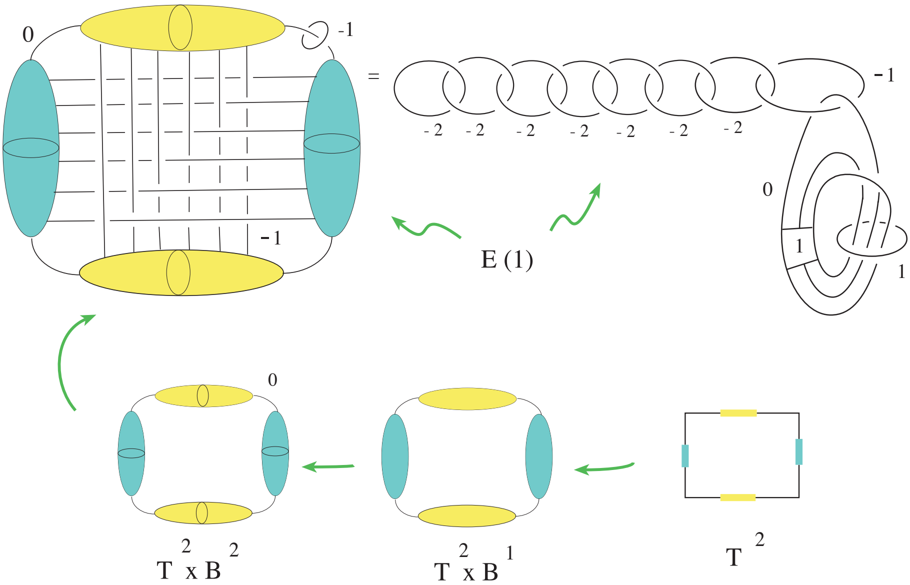
$E(1)_{p,q}$

- Dolgachev surface $E(1)_{p,q}$ is an elliptic complex surface, which is obtained from the standard elliptic surface $E(1) = \mathbf{CP}^2 \# 9\bar{\mathbf{CP}}^2$ by the log transform operation of orders p and q on two disjoint parallel fibers (p and q are relatively prime). p- log transform on any manifold containing $T^2 \times B^2$ is: Remove $T^2 \times B^2$ and glue it back by the nontrivial diffeomorphism of T^3 given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & p \end{pmatrix}$$

- 25 years ago Donaldson gave the first example of an oriented exotic smooth 4-manifold. he proved that Dolgachev surface $E(1)_{p,q}$ is an exotic copy of $E(1)$.
- **Question**: What makes $E(1)_{p,q}$ an exotic copy of $E(1)$? Find a good local explanation, such as finding its cork structure.

$E(1)$



The p-log transform operation: $T^2 \times B^2 \mapsto (T^2 \times B^2)_p$

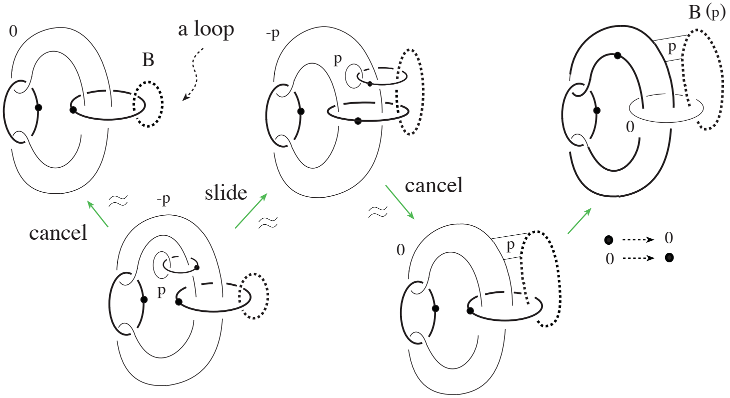
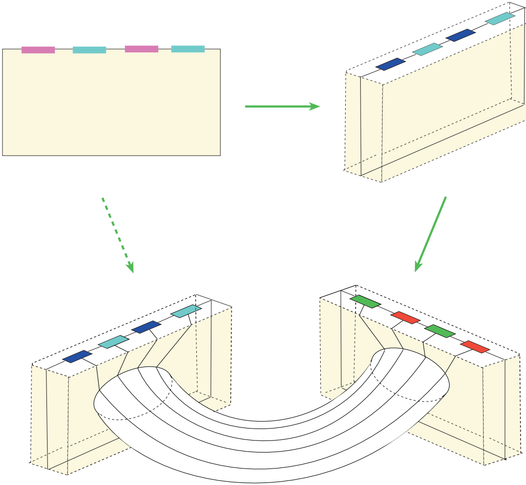
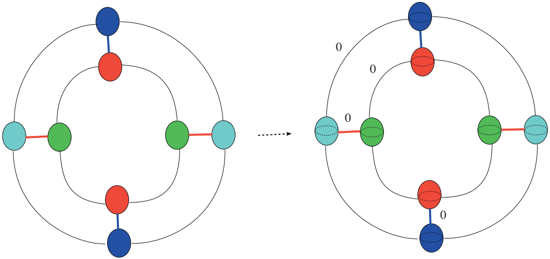


