Notes on Geometric Topology (v.8.1)

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Preface

These notes, which I claim no originality, are compiled from many sources for the purpose of understanding geometric structures on manifolds. In recent years the differential geometric and analytic techniques have become topologists' tool as well. These notes are heavily influenced by many papers and books I red, and wonderful lectures I attended in the course of many decades most notably by S.S. Chern, B. Lawson and C.Taubes. While I have been working on the "4-Manifolds" book project, it has become clear that supplementing it with the notes of this type would be beneficial for understanding some of the constructions there (especially the part about the Seiberg-Witten invariants).

Contents

Chapter 1

Differential forms

1.1 Derivations

Let M be a smooth manifold. Let $T^0_0(M) = C^\infty(M)$ and consider the bundles

$$T^p_q(M) := (\otimes^p TM) \otimes (\otimes^q T^*M) \longrightarrow M$$

$$\Lambda^q(M) := \Lambda^q T^*(M) \longrightarrow M$$

The sections of $T^p_q(M)$, and $\Lambda^q(M)$, are called (p,q)-tensors, and q-forms, respectively.

$$\Gamma^p_q(M) = \Gamma(T^p_q(M))$$
$$\Omega^q(M) = \Gamma(\Lambda^q(M))$$
$$\Gamma_p(M) := \Gamma^0_p(M)$$

We have the projection operator $P : \Gamma_p(M) \longrightarrow \Omega^p(M) \subset \Gamma_p(M)$

$$P(\theta_1 \otimes \cdots \otimes \theta_p) = \frac{1}{p!} \sum_{\sigma \in S_p} \mathrm{Sgn}(\sigma) \theta_{s_1} \otimes \cdots \otimes \theta_{\sigma_p}$$

Define $\theta_1 \wedge \cdots \wedge \theta_k = P(\theta_1 \otimes \cdots \otimes \theta_k)$. P has the properties:

(1) P is linear and onto

(2) $Pf = f$ for $f \in \Omega^k(M)$

(3) $P^2 = P$

Definition 1.1. *A contraction is the map* $c_{ij} : T^p_q(M) \longrightarrow T^{p-1}_{q-1}(M)$ *given by*

$$\begin{aligned} &c_{ij}(X_1 \otimes \cdots \otimes X_p \otimes \omega_1 \otimes \cdots \otimes \omega_q) \\ &\quad = \omega_j(X_i)(X_1 \otimes \cdots \hat{X}_i \cdots \otimes X_p \otimes \omega_1 \cdots \hat{\omega}_j \cdots \otimes \omega_q) \end{aligned}$$

Definition 1.2. *A linear map* $D : \Gamma^*_*(M) \longrightarrow \Gamma^*_*(M)$ *is a derivation if*

(1) $D(\Gamma^p_q(M)) \subset \Gamma^p_q(M)$

(2) $D(T_1 \otimes T_2) = (DT_1) \otimes T_2 + T_1 \otimes (DT_2)$

(3) $D \circ c_{ij} = c_{ij} \circ D$ *for each contraction* c_{ij}

Lemma 1.3. *A derivation is determined by its restrictions to* $\Gamma^0_0(M)$, $\Gamma^1_0(M)$

Proof. D is determined on $\Gamma^0_1(M) = \Gamma(T^*M)$: Let $\omega \in \Gamma^0_1(M)$, then for $Y \in \Gamma^1_0(M)$

$$\begin{aligned} D(\omega(Y)) &= D(c_{11}(Y \otimes \omega)) = c_{11}(D(Y \otimes \omega)) \\ &= c_{11}(D(Y) \otimes \omega + Y \otimes D(\omega)) \\ &= \omega(D(Y)) + D(\omega)(Y) \end{aligned}$$

Hence $D(\omega)(Y) = D(\omega(Y)) - \omega(D(Y))$ is determined, and similarly

$$D(Y_1 \otimes \cdots \otimes Y_r \otimes \theta_1 \otimes \cdots \otimes \theta_s) \quad \text{is determined by (2)}$$

□

Any diffeomorphism $f : M \to M$ induces $f_* : T_p(M) \to T_{f(p)}(M)$. This extends $f^* : \Gamma^0_s(M) \to \Gamma^0_s(M)$ by $\omega \to f^*(\omega)$, and $f^* : \Gamma^*_0(M) \to \Gamma^*_0(M)$ by

$$f^*(Y) = (f^{-1})_*(Y)$$

Hence, any diffeomorphism $f : M \to M$ induces $f^* : \Gamma^r_s(M) \longrightarrow \Gamma^r_s(M)$.

1.2 Lie derivative

Let $X \in \Gamma^1_0(M)$ and let $\varphi_t : M \longrightarrow M$ be the local 1-parameter group of diffeomorphisms. Then the Lie derivative $\mathcal{L}_X : \Gamma^r_s(M) \longrightarrow \Gamma^r_s(M)$ of X is defined by

$$\begin{aligned} \mathcal{L}_X(\alpha)_p &= \lim_{t \longrightarrow 0} \frac{1}{t}[\varphi_t^*(\alpha) - \alpha] \\ &= \frac{d}{dt}\varphi_t^*(\alpha)|_{t=0} \end{aligned}$$

Remark 1.4. *$\tau \in \Gamma^r_s(M)$ is invariant under φ_t, i.e. $\varphi_t^*(\tau) = \tau \Longleftrightarrow \mathcal{L}_X(\tau) = 0$.*

Proposition 1.5. *$\mathcal{L}_X$ satisfies the following*

(1) $\mathcal{L}_X$ is a derivation

(2) $\mathcal{L}_X(f) = X(f)$ for $f \in C^\infty(M)$

(3) $\mathcal{L}_X(Y) = [X, Y]$

Remark 1.6. *By the Lemma 1.3, properties (1), (2), (3) uniquely determine $\mathcal{L}_X$*

Proof. Write $f \circ \varphi_t(x) = f(x) + tg_t(x)$

(2)
$$\begin{aligned} \mathcal{L}_X(f) &= \frac{d}{dt}\varphi_t^*(f)|_{t=0} \\ &= \frac{d}{dt}\varphi_t(f)|_{t=0} \underset{def}{=} X(f) \end{aligned}$$

In particular $g_0 = X(f)$.

(3)
$$\begin{aligned} \mathcal{L}_X(Y)_p &= \lim_{t \longrightarrow 0} \frac{1}{t}[\varphi_t^*(Y)_p - Y_p] \\ &= \lim_{t \longrightarrow 0} \frac{1}{t}[(\varphi_{-t})_* Y_{\varphi_t(p)} - Y_p] \\ &= \lim_{t \longrightarrow 0} \frac{1}{-t}[Y_p - (\varphi_{-t})_* Y_{\varphi_t(p)}] \\ &= \lim_{h \longrightarrow 0} \frac{1}{h}[Y_p - (\varphi_h)_* Y_{\varphi_{-h}(p)}] \end{aligned}$$

$$\begin{aligned}
\mathcal{L}_X(Y)_p(f) &= \lim_{h\longrightarrow 0}\frac{1}{h}[Y_p(f) - Y_{\varphi_{-h}(p)}(\varphi_h \circ f)] \\
&= \lim_{h\longrightarrow 0}\frac{1}{h}[Y_p(f) - Y_{\varphi_{-h}(p)}(f)] - \lim_{h\longrightarrow 0} Y_{\varphi_{-h}(p)}(g_h) \\
&= \lim_{k\longrightarrow 0}\frac{1}{k}[Y_{\varphi_k(p)}(f) - Y_p(f)] - Y_p X(f) \\
&= (XY - YX)(f) = [X,Y](f)
\end{aligned}$$ □

Definition 1.7. *Let $L = X \otimes \omega \in \Gamma_1^1(M)$. Then $\forall\ p \in M\ L_p$ can be considered as a linear map $L_p : T_p(M) \longrightarrow T_p(M)$, and hence it extends uniquely to a derivation*

$$L^* : \Gamma_q^p(M) \longrightarrow \Gamma_q^p(M)$$

Proposition 1.8. *Every derivation $D : \Gamma_s^r(M) \longrightarrow \Gamma_s^r(M)$ can be decomposed as the sum $D = \mathcal{L}_X + L^*$, where $L \in L_1'(M)$, and X is a vector field.*

Proof. On $\Gamma_0^0(M) = C^\infty(M)$ we have

$$D : C^\infty(M) \longrightarrow C^\infty(M)$$

(1) $D(af + bf) = aD(f) + bD(g) \quad a, b \in \mathbb{R}$

(2) $D(fg) = D(f)g + fD(g)$

Hence $D(f) = X(f)$ for some $X \in \Gamma_0^1(M)$. Let $D' = D - \mathcal{L}_X$ then for $Y \in \Gamma^1(M)$

$$D'(fY) = (D'f)Y + f(D'Y) = f(D'Y)$$

So D' is $C^\infty(M)$-linear on $\Gamma^1(M)$. Let $D'|_{\Gamma^1(M)} = L$, then $D' = L^*$ since D' and L^* agree on $\Gamma_0^0(M)$ and $\Gamma_0^1(M)$. □

Definition 1.9. *A linear $D : \Omega^*(M) \longrightarrow \Omega^{*+r}(M)$ is a derivation of degree r if:*

$$D(\omega_1 \wedge \omega_2) = D(\omega_1) \wedge \omega_2 + \omega_1 \wedge D(\omega_2)$$

D is a skew derivation of degree r if for $\omega_1 \in \Omega^p(M)$ it satisfies:

$$D(\omega_1 \wedge \omega_2) = D(\omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge D(\omega_2)$$

Example 1.10. $(i_X\omega)(Y_1,\dots,Y_0) = \omega(X,Y_1,\dots,Y_0)$.

Any (skew) derivation is determined by its action on $\Omega^0(M) = C^\infty(M)$ *and* $\Omega^1(M)$.

