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A NOTE ON A HOMOLOGY SPHERE

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Abstract. Here we give a solution to a problem of Y.Matsumoto which was posed in "Kirby's problem list"

In this note we solve a problem posed by Y.Matsumoto in Kirby's problem list (Problem 4.28 (A) of [K]). This problem should have been solved ten years ago after Donaldson's Theorem-C in [D] which imposed a restriction to intersection forms of certain 4 manifolds because the solution does not use any 4-manifold techniques developed since then. The problem is whether the 4-manifold M obtained by attaching a pair of two handles to a 4-ball B^4 along the two linked left handed trefoil knots, as in Figure 1, contains a smoothly imbedded wedge of 2-spheres representing generators of $H_2(M)$? An affirmative answer implies that ∂M bounds a contractible manifold W. We show that this is not the case. In fact ∂M does not bound a 4-homology ball W with $\pi_1(\partial M) \to \pi_1(W)$ onto.

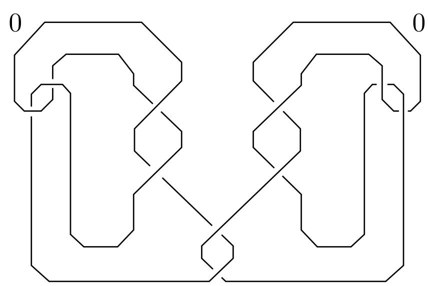

Figure 1

The 4-manifolds represented by Figures 1, 2, 3, 4 have the same boundary: By blowing down the (+1)-framed 2-handle in Figure 4 we get Figure 3; Figure 3 is obtained from Figure 2 by sliding the (−1)-framed handle over one of the 0-framed handles as indicated by the dotted arrow, and then cancelling the 0-framed handle pair; Figure 1 is obtained from Figure 2 by blowing down the two (−1)-framed 2-handles. Now by the usual "blowing up and down" process (e.g. [A] figures 9-19) we can turn two (−1)-framed trefoil knots of Figure 4 into two E_8's and obtain Figure 5. This process turns the (+1)-framed 2-handle into a connected sum of two right-handed trefoil knots. By introducing two hyperbolic pairs as in Figure 6 we can make the (+1)-framed knot of Figure 5 slice, which we can blow down.

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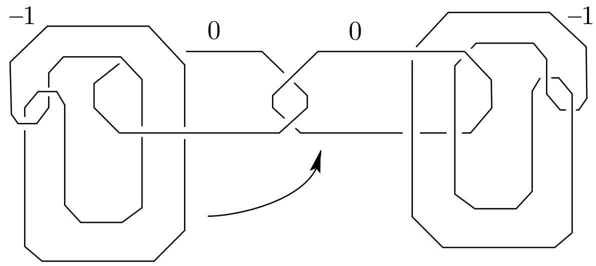

FIGURE 2

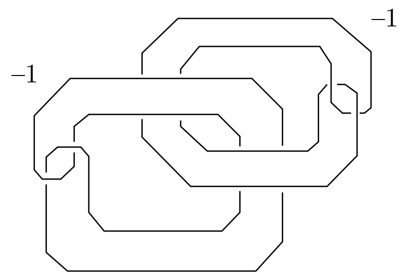

FIGURE 3

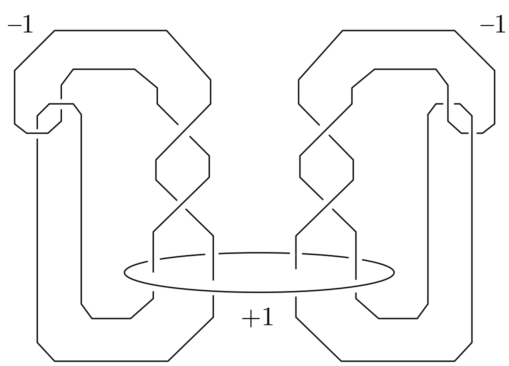

FIGURE 4

This gives a smooth spin manifold Q with intersection form $E_8 \oplus E_8 \oplus 2\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $\partial Q = \partial M$. Hence if ∂M were to bound a contractible manifold W, then $Q \smile (-W)$ would be a manifold violating Donaldson's Theorem-C in [D].

An interesting fact: by blowing down one of the -1 spheres of Figure 3 we see that the manifold ∂M is also obtained by -1 surgery to 0-double of the left handed trefoil knot (Figure 7). Another interesting side fact which the reader can check is that the manifold M is a 2-fold branched covering of the cusp manifold (i.e. 4-ball with a 2-handle attached along the left handed trefoil knot with either (± 1)-framing, e.g. Figure 8) along a properly imbedded 2-disc (the obvious disc bounded by γ in Figure 8).

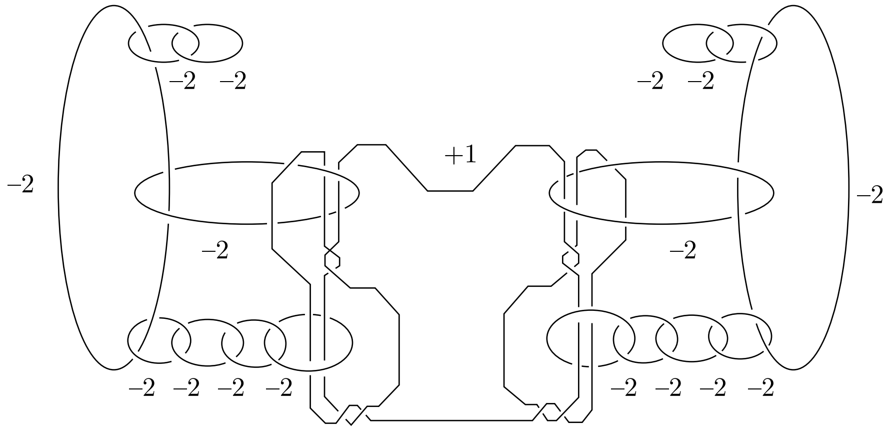

FIGURE 5

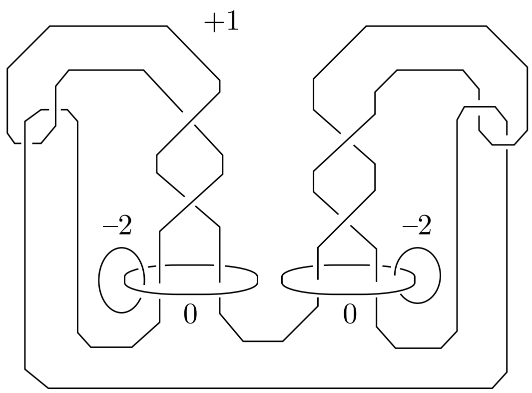

FIGURE 6

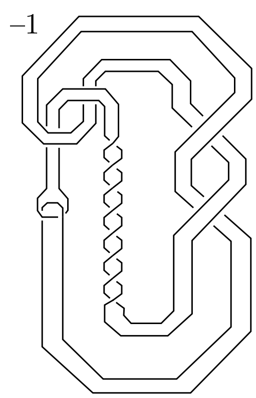

FIGURE 7

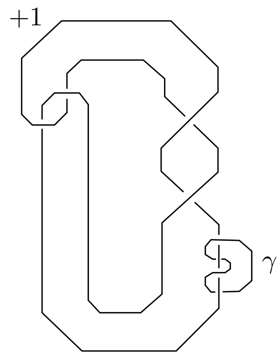

FIGURE 8

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