Figure: How to perform p-log transform operation

$T^2 \mapsto T^2 \times [0,1/2] \cup T^2 \times [1/2,1]$



$T^2 \times B^1_- \cup T^2 \times B^1_+ \;\mapsto\; T^2 \times B^2_- \cup T^2 \times B^2_+$



$T^2 \times B^2_- \cup T^2 \times B^2_+$ in circle-with-dot notation

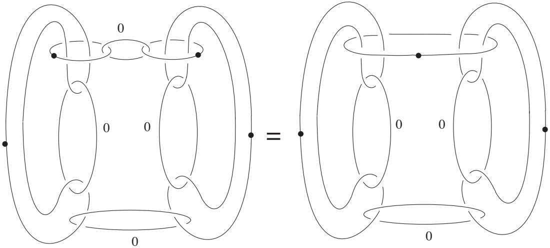
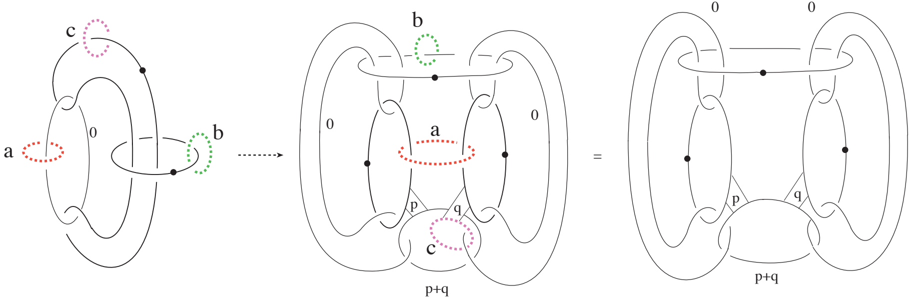
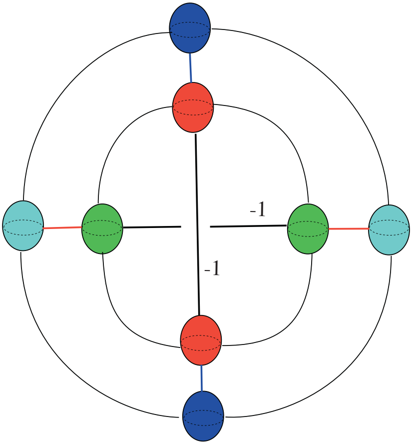


Figure: $T^2 \times B^2 = T^2 \times B^2_- \cup T^2 \times B^2_+$

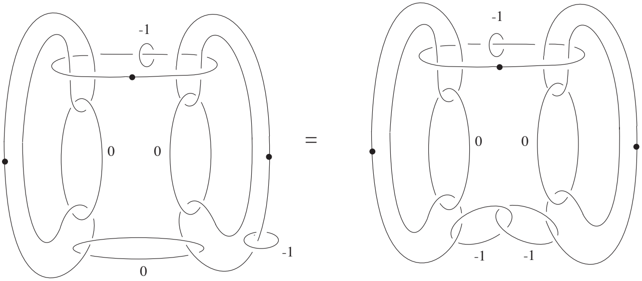
The operation $T^2 \times B^2 \mapsto (T^2 \times B^2)_{p,q}$



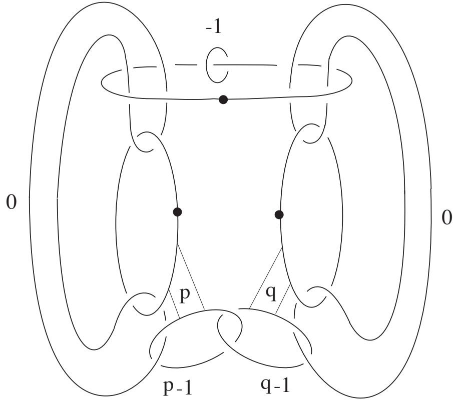
The cusp C (with a view of two disjoint tori inside!)