Proposition 1.11. *Let* D, D' *be derivations of degree* d, d' *and* S, S' *be skew-derivatives of degree* s, s'. *Then*

(a) $[D,D'] = D\circ D' - D'\circ D$ *is a derivation of* $\deg\ d+d'$

(b) $[D,S]$ *is a skew derivation of* $\deg\ d+s$

(c) $S\circ S' + S'\circ S$ *is a derivation of* $\deg\ s+s'$

Example 1.12. *Three natural derivations of* $\Omega^*(M)$ *are:*

(1) *Exterior derivative* $d =$ *skew derivation of degree* 1

(2) *Lie derivative* $\mathcal{L}_X =$ *derivation of degree* 0

(3) *Interior differentiation* $i_X = X\lrcorner =$ *skew derivation of degree* -1

Proposition 1.13. *There is a unique skew derivation* $d : \Omega^*(M) \longrightarrow \Omega^*(M)$ *of degree* 1 *satisfying*

(1) $d(f) = df\ \forall\ f \in \Omega^0(M) = C^\infty(M)$

(2) $d^2 = 0$

Proposition 1.14. *The derivation* $\mathcal{L}_X : \Gamma_k(M) \longrightarrow \Gamma_k(M)$ *descends to a derivation* $\mathcal{L}_X : \Omega^k(M) \longrightarrow \Omega^k(M)$ *such that the following commutes:*

$$\begin{array}{ccc} \Gamma_k(M) & \xrightarrow{\mathcal{L}_X} & \Gamma_k(M) \\ \downarrow P & & \downarrow P \\ \Omega^k(M) & \xrightarrow{\mathcal{L}_X} & \Omega^k(M) \end{array}$$

where P *is the usual projection,* $P^2 = P$. *Also* $d\circ\mathcal{L}_X = \mathcal{L}_X\circ d$, *i.e. it commutes with* d.

Proof. For $\omega \in \Omega^k(M)$ we claim:

$$(\mathcal{L}_X\omega)(X_1,..,X_k) = X(\omega(X_1,..,X_k)) - \sum \omega(X_1,..,X_{i-1},[X,X_i],X_{i+1},..,X_k) \quad (1.1)$$

It suffices to check this on $\Omega^0(M)$ and $\Omega^1(M)$ and show both side is a derivation

$$\text{On} \quad \Omega^0(M) = C^\infty(M), \;\; \mathcal{L}_X(f) = X(f)$$

$$\text{On} \;\; \Omega^1(M), \;\; (\mathcal{L}_X\omega)(Y) = X(\omega(Y)) - \omega[X,Y]$$

The last line follows from $c_{11}(\mathcal{L}_X(Y \otimes \omega)) = \mathcal{L}_X c_{11}(Y \otimes \omega)$

$$\begin{aligned} \text{Hence} \;\; \mathcal{L}_X(\omega(Y)) &= c_{11}(\mathcal{L}_X(Y) \otimes \omega + Y \otimes \mathcal{L}_X(\omega)) \\ &= \omega(\mathcal{L}_X(Y)) + \mathcal{L}_X(\omega)(Y) \end{aligned}$$

It is a derivation so the result follows. Also $d \circ \mathcal{L}_X = \mathcal{L}_X \circ d$. since $\varphi_t^* \circ d = d \circ \varphi_t^*$ and

$$\mathcal{L}_X(\omega) = \lim_{t \longrightarrow 0} \frac{1}{t}[\varphi_t^*(\omega) - \omega] \qquad \square$$

Proposition 1.15. *$\mathcal{L}_X : \Omega^*(M) \longrightarrow \Omega^*(M)$ is the unique derivation which commutes with d.*

Proof. If $D : \Omega^*(M) \longrightarrow \Omega^*(M)$ is any such derivation then $D : C^\infty(M) \longrightarrow C^\infty(M)$ is a derivation $\therefore \exists\, X \in \Gamma^1(M)$ with $Df = X(f)$ for all $f \in C^\infty(X)$. Let $D' = D - \mathcal{L}_X$

$$D' \equiv 0 \;\; \text{on} \;\; \Omega^0(M)$$

It suffices to show $D' \equiv 0$ on $\Omega^1(M)$. Let $\omega \in \omega^1(M)$, $\omega = f \cdot dg$ (locally $\omega = \sum a_i dX^i$) where $f, g \in C^\infty(M)$

$$D'(\omega) = D'(f \cdot dg) = D'(f)dg + fD'(dg) = fD'(dg) = fdD'(g) = 0 \quad \square$$

Proposition 1.16. *(1) $\mathcal{L}_X = d \circ i_X + i_X \circ d$ (Cartan formula)*

(2) $d\alpha(X_0,..,X_p) = \sum(-1)^i \mathcal{L}_{X_i}(\alpha(X_0,\ldots,\hat{X}_i,..,X_p))$

$$+ \sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j],X_0,..,\hat{X}_i,..,\hat{X}_j,..,\hat{X}_p)$$

Example 1.17. $d\alpha(X,Y) = X_0(\alpha(X_1)) - X_1(\alpha(X_0)) - \alpha[X_0, X_1]$

Proof. Check that both sides are derivations, and evaluate them on $\Omega^0(M)$ and $\Omega^1(M)$, by recalling $i_X(f) = 0$ and $i_X(\omega) = \omega(X)$ □

Remark 1.18. *We can alternatively define $\Gamma^1(M)$ as the $C^\infty(M)$-module:*

$$\Gamma^1(M) = \{\text{Linear maps } X : C^\infty(M) \longrightarrow C^\infty(M) \mid X(fg) = X(f)g + fX(g)\}$$

$$\Gamma_1(M) = \{C^\infty(M)\text{-linear maps } \Gamma^1(M) \longrightarrow C^\infty(M)\}$$

For example $\omega \in \Gamma_1(M)$ gives the map $X \longmapsto \{p \in M \longmapsto \omega_p(X_p)\}$

$$\Gamma^r_s(M) = \{C^\infty(M)\text{-multilinear maps } \underbrace{\Gamma_1(M) \times \cdots \times \Gamma_1(M)}_{r} \times \underbrace{\Gamma^1(M) \times \cdots \times \Gamma^1(M)}_{s} \longrightarrow C^\infty(M)\}$$

$$\Omega^p(M) = \{\text{skew symmetric } C^\infty(M)\text{-multilinear maps } \underbrace{\Gamma^1(M) \times \cdots \times \Gamma^1(M)}_{p-times} \longrightarrow C^\infty(M)\}$$

$$\Gamma((T^p_q M)^* \otimes T^r_s(M)) = \{C^\infty(M)\text{-multilinear maps } \Gamma^p_q(M) \longrightarrow \Gamma^r_s(M)\}$$

Remark 1.19. *The map $\Gamma^1(M) \times \Gamma^1(M) \longrightarrow \Gamma^1(M)$ given by*

$$(X,Y) \longmapsto [X,Y]$$

is not a tensor since $[fX, gY] = fg[X,Y] + f(Xg)Y - g(Yf)X \neq fg[X,Y]$

Definition 1.20. *A metric on a smooth manifold M is a 2-tensor $g \in \Gamma^0_2(M)$, which is positive definite and symmetric i.e. $g(X,Y) = g(Y,X)$, and $g(X,X) > 0$ whenever $X \neq 0$. Evaluation of g on vector fields X, Y is denoted by*

$$g(X,Y) = \langle X, Y\rangle$$

Evey smooth M admits a metric g (define it locally piece it together with partition of unity). A manifold with a metric (M,g) is called a Riemanian manifold.

1.3 Covariant differentiation

Definition 1.21. *A connection on $TM \to M$ is an $\mathbb{R}$-bilinear map*

$$\Gamma(TM) \times \Gamma(TM) \longrightarrow \Gamma(TM)$$

$$(X, Y) \mapsto \nabla_X(Y) \;\; \textit{satisfying}:$$

(1) $\nabla_{fX}(Y) = f\nabla_X(Y)$

(2) $\nabla_X(fY) = (Xf)Y + f(\nabla_X Y)$

where $f \in C^\infty(M)$. Therefore a connection is a $C^\infty(M)$-linear map

$$\begin{array}{ccc} \Gamma^1(M) & \longrightarrow & \{\text{Derivations of } \Gamma^1(M)\} \\ X & \longmapsto & \nabla_X \end{array}$$

Definition 1.22. *A connection is called torsion free if the following tensor is zero*

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

Theorem 1.23. *(Levi-Civita connection) Every Riemannian manifold (M, g) has a unique torsion free connection consistent with the metric, i.e.*

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

Proof.

$$\begin{aligned} X\langle Y, Z\rangle &= \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ Y\langle Z, X\rangle &= \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle = \langle \nabla_Y Z, X\rangle + (\langle Z, \nabla_X Y\rangle + \langle Z, [Y, X]\rangle) \\ Z\langle X, Y\rangle &= \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle = (\langle \nabla_X Z, Y\rangle + \langle [Z, X], Y\rangle) + (\langle X, \nabla_Y Z\rangle + \langle X, [Z, Y]\rangle) \end{aligned}$$

Subtracting the third line from the sum of first two lines gives

$$X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle = 2\,\langle \nabla_X Y, Z\rangle + \langle Z, [Y, X]\rangle - \langle X, [Z, Y]\rangle - \langle Y, [Z, X]\rangle$$

So $2\,\langle \nabla_X Y, Z\rangle = X\langle Y, Z\rangle + Y\langle Z, X\rangle - Z\langle X, Y\rangle - \langle X, [Y, Z]\rangle + \langle Y, [Z, X]\rangle + \langle Z, [X, Y]\rangle$

□

Proposition 1.24. *Let (M,g) be a Riemannian manifold. In local coordinates write:*

$$g = \sum g_{ij}\, dx^i \otimes dx^j, \;\; X = \sum a^j \frac{\partial}{\partial x^j}, \;\; Y = \sum b^k \frac{\partial}{\partial x^k} \quad \Longrightarrow$$

$$\nabla_X(Y) = \sum_{i=1}^{n} (\sum_{j=1}^{n} a^j \frac{\partial b^i}{\partial x^j} + \sum_{j,k=1}^{n} \Gamma^i_{jk} a^j b^k) \frac{\partial}{\partial x^i}$$

where Γ^i_{jk} *are defined by* $D_{\frac{\partial}{\partial x^j}}(\frac{\partial}{\partial x^k}) = \sum_i \Gamma^i_{jk} \frac{\partial}{\partial x^i}$. *If we call* $\partial_k g_{ij} = \frac{\partial}{\partial x^k}(g_{ij})$ *then*

$$\Gamma^i_{jk} = \frac{1}{2} \sum_{\ell=1}^{n} g^{li}(\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk}) \; , \; where \;\; (g^{jl}) = (g_{jl})^{-1}$$

Proof. $\Gamma^i_{jk} = \Gamma^i_{kj}$ since $[\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}] = 0$ By the previous proposition

$$2g\langle \nabla_{\frac{\partial}{\partial x^j}}(\frac{\partial}{\partial x^k}), \frac{\partial}{\partial x^i} \rangle = \frac{\partial}{\partial x^j} g\langle \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^i} \rangle + \frac{\partial}{\partial x^k} g\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle - \frac{\partial}{\partial x^i} g\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k} \rangle$$

$$2g_{\ell i}\Gamma^\ell_{jk} = \partial_j g_{ki} + \partial_k g_{ij} - \partial_i g_{jk}$$

$$\Gamma^i_{jk} = \frac{1}{2} \sum_{\ell=1}^{n} g^{li}(\partial_j g_{kl} + \partial_k g_{lj} - \partial_l g_{jk}) \quad \square$$

Let $\nabla : \Gamma(TM) \longrightarrow \Gamma(TM \otimes T^*M)$ be a connection on TM, and $X \in \Gamma^1(M)$ then

$$\nabla_X : \Gamma(TM) \longrightarrow \Gamma(TM)$$

extends by the rules of derivation to any tensor field: $\nabla_X : \Gamma^r_s(M) \longrightarrow \Gamma^r_s(M)$

(1) $\nabla_X \circ c_{ij} = c_{ij} \circ \nabla_X$

(2) $\nabla_X(S \otimes T) = \nabla_X(S) \otimes T + S \otimes \nabla_X(T)$

For example, $\nabla_X : \Gamma_p(M) \longrightarrow \Gamma_p(M)$ then

$$(\nabla_X S)(X_1, .., X_p) = X(S(X_1, .., X_p)) - \sum_{i=1}^{p} S(X_1, .., X_{i-1}, \nabla_X(X_i), .., X_p) \tag{1.2}$$

In paticular if $\theta \in \Gamma_1(M)$, then $(\nabla_X \theta)(X_1) = X\theta(X_1) - \theta(\nabla_X(X_1))$ because

$$c_{11}\nabla_X(X_1 \otimes \theta) = c_{11}(X_1 \otimes \nabla_X \theta + \nabla_X(X_1) \otimes \theta) = (\nabla_X \theta)(X_1) + \theta(\nabla_X(X_1))$$
$$\nabla_X c_{11}(X_1 \otimes \theta) = \nabla_X(\theta(X_1)) = X\theta(X_1)$$

Define $\nabla : \Gamma^p_q(M) \longrightarrow \Gamma^p_{q+1}(M)$ by

$$(\nabla T)(\omega^1, \dots, \omega^p_1, X_1, \dots, X_{q+1}) = (\nabla_{X_1} T)(\omega^1, \dots, \omega^p_1, X_2, \dots, X_{q+1})$$

Proposition 1.25. *Let (M, g) be a Riemannian manifold and $X, X_1, \dots, X_p \in \Gamma^1(M)$, and $\omega \in \Omega^p(M)$. Then for $S \in \Gamma^0_p(M)$ we have:*

(a) $(\mathcal{L}_X S)(X_1, .., X_p) = (\nabla_X S)(X_1, .., X_p) \; + \; \sum_{i=1}^p S(X_1, .., X_{i-1}, \nabla_{X_i}(X), .., X_p)$

(b) $d\omega(X_0, X_1, .., X_p) = \sum_{i=0}^p (-1)^i \nabla_{X_i}\omega(X_0, .., X_{i-1}, \hat{X}_i, X_{i+1}, .., X_p)$

Proof. (a) follows from (1.1) and (1.2), and (b) follows from Proposition 1.16, or from a direct calculation. □

Lemma 1.26. *Let ∇ be the connection on TM^* induced by the Levi-Cevita connection, then the following commutes*

$$\begin{array}{ccc} \Gamma(T^*M) & \xrightarrow{\nabla} & \Gamma(T^*M \otimes T^*M) \\ d \searrow & & \downarrow P \text{ (projection)} \\ & & \Gamma(\Lambda^2 T^*M) \end{array}$$

Proof.

$$\begin{aligned} \nabla_Y c_{11}(X \otimes \theta) &= c_{11}\nabla_Y(X \otimes \theta) \\ &= c_{11}(\nabla_Y(X) \otimes \theta + X \otimes \nabla_Y(\theta)) \\ \nabla_Y(\theta(X)) &= \theta(\nabla_Y X) + (\nabla_Y \theta)(X) \\ \text{So } (\nabla_Y \theta)(X) &= Y(\theta(X)) - \theta(\nabla_Y(X)) \end{aligned}$$

$$
\begin{aligned}
d\theta(Y,X) &= Y(\theta(X)) - X(\theta(Y)) - \theta[Y,X] \\
&= Y(\theta(X)) - X(\theta(Y)) - \theta(\nabla_Y X - \nabla_X Y) \\
&= Y(\theta(X)) - \theta(\nabla_Y X) - (X(\theta(Y)) - \theta(\nabla_X Y)) \\
&= (\nabla_Y \theta)(X) - (\nabla_X \theta)(Y) \\
&= \nabla\theta(Y,X) - \nabla\theta(X,Y) \\
&= P\nabla(\theta)(Y,X) \qquad \square
\end{aligned}
$$

Definition 1.27. *Let (M,g) be a Riemaninan manifold, define the "adjoint"*

$$d^* : \Omega^{p+1}(M) \to \Omega^p(M)$$

by the equation $\langle d\alpha, \beta\rangle = \langle \alpha, d^*\beta\rangle$, for all $\alpha \in \Omega^p(M)$ and $b \in \Omega^{p+1}(M)$

Remark 1.28. *Let (M,g) be a Riemannian manifold and $\{e_0, .., e_m\}$ be local orthonormal basis of vector fields in $\Gamma^1(M)$ and $\{e^1, .., e^m\}$ the dual basis of 1-forms, then*

$$d\omega = \sum_{i=1}^{m} e^i \wedge \nabla_{e_i} \tag{1.3}$$

$$d^*\omega = -\sum_{i=1}^{m} e_i \lrcorner \nabla_{e_i} \tag{1.4}$$

1.4 Curvature of a Riemanninan manifold

Definition 1.29. *Let (M,g) be a Riemanian manifold, and ∇ be the LeviCevita connection. For vector fields $X, Y, Z \in \Gamma^1(M)$ we define the curvature R by*

$$
\begin{aligned}
R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\
R(X,Y) &= [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}
\end{aligned}
$$

It follows from the definition that, R is a $C^\infty(M)$-multilinear map

$$\Gamma^1(M) \times \Gamma^1(M) \times \Gamma^1(M) \longrightarrow \Gamma^1(M)$$

We can view $(X, Y) \mapsto R(X, Y)$ as an endemorphism valued 2-form $R \in \Omega^2(End(TM))$

Chapter 2

Principal Bundles

Definition 2.1. *Let G be a Lie group. A principal G-bundle $P \stackrel{\pi}{\longrightarrow} X$, over a smooth manifold X, is a smooth manifold P along with a map π such that, there is an open covering $\mathcal{U} = \{U_\alpha\}$ of M satisfying:*

(a) For each $x \in X$, $\pi^{-1}(x)$ is a lie group which is isomorphic to G.

(b) There is a free action $P \times G \to P$, $(p, g) \longmapsto pg$ (free means $pg = p \Leftrightarrow g = 1$). This action commutes with π, i.e. $\pi(pg) = \pi(p)$.

(c) For each α, there is a diffeomorphism h_α, making the following commute:

$$\begin{array}{ccccc} P|_{U_\alpha} := \pi^{-1}(U_\alpha) & & \underset{\approx}{\stackrel{h_\alpha}{\longrightarrow}} & & U \times G \\ & \pi \searrow & & \swarrow \pi_1 & \\ & & U & & \end{array}$$

(π_1 is the projection to the first factor). Furthermore h_α commutes with G-action, that is if $h_\alpha(p) = (\pi(p), \bar{h}_\alpha(p))$ then $h_\alpha(pg) = (\pi(p), \bar{h}_\alpha(p)g)$, i.e., $\bar{h}_\alpha(pg) = \bar{h}_\alpha(p)g$.

In short we can denote the principal bundle by $P = \{U_\alpha, h_\alpha\}$. For example, $M \times G \to M$ is a principal G bundle by the obvious left multiplication of the second factor. Also, if H is a closed subgroup of G, then the quotient map $G \to G/H$ gives a principal H-bundle, by the obvious left action by H. The following are more nontrivial examples.