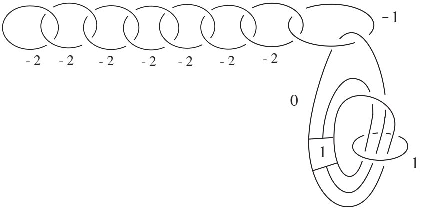
The cusp C in circle-with-dot notation



The operation $C \mapsto C_{p,q}$



$E(1)$ (with two disjoint tori inside of the cusp C)



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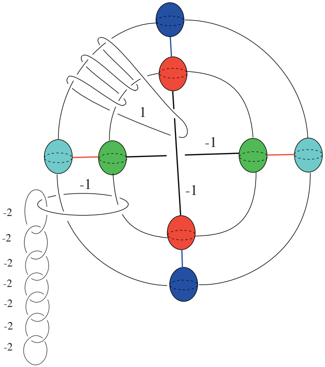
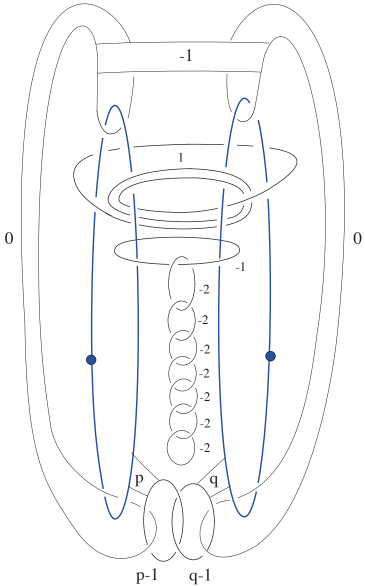
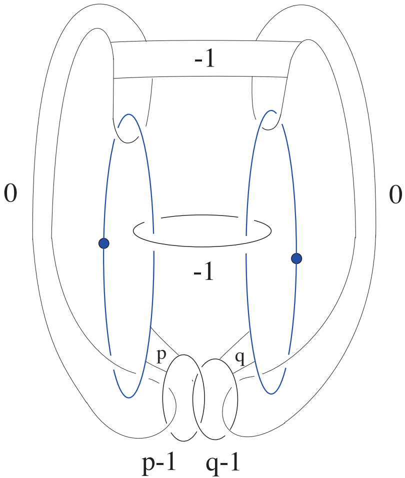


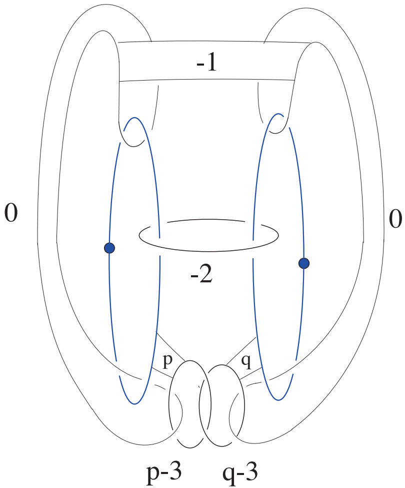
Finally we have the picture of $E(1)_{p,q}$



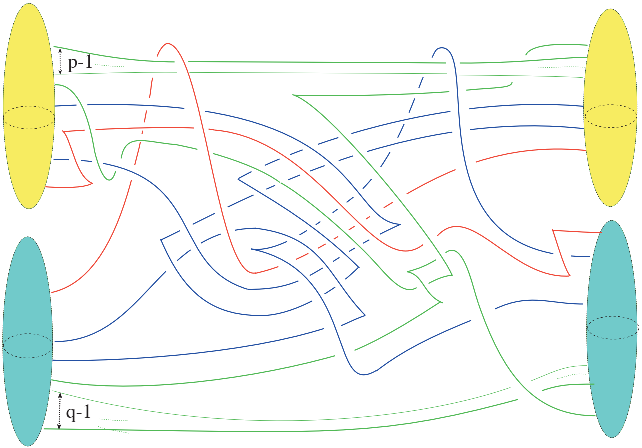
$Q_{p,q} \subset E(1)_{p,q}$ (Nucleus responsible for exoticness)