Example 2.2. *The Hopf fibration* $S^3 \xrightarrow{\pi} S^2$ *is a principle* $G = S^1$*-bundle, where*

$$\begin{aligned} S^3 &= \{(u,v) \in \mathbb{C}^2 \mid |u|^2 + |v|^2 = 1\} \\ S^1 &= \{e^{i\theta} \in \mathbb{C} \mid 0 \le \theta \le 2\pi\} \end{aligned}$$

$$S^1 \text{ acts by } ((u,v), e^{i\theta}) \longmapsto (ue^{i\theta}, ve^{i\theta}) \text{ and}$$

$$\pi(u,v) = (2u\bar{v}, |u|^2 - |v|^2) \in \mathbb{C} \times \mathbb{R} = \mathbb{R}^3$$

Notice in particular, π *takes* S^3 *to the unit sphere* S^2 *in* $\mathbb{R}^3$*, and* $\pi(ue^{i\theta}, ve^{i\theta}) = \pi(u,v)$.

$$h_\pm^{-1} : \mathbb{C} \times S^1 \longrightarrow S^3$$

$$h_+^{-1}(z, e^{i\theta}) = \frac{1}{(1+|z|^2)^{1/2}}(ze^{i\theta}, e^{i\theta})$$

$$h_-^{-1}(w, e^{i\theta}) = \frac{1}{(1+|w|^2)^{1/2}}(e^{i\theta}, we^{i\theta})$$

From this we can check that the following commutes (the same holds for h_-*).*

$$\begin{array}{cccc} (z, e^{i\theta}) & \xrightarrow{h_+^{-1}} & \left(\dfrac{z}{\sqrt{1+|z|^2}}e^{i\theta},\ \dfrac{1}{\sqrt{1+|z|^2}}e^{i\theta}\right) \in S^3 & \\ \downarrow & \swarrow & \downarrow \pi & \\ z & \underset{\text{stereographic projection}}{\longleftarrow} & \left(\dfrac{2z}{|z|^2+1},\ \dfrac{|z|^2-1}{|z|^2+1}\right) \in S^2 & \end{array} \tag{2.1}$$

The two stereographic projections and their inverses give two charts of S^2*:*

$$\left.\begin{aligned} (u,v,w) &\longmapsto \left(\frac{u}{1-w}, \frac{v}{1-w}\right) = z \\ z = (x,y) &\longmapsto \frac{1}{1+|z|^2}(2z,\ |z|^2 - 1) \end{aligned}\right\} +$$

$$\left.\begin{array}{rcl}(u,v,w) & \longmapsto & \left(\dfrac{u}{1+w},\ \dfrac{-v}{1+w}\right) = z \\ z=(x,y) & \longmapsto & \dfrac{1}{1+|z|^2}(2\bar{z},\ 1-|z|^2)\end{array}\right\}-$$

Hence we have $S^2 \approx \mathbb{C}\cup\mathbb{C}/\sim$*, with the identification* $z\sim 1/z$ *on* $\mathbb{C}-\{0\}$*.*

When $h_+^{-1}(z,e^{i\theta_+}) = h_-^{-1}(w,e^{i\theta_-})$*, we compute from the identities*

$$\frac{z}{\sqrt{1+|z|^2}}e^{i\theta_+} = \frac{1}{\sqrt{1+|w|^2}}e^{i\theta_-}$$
$$\frac{1}{\sqrt{1+|z|^2}}e^{i\theta_+} = \frac{w}{\sqrt{1+|w|^2}}e^{i\theta_-}$$

$$\Rightarrow \quad (z,e^{i\theta_+}) = (\frac{1}{w},\ \frac{w}{|w|}e^{i\theta_-})$$

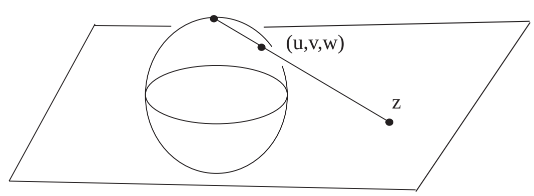


Figure 2.1: Stereographic projection

Example 2.3. *Hopf fibering* $S^7 \xrightarrow{\pi} S^4$*, with* $G=SU(2)=Sp(1)=S^3$*:*

$$\begin{array}{rcl} S^7 & = & \{(u,v)\in\mathbb{H}^2 \mid |u|^2+|v|^2=1\} \\ S^3 & = & \{g\in\mathbb{H} \mid |g|=1\}\end{array}$$

where $\mathbb{H}\cong\mathbb{R}^4$ *denotes the quaternions, and the action of* S^3 *is given by:*

$$(g,u,v)\longmapsto(ug,vg)$$

$$\pi(u,v) = (2u\bar{v},\ |u|^2-|v|^2)\in S^4\subset\mathbb{H}\times\mathbb{R}$$

where $v=v_0+v_1i+v_2j+v_3k$ *, and* $\bar{v}=v_0-v_1i-v_2j-v_3k$*.*

As in the previous example the following maps give trivializations:

$$\begin{aligned} h_\pm^{-1} : \mathbb{H} \times S^3 &\longrightarrow S^7 \\ h_+^{-1}(x,g) &= \frac{1}{(1+|x|^2)^{1/2}}(xg,\ g) \\ h_-^{-1}(y,g) &= \frac{1}{(1+|y|^2)^{1/2}}(g,\ yg) \end{aligned} \tag{2.2}$$

Also $h_\pm$ *and the steorographic projection give the commuting diagrams:*

$$\begin{array}{ccc} (x,g) & \overset{h_+^{-1}}{\longmapsto} & \frac{1}{(1+|x|^2)^{1/2}}(xg,\ g) \in S^7 \\ \downarrow & & \downarrow \pi \\ x & \longleftarrow & \left(\frac{2x}{1+|x|^2}, \frac{|x|^2-1}{|x|^2+1}\right) \in S^4 \end{array}$$

and as before when $h_+^{-1}(x,g_+) = h_-^{-1}(y,g_-)$ *we get:*

$$(x, g_+) \quad = \quad \left(\frac{1}{y},\ \frac{y}{|y|}\ g_-\right)$$

2.1 Cocycle representation of principal bundles

A nice way of describing a G-principal bundle $G \to P \xrightarrow{\pi} X$ is by cocyles: Let $\{U_\alpha\}$ be a covering of X and $\pi^{-1}(U_\alpha)$ trivial with trivializations $h_\alpha(p) = (\pi(p), \bar{h}_\alpha(p))$.

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{h_\alpha} & U_\alpha \times G \\ \pi \searrow & & \swarrow \pi_1 \\ & U_\alpha & \end{array}$$

If $x \in U_\alpha \cap U_\beta$ and $x = \pi(p)$ we can consider the G-equivariant composition:

$$(U_\alpha \cap U_\beta) \times G \xrightarrow{h_\alpha^{-1}} \pi^{-1}(U_\alpha \cap U_\beta) \xrightarrow{h_\beta} (U_\alpha \cap U_\beta) \times G$$

$$\begin{aligned} h_\beta \circ h_\alpha^{-1}(x, g_\alpha) &:= (x, \theta(x, g_\alpha)) \\ &= (x, \theta(x, 1) g_\alpha) \\ &:= (x, g_{\beta\alpha}(x) g_\alpha) \end{aligned}$$

The definitions imply that $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ satisfy the "cocycle" conditions:

(1) $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$

(2) $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \in G$ on $U_\alpha \cap U_\beta \cap U_\gamma$

Conversely given $\{U_\alpha \mid g_{\alpha\beta} : U_\alpha \cap U_\beta \to G\}$ satisfying (1) and (2), we can construct a principle G-bundle P by making the followingt identifications:

$$P = \coprod_\alpha (U_\alpha \times G)/\sim$$

$$(x, g_\alpha) \sim (y, g_\beta) \Longleftrightarrow x = y \text{ and } g_\beta = g_{\beta\alpha}(x) g_\alpha$$

We denote this bundle by $P = \{U_\alpha, g_{\alpha\beta}\}$, and its elements by $[x_\alpha, g_\alpha] \in P$. Cearly the sections $s : M \to P$ of this bundle P are given by the local data:

$$s = \{s_\alpha : U_\alpha \to G \mid s_\beta(x) = g_{\beta\alpha}(x) s_\alpha(x)\}$$

For example, for any $n \in \mathbb{Z}$ we can construct a principal $SU(2)$ bundle

$$P_n \to S^4 = (S^4_+, S^4_-)$$

where $S^4_\pm$ are the complements of the north and south poles of S^4 (which we identify with $\mathbb{H}$), with the transition functions $g_\pm(y) = \left(\frac{y}{|y|}\right)^n$, $y \in \mathbb{H} - \{0\}$.

2.2 Principal bundle of a vector bundle

Let $E \xrightarrow{\pi} X$ be a k^n-bundle, where $k = \mathbb{R}$ or $\mathbb{C}$. A frame on E is a vector space isomorphism $k^n \to E_x$. We can associate this vector bundle a principle $GL(n, k)$ bundle consisting of all frames ($n \times n$ nonsingular matrices over k) bundle:

$$P_E = \{(x, v) \mid x \in M, \ \ v = \{v_1, \ldots, v_n\} \text{ is a frame of } E|_x\} \to X$$

with the projection $(x, v) \mapsto x$. If $\{U_\alpha \mid h_\alpha : E_\alpha \longmapsto U_\alpha \times k^n\}$ are trivialization of E, by picking a fixed frame $\{e_1, \dots, e_n\}$ in k^n we can define trivializations of P_E by

$$\{U_\alpha \mid \hat{h}_\alpha : P_\alpha \longmapsto U_\alpha \times GL(n, k)\ \}$$

$$\hat{h}_\alpha(x, v) = (x, (v_{ij})_{i,j=1}^n)$$

where v_{ij} are the components of the vector $\bar{h}_\alpha(v_i)$ with respect to frame $\{e_1, \dots, e_n\}$. Hence if E has transition functions $\{g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow GL(n, k)\}$, then P_E is the principal $GL(n, k)$-bundle constructed from the cocycle data $\{U_\alpha, g_{\alpha\beta}\}$. If $E \longrightarrow M$ has a Euclidean (or Hermitian) metric $g(\ ,\ ) : E \longrightarrow [0, \infty)$, then $P_E = \{(x, v) \mid x \in M,\ v = \{v_1, \dots, v_n\}$ an orthogonal frame of $E|_x\}$ becomes an $O(n)$ (or $U(n)$) bundle. To a vector bundle $\pi : E \to M$ given by $\{U_\alpha \mid g_{\alpha\beta} : U_\alpha \cap U_\beta \to GL(n, \mathbb{R})\}$ we can associate a line bundle $L_E \to M$ by $\{\det g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{R}\}$. We call E *orientable* if L_E is trivial, and an *orientation* is a choice its trivialization. If $\pi : E \longrightarrow M$ is an oriented $\mathbb{R}^n$ bundle

$$P_E = \{(x, v) \mid x \in M,\ \ v \text{ an oriented frame in } \ E|_x\}$$

defines on $GL_+(n, \mathbb{R}) = \{g \in GL(n, \mathbb{R}) \mid \det(g) > 0\}$ bundle. An *oriented* frame $v(x) = \{v_1(x), \dots, v_n(x)\} \in E_x$ defines a section of $L_E \to X$ by

$$x \longmapsto (x,\ \ v_1(x) \wedge \cdots \wedge v_n(x))$$

2.3 Associated vector bundles of a principal bundle

Let G be a Lie group, $\pi : P \longrightarrow X$ a principal G-bundle with representation

$$\rho : G \longrightarrow GL(n, k)$$

Definition 2.4. *The vector bundle associated to P is $\tilde{\pi} : E(P, \rho) \to X$, where*

$$E(P, \rho) := P \times_\rho k^n = P \times k^n / \sim$$

$$\begin{array}{ccc} (p, v) & \sim & (pg^{-1}, \rho(g)v) \\ \tilde{\pi} \searrow & & \swarrow \tilde{\pi} \\ & \pi(p) & \end{array}$$

Let $x \in X$ and $p \in \pi^{-1}(x) \cong G$ with $p = (x, g)$, since $(x, g, v) \sim (x, 1, \rho(g)v)$, hence

$$\tilde{\pi}^{-1}(x) = \{\rho(g)v \mid v \in k^n,\ \rho \in G\} \ \cong\ k^n$$

Therefore if $\{U_\alpha, g_{\alpha\beta}\}$ describes P, then $\{U_\alpha, \rho(g_{\alpha\beta})\}$ is the cocyce describing $E(P, \rho)$

$$\rho(g_{\alpha\beta}) : U_\alpha \cap U_\beta \longrightarrow GL(n, k)$$

Example 2.5. *Let $\pi : P_n \to S^4$ be the $SU(2)$ bundle with transition functions $(y/|y|)^n$. We have seen before that P_1 is the Hopf bundle $S^7 \xrightarrow{\pi} S^4$. Now define $P = P_1 \oplus P_{-1} = \{(p, q) \in P_1 \times P_{-1} \mid \pi(p) = \pi(q)\}$ be the $SU(2) \times SU(2)$ bundle with transition function:*

$$g_{+,-}(x) = \left(\frac{x}{|x|}, -\frac{\bar{x}}{|x|}\right).$$

If $\rho : SU(2) \times SU(2) \to GL(4, \mathbb{R})$ is $\rho(g, f)v = g\,vf^{-1}$, then we claim

$$P \times_\rho \mathbb{H} \cong TS^4$$

Recall that in Example 2.3, the local trivializations of P_1 are given by $h_\pm$. Similarly we can describe the local trivializations of P_{-1} : $S^3 \longrightarrow S^7 \xrightarrow{\pi'} S^4 \subset \mathbb{R}^5$ by

$$\pi'(u, v) = (2\bar{v}u, |u|^2 - |v|^2)$$

$$(h'_\pm)^{-1} : \mathbb{H} \times SU(2) \longrightarrow S^7$$

$$(h'_+)^{-1}(x, g) = \frac{1}{\sqrt{1 + |x|^2}}(g^{-1}x, g^{-1})$$

$$(h'_-)^{-1}(y, f) = \frac{1}{\sqrt{1 + |y|^2}}(-f^{-1}, -f^{-1}y)$$

$$\begin{array}{ccc} (x, g) & \longmapsto & \dfrac{1}{\sqrt{1 + |x|^2}}(g^{-1}x, g^{-1}) \\ \downarrow & & \downarrow\ \pi' \\ x & \longmapsto & \left(\dfrac{2x}{|x|^2 + 1}, \dfrac{|x|^2 - 1}{|x|^2 + 1}\right) \end{array}$$

$$\begin{array}{ccc} (y,f) & \longmapsto & \dfrac{1}{\sqrt{1+|y|^2}}(-f^{-1},-f^{-1}y) \\ \downarrow & & \downarrow\ \pi' \\ y & \longmapsto & \left(\dfrac{2\bar{y}}{1+|y|^2},\ \dfrac{1-|y|^2}{1+|y|^2}\right) \end{array}$$

As before, $(h'_+)^{-1}(x,g) = (h'_-)^{-1}(y,f)$ *implies:*

$$(x,g) = (\frac{1}{y}, -\frac{\bar{y}}{|y|}f)$$

Then the trivializations $H_\pm : \mathbb{H} \times SU(2) \times SU(2) \to P_1 \oplus P_{-1}$ *are:*

$$H_+^{-1}(x,g_1,g_2) = (h_+^{-1}(x,g_1),\ (h'_+)^{-1}(x,g_2))$$

$$H_-^{-1}(y,f_1,f_2) = (h_-^{-1}(y,f_1),\ (h'_-)^{-1}(y,f_2))$$

and $(H_+)^{-1}(x,g_1,g_2) = (H_-)^{-1}(y,f_1,f_2)$ *gives*

$$(x,g_1,g_2) = (\frac{1}{y},\ \frac{y}{|y|}f_1,\ -\frac{\bar{y}}{|y|}f_2)$$

So the transition function of $P \times_\rho \mathbb{H}$ *is given by* $\rho(g_\pm)(y)v = -\frac{y}{|y|}\ v\ \frac{y}{|y|}$. *We can check that* $P \times_\rho \mathbb{H} \cong TS^4$ *by writing* $S^4 = \mathbb{H} \cup \mathbb{H}$ *with identification:*

$$\mathbb{H} - 0 \longrightarrow \mathbb{H} - 0 \quad x \sim 1/x$$

From $\frac{d}{dt}x(t)x^{-1}(t) = 0$ *we see that:*

$$\frac{dx(t)}{dt}x^{-1}(t) + x(t)\frac{dx^{-1}(t)}{dt} = 0$$

$$\frac{dx^{-1}}{dt}(t) = -x(t)^{-1}\frac{dx(t)}{dt}x^{-1}(t)$$

Hence $TS^4 = \mathbb{H} \times \mathbb{H} \cup \mathbb{H} \times \mathbb{H}$ *is given by the identifications*

$$(x,v) \sim (1/x, -x^{-1}vx^{-1})$$

Example 2.6. *Let $\mathbb{P}^n_{\mathbb{C}} = \mathbb{C}P^n$ be the complex projective space. The tangent bundle $T\mathbb{C}P^n \to \mathbb{C}P^n$ can be described as follows: Let L be a complex line in $\mathbb{C}^{n+1}$ corresponding to a point of $\mathbb{C}P^n$, and let $L^\perp$ be the dual hyperplane in $\mathbb{C}^{n+1}$. A nearby complex line to L_0 in $\mathbb{C}^{n+1}$ corresponds to a linear map $L_0 \to L_0^\perp$. Hence as a bundle $T\mathbb{C}P^n = Hom(L, L^\perp)$, where $L \to \mathbb{C}P^n$ is the canonical complex line bundle (fiber over L_0 are vectors in L_0), and $L^\perp \to \mathbb{C}P^n$ is the dual $\mathbb{C}^n$-bundle (fiber over L_0 are vectors in $L_0^\perp$) . By adding the trivial line bundle $\epsilon = Hom(L, L)$ to both sides we get ($\bar{L}$ is the dual bundle $Hom(L, \epsilon)$)*

$$\begin{aligned} T\mathbb{C}P^n \oplus \epsilon &= Hom(L, L^\perp) \oplus Hom(L, L) = Hom(L, L \oplus L^\perp) \\ &= Hom(L, \epsilon^{n+1}) = \bar{L} \oplus .. \oplus \bar{L} \quad (n+1 \text{ copies}) \end{aligned}$$

Definition 2.7. *For any principal G-bundle $P \to X$, we denote the bundle associated to the adjoint representation $\rho : G \to Aut(\mathfrak{g})$ by $ad(P) \to X$.*

2.4 Reducing the structure group

If $\rho : H \hookrightarrow G$ is a monomorphism between Lie groups, then any principal H-bundle $P_H \to M$ extends to a principal G-bundle $P_G = P_H \times_\rho G$

$$(p, g) \sim (ph^{-1}, \rho(h)g)$$

Conversely if $P_G \to M$ is given and $\rho : H \hookrightarrow G$ a closed subgroup. We can ask when $P_G = P_H \times_\rho G$? Let $\{U_\alpha, g_{\alpha\beta}\}$ be cocycle data for $P_G \to M$, then if such P_H exists there is a cocycle data $\{U_\alpha, h_{\alpha\beta}\}$ describing it. Then if $x_\alpha \in U_\alpha \cap U_\beta$

$$\begin{aligned} ([x_\alpha, h_\alpha], g) &\sim ([x_\alpha, 1], \rho(h_\alpha)g) = (x_\alpha, \bar{g}_\alpha) \\ &\sim ([x_\alpha, 1], \rho(h_{\alpha\beta}h_\beta)g) \\ &= ([x_\alpha, 1], \rho(h_{\alpha\beta})\rho(h_\beta)g) \\ &= ([x_\alpha, 1], \rho(h_{\alpha\beta})\bar{g}_\beta) = (x_\alpha, \rho(h_{\alpha\beta})\bar{g}_\beta) \end{aligned}$$

$$\text{Hence} \quad g_{\alpha\beta} = \rho(h_{\alpha\beta}) : U_\alpha \cap U_\beta \underset{h_{\alpha\beta}}{\longrightarrow} H \underset{\rho}{\longrightarrow} G$$

So $P_H = \{U_\alpha, h_{\alpha\beta}\}$ and $P_G = \{U_\alpha, \rho(h_{\alpha\beta})\}$, i.e. we must have a lifting

$$\begin{array}{ccc} & & H \\ & {}^{h_{\alpha\beta}}\nearrow & \downarrow \rho \\ U_\alpha \cap U_\beta & \xrightarrow{g_{\alpha\beta}} & G \end{array}$$