$M_{p,q} \subset E(1)_{p,q} \# 5\bar{CP}^2$ (Stein submanifold responsible for exoticness)



$M_{p,q}$ as a Stein handlebody



To sum up:

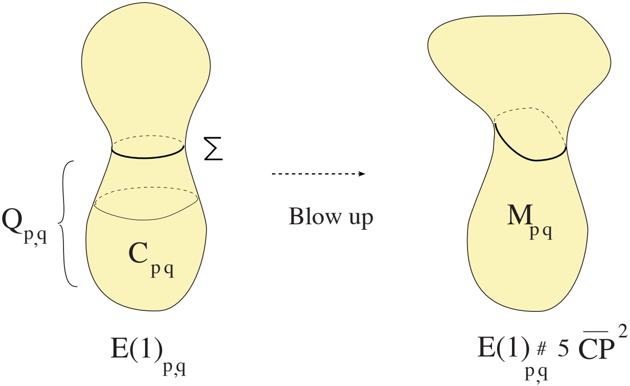
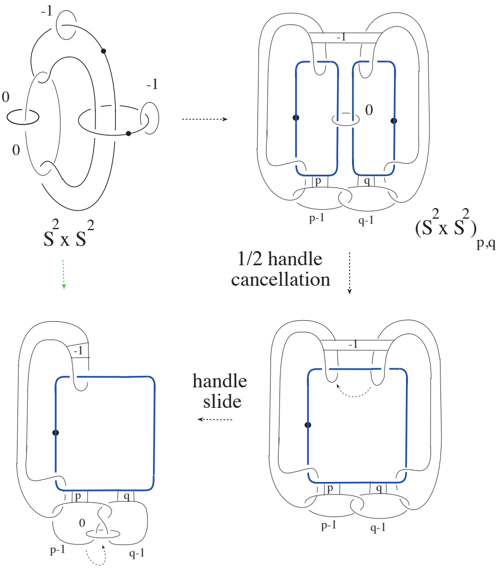
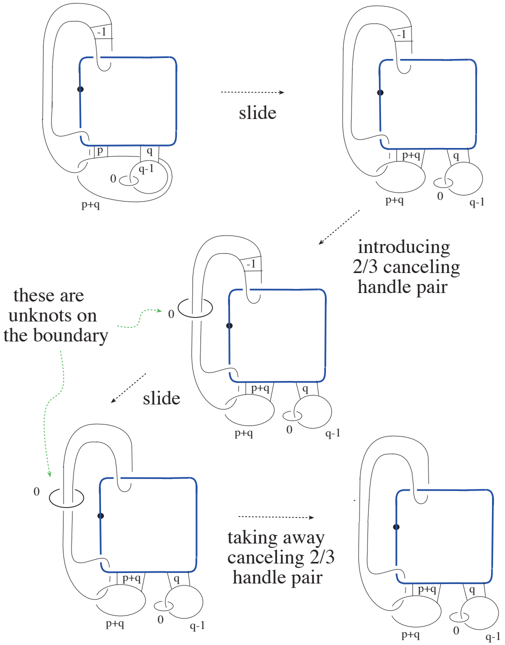
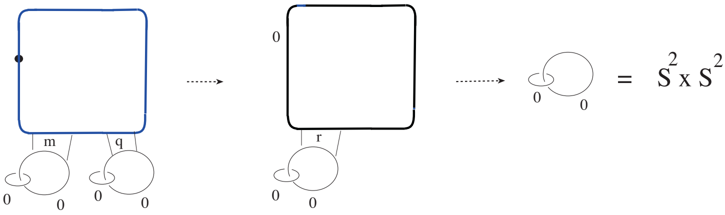


Figure: $M_{p,q}$ is Stein

Proving $(S^2 \times S^2)_{p,q} = S^2 \times S^2$




-1
p
q
q-1
0
p+q
slide
-1
p+q
q
p+q
0
q-1
introducing
2/3 canceling
handle pair
these are
unknots on
the boundary
-1
0
p+q
q
p+q
0
q-1
slide
0
p+q
q
p+q
0
q-1
taking away
canceling 2/3
handle pair
p+q
q
p+q
0
q-1



(m and q are relatively prime)