If $P \xrightarrow{\pi} M$, $P' \xrightarrow{\pi'} M$ are two isomorphic principal G-bundles given by the cocycle datas $\{U_\alpha, g_{\alpha\beta}\}$ and $\{U_\alpha, h_{\alpha\beta}\}$, then there are $\eta_\alpha : U_\alpha \to G$ such that $h_{\alpha\beta} = \eta_\alpha g_{\alpha\beta} \eta_\beta^{-1}$. This is because any bundle isomorphism θ

$$\begin{array}{ccc} P & \xrightarrow{\theta} & P' \\ & \searrow \quad \swarrow & \\ & M & \end{array}$$

gives $\bar{\theta}_\alpha : U_\alpha \times G \xrightarrow{\cong} U_\alpha \times G$ (local trivializations) and a commuting diagram

$$\begin{array}{ccccc} & (U_\alpha \cap U_\beta) \times G & \xrightarrow{\bar{\theta}_\alpha} & (U_\alpha \cap U_\beta) \times G & \\ \nearrow & & & & \nwarrow \\ P \mid U_\alpha \cap U_\beta & \downarrow & & \downarrow & P' \mid U_\alpha \cap U_\beta \\ \searrow & & & & \swarrow \\ & (U_\alpha \cap U_\beta) \times G & \xrightarrow{\bar{\theta}_\beta} & (U_\alpha \cap U_\beta) \times G & \end{array}$$

$$\begin{array}{ccc} (x, g_\alpha) & \longmapsto & (x, \theta_\alpha(x) g_\alpha) \\ \downarrow & & \downarrow \\ & & (x, h_{\beta\alpha}(x)\theta_\alpha(x) g_\alpha) \\ (x, g_{\beta\alpha}(x) g_\alpha) & & \\ & \searrow & \Vert \\ & & (x, \theta_\beta(x) g_{\beta\alpha}(x) g_\alpha) \end{array}$$

Therefore $P_G = \{U_\alpha, g_{\alpha\beta}\} \cong P_H \times_\rho G$ for a homomorphism $\rho : H \to G$, if there is $\theta_\alpha : U_\alpha \longrightarrow G$ such that $\theta_\alpha g_{\alpha\beta} \theta_\beta^{-1}$ lifts to H

$$\begin{array}{ccc} & & H \\ & \nearrow & \downarrow \rho \\ U_\alpha \cap U_\beta & \xrightarrow{\theta_\alpha g_{\alpha\beta} \theta_\beta^{-1}} & G \end{array}$$

Example 2.8. *The tangent frame bundle of $M^m \times N^n$ is a $GL(m+n)$-bundle which reduces to a $GL(m) \times GL(n)$ bundle.*

Example 2.9. *If M^{2n} is a complex manifold given by charts*

$$\{U_\alpha \mid (z_\alpha^i) : U_\alpha \to \mathbb{C}^n\}$$

with transition functions $z_\alpha^j = g_{\alpha\beta}^j(z_\beta^1, .., z_\beta^n)$ on $U_\alpha \cap U_\beta$, $z_\alpha^j = x_\alpha^j + i\, y_\alpha^j$, then

$$\begin{pmatrix} \dfrac{\partial}{\partial z_\beta^i} \\ \\ \dfrac{\partial}{\partial \bar{z}_\beta^i} \end{pmatrix} = \left(\begin{array}{c|c} \dfrac{\partial}{\partial z_\beta^i} g_{\alpha\beta}^j & 0 \\ \hline 0 & \dfrac{\partial}{\partial \bar{z}_\beta^i} \bar{g}_{\alpha\beta}^j \end{array}\right) \begin{pmatrix} \dfrac{\partial}{\partial z_\alpha^j} \\ \\ \dfrac{\partial}{\partial \bar{z}_\alpha^j} \end{pmatrix}$$

$$\det(g_{\alpha\beta}) = |\det(dg_{\beta\alpha})|^2 > 0$$

A manifold M^{2n} is almost complex if its $SO(2n)$ frame bundle reduces to $U(n)$-bundle. A manifold is almost quaternionic if $SO(4n)$ frame bundle reduces to $Sp(n)$-bundle.

2.5 Classifying Principal Bundles

Recall that principal G-bundles $P = \{U_\alpha, g_{\alpha\beta}\}$ and $P' = \{U_\alpha, g'_{\alpha\beta}\}$ are isomorphic, then $\exists\ \eta_\alpha : U_\alpha \to G$ such that $g_{\alpha\beta} = \eta_\alpha g'_{\alpha\beta} \eta_\beta^{-1} \Leftrightarrow \{g_{\alpha\beta}\} = \{g'_{\alpha\beta}\}$, as Čech cohomology clases. If G is abelian, then the set of isomorphism classes of principal G bundles $Iso_G(M)$ is an abelian group group by the multiplication rule $[P][P'] := \{U_\alpha, g_{\alpha\beta} g'_{\alpha\beta}\}$. Since G is abelian, we still have $(g_{\alpha\beta} g'_{\alpha\beta})(g_{\beta\gamma} g'_{\beta\gamma})(g_{\gamma\alpha} g'_{\gamma\alpha}) = 1$, and $-[P] = \{U_\alpha, g_{\alpha\beta}^{-1}\}$. If $G = U(1)$.

Theorem 2.10. *$Iso_{S^1}(M) \cong H^2(M; \mathbf{Z})$.*

Proof. Choose $P = \{U_\alpha, g_{\alpha\beta}\}$ with $U_\alpha \cap U_\beta$, contractible $g_{\alpha\beta} : U_\alpha \cap U_\beta \to S^1$

$$\text{Write} \quad g_{\alpha\beta} = e^{2\pi i \theta_{\alpha\beta}}, \ \text{where} \ \ \theta_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{R}$$

From the cocycle condition, $\theta_{\alpha\beta}$ is defined up to $\mathbb{Z}$, and $\theta_{\alpha\beta} + \theta_{\beta\gamma} + \theta_{\gamma\alpha} := \eta_{\alpha\beta\gamma} \in \mathbb{Z}$, where $\eta_{\alpha\beta\gamma}$ is defined on $U_\alpha \cap U_\beta \cap U_\gamma$. The coboundary of the cycle $\eta = \{\eta_{\alpha\beta\gamma}\}$ is

$$(\delta\eta)_{\alpha\beta\gamma\delta} = \eta_{\beta\gamma\delta} - \eta_{\alpha\gamma\delta} + \eta_{\alpha\beta\delta} - \eta_{\alpha\beta\gamma}$$

on $U_\alpha \cap U_\beta \cap U_\gamma \cap U_\delta$. From the definitions it follows that $\delta\eta = 0$, i.e. η is a cocyle. Hence $\varphi(\{g_{\alpha\beta}\}) = \{\eta_{\alpha\beta\gamma}\}$ defines a map to Čech cohomology group.

$$Iso_{U(1)}(M) \overset{\varphi}{\longmapsto} \check{H}^2(M : \mathbb{Z}) \cong H^2(M : \mathbb{Z})$$

Changing P by an isomorphism, corresponds to changing $\{\theta_{\alpha\beta}\} \longmapsto \{\theta_{\alpha\beta} + \theta_\alpha - \theta_\beta\}$ for some $\theta_\alpha : U_\alpha \longrightarrow \mathbb{R}$. This does not change this map φ. Clearly φ is a homomorphism. Recall as Čech cohomology classes $\{\eta_{\alpha\beta\gamma}\} = \{\eta'_{\alpha\beta\gamma}\}$ means that for some functions $a_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{Z}$ we have $\eta'_{\alpha\beta\gamma} = \eta_{\alpha\beta\gamma} + a_{\alpha\beta} + a_{\beta\gamma} + a_{\gamma\alpha}$.

Checking φ is onto: Let $\eta = \{\eta_{\alpha\beta\gamma}\}$ with $\delta\eta = 0$, let $\{\psi_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$, i.e. $\psi_\alpha : U_\alpha \to [0,1]$ smooth, $\psi_\alpha \equiv 0$ outside of U_α, $\sum \psi_\alpha \equiv 1$.

$$\text{Define} \quad g_{\alpha\beta} = \exp(2\pi i \sum_\delta \psi_\delta \eta_{\alpha\beta\delta}) \quad \Rightarrow$$

$$\begin{aligned} g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} &= \exp(2\pi i \sum_\delta \psi_\delta \eta_{\alpha\beta\delta} + \psi_\delta \eta_{\beta\gamma\delta} + \psi_\delta \eta_{\gamma\alpha\delta}) \\ &= \exp(2\pi i \sum \psi_\delta(\eta_{\alpha\beta\delta} + \eta_{\beta\gamma\delta} + \eta_{\gamma\alpha\delta})) \\ &= \exp(2\pi i \eta_{\alpha\beta\gamma}) = 1 \end{aligned}$$

It follows $\varphi(\{g_{\alpha\beta}\}) = \{\eta_{\alpha\beta\gamma}\}$. We can check that if we change $\{\eta_{\alpha\beta\gamma}\}$ by coboundary the corresponding bundle $\{g_{\alpha,\beta}\}$ changes by isomorphism as follows: $\{\eta_{\alpha\beta\gamma}\} = \{\eta'_{\alpha\beta\gamma}\}$ $\Rightarrow$ there are $a_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathbb{Z}$, $a := \{a_{\alpha\beta}\}$ with

$$\begin{aligned} \eta'_{\alpha\beta\gamma} &= \eta_{\alpha\beta\gamma} + (\delta a)_{\alpha\beta\gamma} = \eta_{\alpha\beta\gamma} + a_{\alpha\beta} + a_{\beta\gamma} + a_{\gamma\alpha} \quad \Rightarrow \\ g'_{\alpha\beta} &= g_{\alpha\beta} \exp(2\pi i \sum \psi_\delta(a_{\alpha\beta} + a_{\beta\delta} + a_{\delta\alpha})) \\ &= g_{\alpha\beta} \exp(2\pi i \sum \psi_\delta a_{\beta\delta}) \exp(-2\pi i \sum \psi_\delta a_{\alpha\delta}) \\ &= \eta_\alpha g_{\alpha\beta} \eta_\beta^{-1} \end{aligned}$$

where $\eta_\alpha = \exp(-2\pi i \sum_\delta \psi_\delta a_{\alpha\delta})$ so $\{g'_{\alpha\beta}\} = \{g_{\alpha\beta}\}$

Checking φ is injective: If $\eta_{\alpha\beta\gamma} = \theta_{\alpha\beta} + \theta_{\beta\gamma} + \theta_{\gamma\alpha} = 0$, then

$$g_{\alpha\beta} = e^{2\pi i \theta_{\alpha\beta}} = e^{2\pi i(\theta_{\alpha\gamma} - \theta_{\beta\gamma})}$$

Let $\eta_\alpha : U_\alpha \to S^1$ defined by $\eta_\alpha = \exp(2\pi i \sum \psi_\gamma \theta_{\alpha\gamma})$, then

$$\begin{aligned} \eta_\alpha \eta_\beta^{-1} &= \exp(2\pi i \sum_\gamma \psi_\gamma(\theta_{\alpha\gamma} - \theta_{\beta\gamma})) \\ &= \exp(2\pi i \sum_\gamma \psi_\gamma \theta_{\alpha\beta}) \\ &= \exp(2\pi i \, \theta_{\alpha\beta}) = g_{\alpha\beta} \ \Rightarrow \ \{g_{\alpha.\beta}\} = 0 \quad \square \end{aligned}$$

Let $\mathbb{A}$ be the sheaf of $\mathbb{C}$- valued smooth functions, and $\mathbb{A}^*$ be the sheaf of S^1- valued nonzero smooth functions on X. The exact sequence

$$0 \to \mathbb{Z} \to \mathbb{A} \xrightarrow{\alpha} \mathbb{A}^* \to 0$$

with $\alpha(f) = e^{2\pi i f}$, induces the long exact exact sequence in sheaf cohomology:

$$\check{H}^1(X; \mathbb{A}) \longrightarrow \check{H}^1(X; A^*) \longrightarrow \check{H}^2(X, \mathbb{Z}) \longrightarrow \check{H}^2(M, \mathbb{A})$$

Since $\mathbb{A}$ is a fine sheaf, the first and last groups are zero, and hence the middle map is an isomorphism. Theorem 2.10 specifically demonstrates this isomorphism. Also note that when $\mathfrak{U} = \{U_\alpha\}$ is a covering of X with property that all $U_\alpha \cap U_\beta$ are contractible then we can identify $\check{H}^2(X, \mathbb{Z}) = H^2(X, \mathbb{Z})$, and $Iso_{S^1}(M) = \check{H}^1(X; A^*)$ by definition.

2.6 Constructing Universal Principal G-bundles

For a compact Lie group G, we will construct the universal principal G-bundle $E_G \to B_G$. This means that every principal bundle over X is obtained by pulling back this bundle by some some map $X \to B_G$. Our method is basically Milnor's join construction as interpreted by Taubes. To motivate the general case treat the simple case of $G = S^1$.

2.6.1 Construction of B_{S^1}

Let X be compact smooth manifold, $P \xrightarrow{\pi} X$ be a principal S^1-bundle, $\{U_\alpha\}_{\alpha=1}^N$ be an open cover of X, $\{\psi_\alpha\}$ be a partions of unity sobordinate to this cover, h_α be trivializations of $P|_{U_\alpha}$, and p be the obvious projection, and ρ_α be the following composition:

$$\rho_\alpha : P|_{U_\alpha} \xrightarrow{h_\alpha} U_\alpha \times S^1 \xrightarrow{p} S^1 \subset \mathbb{C}$$

Then we can define the map $\Psi : P \longrightarrow ES^1_N := \mathbb{C}^N - \{0\} \simeq S^{2N-1}$ by

$$\Psi(p) = (\psi_1(\pi(p))\rho_1(p), \ldots, \psi_N(\pi(\rho))\rho_N(p))$$

$\mathbb{C}_*$ acts freely on ES^1_N by diagonal action $((z_1, \ldots, z_N)z) \to (z_1 z, \ldots, z_N z)$ with the quotient $\mathbb{P}^{N-1}_{\mathbb{C}}$. This induces a free action of S^1 on ES^1_N with quotient $BS^1_N := \mathbb{P}^{N-1}_{\mathbb{C}} \times \mathbb{R}_+$

$$S^1 \longrightarrow ES^1_N \quad \longrightarrow \quad BS^1_N, \ \ (z_1, \ldots, z_N) \longmapsto [z_1, \ldots, z_N]$$

$BS^1(N)$ has coordinate charts $U_j = \{ \ [z_1, \ldots, z_N] \mid z_j \in \mathbb{R}_+ \ \}$, for example

$$h_1 : ES^1_N \ |_{U_1} \xrightarrow{\cong} U_1 \times S^1$$

$$h_1^{-1}((z_2, \ldots, z_N), z) \longmapsto (z_1 z, z_2 z, \ldots, z_N z)$$

Since $\rho_1(pq) = \rho_1(p)q \Rightarrow \Psi$ is S^1-equivariant and the following commutes:

$$\begin{array}{ccccc} P & \xrightarrow{\Psi} & ES^1_N & \simeq S^{2N-1} \\ \downarrow \pi & & \downarrow \pi & \\ X & \xrightarrow{\psi} & BS^1_N & \simeq \mathbb{P}^{N-1}_{\mathbb{C}} \end{array}$$

where we define $\psi(x) = \pi\Psi(p)$ by picking any $p \in P|_x$. Also since Ψ is an isomorphism on each fiber we have an isomorphism:

$$P \cong \psi^* ES^1_N$$

Therefore when X compact, then any principal S^1 bundle $P \to X$ is isomorphic to $f^* ES^1_N$ for some $f : X \to BS^1_N$. We have the inclusions:

$$\begin{array}{ccccccccc} ES^1_N & \hookrightarrow & ES^1_{N+1} & \hookrightarrow & \cdots & \hookrightarrow & E_{S^1} & \simeq & S^\infty \\ \downarrow & & \downarrow & & & & \downarrow & & \downarrow \\ BS^1_N & \hookrightarrow & BS^1_{N+1} & \hookrightarrow & \cdots & \hookrightarrow & B_{S^1} & \simeq & \mathbb{P}^\infty_{\mathbb{C}} \end{array}$$

where $S^\infty = \{(z_1, z_2, \ldots) \mid$ finitely many z_i are nonzero, and $\sum |z_i|^2 = 1\}$, with the topology defined by $U \subset S^\infty$ is open $\iff U \cap S^{2N-1}$ open $\forall \ n$.

Similarly we can construct the universal S^3 bundle $E_{S^3} \to B_{S^3}$ by viewing $S^3 \subset \mathbb{H}$ as the unit quaternions, and writing any principal S^3-bundle $P \to X$ as a pullback of the quaternionic Hopf bundle $S^3 \to S^{4n-1} \to \mathbb{P}_{\mathbb{H}}^{N-1}$. In fact this generalizes to $GL(n, \mathbb{C})$.

2.6.2 Construction of $B_{GL(n,\mathbb{C})}$

As above given X, $\{U_\alpha\}_{\alpha=1}^N$, $\{\psi_\alpha\}_{\alpha=1}^N$, and $GL(n, \mathbb{C}) \subset M_{\mathbb{C}}(n) \cong \mathbb{C}^{n^2}$ ($n \times n$ complex matrices). Let h_α be trivializations of $P|_{U_\alpha}$, and ρ_α be the composition:

$$\rho_\alpha : P|_{U_\alpha} \longrightarrow U_\alpha \times GL(n, \mathbb{C}) \longrightarrow GL(n, \mathbb{C}) \hookrightarrow M_{\mathbb{C}}(n)$$

As before we can define $\Psi : P \longrightarrow \prod_N M_{\mathbb{C}}(n)$ by

$$\Psi(p) = (\psi_1(\pi(p))\rho_1(p), \ldots, \psi_N(\pi(p))\rho_N(p))$$

If $x = \pi(p)$, then $\psi_i(x) \neq 0$ for some $i \implies \psi_i(x)\rho_i(p)$ is an invertible matrix, so we can define a map $\Psi : P \longrightarrow E_N(n)$, where

$$\begin{aligned} E_N(n) &= \{(y_1, \ldots, y_N) \in \prod_N M_{\mathbb{C}}(n) \mid \exists\ j \text{ with } y_j \in GL(n, \mathbb{C})\} \\ &= \{(y_1, \ldots, y_N) \in \prod_N M_{\mathbb{C}}(n) \mid \sum_{j=1}^N |\det y_i|^2 > 0\} \end{aligned}$$

As above, there is a free action $GL(n, \mathbb{C}) \times E_N(n) \longrightarrow E_N(n)$

$$((y_1, \ldots, y_N), g) \longmapsto (y_1 g, \ldots, y_N g)$$

Let $BGL_N(n) = E_N(n)/GL(n, \mathbb{C})$. $BGL_N(n)$ is a manifold with local charts given by $U_j = \{[y_1, \ldots, y_N] \mid y_j \in GL(n, \mathbb{C})\}$ given by:

$$\prod_{N-1} M_{\mathbb{C}}(n) \underset{\theta_i}{\overset{\cong}{\longrightarrow}} \mathbf{U}_j\ , \quad (y_1, \ldots, \check{y}_i, \ldots, y_N) \longmapsto [y_1, \ldots, 1, \ldots, y_N]$$

These charts give trivializations, for example for $j = 1$

$$(\prod_{N-1} M_{\mathbb{C}}(n)) \times GL(n, \mathbb{C}) \longrightarrow E_N|_{U_1}$$

$$((y_2, \ldots, y_N), g) \longmapsto (g, y_2 g, \ldots, y_N g)$$

Ψ is equivariant $\Longrightarrow$ it induces ψ such that the following commutes

$$\begin{array}{ccc} P & \xrightarrow{\Psi} & E_N(n) \\ \downarrow \pi & & \downarrow \pi \\ X & \xrightarrow{\psi} & BGL_N(n) \end{array}$$

We have $\psi^* E_N(n) \cong P$, and there are inclusions:

$$\begin{array}{ccccccc} E_N(n) & \hookrightarrow & E_{N+1}(n) & \hookrightarrow & \cdots & \hookrightarrow & E_{GL(n)} \\ \downarrow & & \downarrow & & & & \downarrow \\ BGL_N(n) & \hookrightarrow & BGL_{N+1}(n) & \hookrightarrow & \cdots & \hookrightarrow & B_{GL(n)} \end{array}$$

If M compact then every $P \to X$ is a pull back by a map $f : X \longrightarrow BGL(n)$, i.e.

$$P \cong f^* E_{GL(n)}$$

To sum up, we have the description of the classifying space $B_{GL(n)}$ with charts $\{U_j\}$.

$$B_{GL(n)} = \left\{ [z_1, z_2, \ldots] \middle| \begin{array}{l} \text{at least one } \ z_i \in GL(n) \\ \text{finitely many } z_j \text{ are nonzero} \end{array} \right\}$$

$$U_j = \{[z_1, z_2, \ldots] \mid z_j \in GL(n)\}$$

$$U_j \approx \prod_{\infty} M_{\mathbb{C}}(n) \ \text{ maps to } B_{GL(n)} \text{via}$$

$$[z_1, z_2, \ldots] \longmapsto [z_1, \ldots, z_{j-1}, 1, z_{j+1}, \ldots]$$

2.6.3 Construction of B_G

Let G be a compact connected Lie group, and let $\rho : G \longrightarrow GL(n, \mathbb{C})$ be a faithful representation (it always exits). As above for a given a principal G-bundle $P \to X$, and $\{U_\alpha\}$, $\{\psi_\alpha\}, \{h_\alpha\}$, we have compositions:

$$\rho_\alpha : P|_{U_\alpha} \xrightarrow{h_\alpha} U_\alpha \times G \longrightarrow G \xrightarrow{\rho} GL(n, \mathbb{C})$$

As before we can define the following maps making the diagram commute:

$$\begin{array}{ccccc} P & \xrightarrow{\Psi_P} & E_N(n) & \subset & M_{\mathbb{C}}(n) \\ \downarrow & & \downarrow & & \\ X & \xrightarrow{\psi_P} & BG_N & = & E_N(n)/G \end{array}$$

$$\begin{array}{ccccccc} E_N & \hookrightarrow & E_{N+1}(n) & \hookrightarrow & \cdots & \hookrightarrow & E_G \\ \downarrow & & \downarrow & & & & \downarrow \\ BG_N & \hookrightarrow & BG_{N+1} & \hookrightarrow & \cdots & \hookrightarrow & B_G \end{array}$$

So we have shown that If X is compact, $P_G \longrightarrow X$ a principal G-bundle then $\exists\ f : X \longrightarrow B_G$ with $f^*E_G \cong P$. Next we prove the following.

Claim: If $f_0, f_1 : X \longrightarrow B_G$ with $f_0^*E_G \cong f_1^*E_G \Longrightarrow f_0 \simeq f_1$.

Proof. Consider $f_0, f_1 : X \longrightarrow BG_N \hookrightarrow \cdots \hookrightarrow BG_{2N}$, with

$$\begin{array}{rcl} f_0^*E_N & = & \{(x, y_1, \ldots, y_N) \in M \times EG_N \mid f_0(x) = [y_1, \ldots, y_N]\} \\ f_1^*E_N & = & \{(x, y_1, \ldots, y_N) \in M \times EG_N \mid f_1(x) = [y_1, \ldots, y_N]\} \end{array}$$

$$\begin{array}{ccc} f_0^*E_N & \xrightarrow[\cong]{\eta} & f_1^*E_N \\ \pi \searrow & & \swarrow \pi \\ & X & \end{array}$$

η is an isomorphism on each fiber. So we can write:

$$\eta(x, y_1, \ldots, y_N) = (x, \eta_1(x, y), \ldots, \eta_N(x, y))$$

$$f_0(x) = [y_1, ..., y_N]$$

$$f_1(x) = [\eta_1(x, y), \ldots, \eta_N(x, y)]$$

$$\eta_j(x, yg) = \eta_j(x, y) \cdot g$$

Notice the following maps $i,\ i',\ i'' : BG_N \hookrightarrow BG_{2N}$ are homotopic to each other

$$i[y_1, \ldots, y_N] = [y_1, y_2, .., y_N, 0, \ldots, 0]$$

$$i'[y_1, \ldots, y_N] = [y_1, 0, y_2, 0, \ldots, y_N, 0]$$

$$i''[y_1, \ldots, y_N] = [0, y_1, 0, y_2, \ldots, 0, y_N]$$

For example $[y_1, \ldots, y_N] \longmapsto [ty_1, (1-t)y_1, \ldots, ty_N, (1-t)y_N]$ describes $i' \simeq i''$

Hence $i \circ f_0 \simeq i \circ f_1 \Longleftrightarrow i' \circ f_0 \simeq i'' \circ f_1$. Now we will show $i' \circ f_0 \simeq i'' \circ f_1$:

$$(i' \circ f_0)^* E_{2N} = \{(x, y_1, 0, \ldots, y_N, 0) \mid f_0(x) = [y_1, \ldots, y_N]\} \text{ . Define}$$

$$\begin{array}{ccc} [0,1] \times (i' \circ f_0)^* E_{2N} & \xrightarrow{F} & E_{2N} \quad \text{(equivariant)} \\ \downarrow & & \downarrow \\ [0,1] \times X & \xrightarrow{f} & B_{G_{2N}} \end{array}$$

$$\begin{array}{ccc} (t, x, y_1, 0, \ldots, y_N, 0) & \xrightarrow{F} & ((1-t)y_1, t\eta_1(x,y), \ldots, (1-t)y_N, t\eta_N(x,y)) \\ \downarrow & & \downarrow \\ (t, x) & \xrightarrow{f} & [(1-t)y_1, t\eta_1(x,y), \ldots, (1-t)y_N, t\eta_N(x,y)] \end{array}$$

Then $f(t,x) = \varphi_t(x)$ gives the desired homotopy

$$\begin{aligned} \varphi_0(x) &= i' \circ f_0(x) \\ \varphi_1(x) &= i'' \circ f_1(x) \end{aligned}$$

□

Remark 2.11. *If $P \xrightarrow{G} X$ is a principal G-bundle, $\{U_\alpha\}_{\alpha=1}^N$ an open cover with two trivializations $h_\alpha, h'_\alpha : P|_{U_\alpha} \longrightarrow U_\alpha \times G$. Then if U_α is contractible and G is connected then the induced maps are homotopic $\psi_P \simeq \psi'_P : X \longrightarrow BG_N$. Also we get E_G is contractible and B_G is unique up to homotopy, for example given the classifying maps*

$$\begin{array}{ccccc} E_G & \xrightarrow{\check{f}_0} & E'_G & \xrightarrow{\check{f}_1} & E_G \\ \downarrow & & \downarrow & & \downarrow \\ B_G & \xrightarrow{f_0} & B'_G & \xrightarrow{f_1} & B_G \end{array}$$

$$\left.\begin{aligned} E_G &\cong (f_1 \circ f_0)^* E_G \\ &\cong (id)^* E_G \end{aligned}\right\} \Longrightarrow f_1 \circ f_0 \simeq id, \quad \textit{similarly} \;\; f_0 \circ f_1 \simeq id$$

Example 2.12. *Consider the $O(n)$-bundle $O(n) \to V_n(\mathbf{R}^{n+N}) \to G_n(\mathbf{R}^{n+N})$, which is given by the quotient:*

$$\begin{array}{c} O(n) \\ \downarrow \\ O(n+N)/O(N) = V_n(\mathbf{R}^{n+N}) \\ \downarrow \\ O(n+N)/O(n) \times O(N) = G_n(\mathbf{R}^{n+N}) \end{array}$$

Clearly $O(n)$ acts freely on the total space of this bundle giving the base as the quotient. Also it is known that, $V_n(\mathbf{R}^{n+N})$ is $N-1$ connected. Hence as $N \to \infty$ we can idntify $G_n(\mathbf{R}^{n+N})$ with $B_{O(n)}$.

Chapter 3

Connections

Let G be a Lie group, and $T_eG = \mathfrak{g}$ be its Lie algebra, and $\exp : \mathfrak{g} \to G$ be the exponential map. Then any $X \in \mathfrak{g}$ gives a one parameter subgroup $t \longmapsto \exp(tX)$ of G.

Example 3.1. *If $G \hookrightarrow GL(n)$ then $\mathfrak{g} \subseteq g\ell(n) \cong M_{\mathbb{R}}(n)$ then*

$$\exp(tX) = e^{tX} = I + tX + \frac{1}{2}t^2X^2 + \dots$$

Each $X \in \mathfrak{g}$ defines a left invariant vector field on G: $X_g = (\ell_g)_*X$, where ℓ_g is the left multiplication $\ell_g : x \mapsto g.x$

$$X_{g_1g_2} = (\ell_{g_1g_2})_*X_e = (\ell_{g_1})_*(\ell_{g_2})_*X_e = (\ell_{g_1})_*X_{g_2}$$

So $X_h = (\ell_{hg^{-1}})_*X_g$. There are also similarly defined left invariant differential forms $w_g = (\ell_{g^{-1}})^*\omega$ obtained by pulling back $\omega \in T_e^*(G)$ via $\ell_{g^{-1}} : g.x \mapsto x$

Recall the canonical map, which associates $x \in X$ to the identity endemorphism I

$$x \mapsto dx \in T_x^* \otimes T_x \cong Hom(T_x, T_x)$$

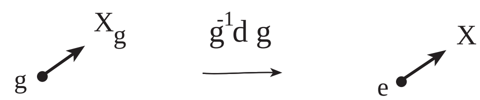
By varying $x \in X$ we can think of $x \mapsto dx$ as the canonical tangent valued 1-form θ on X defined by $\theta(x) = dx$. Now if $g \in G$ we define the *Cartan-Maurer form* on G:

$$\omega_g := g^{-1}dg \in T_g^*(G) \otimes T_e(G) = L(T_g(G), \mathfrak{g})$$
$$g^{-1}dg(X_g) := (\ell_{g^{-1}})^*dg(X_g) = dg((\ell_{g^{-1}})_*X_g) = (\ell_{g^{-1}})_*(X_g)$$

By varying g we can view $g^{-1}dg$ as a $\mathfrak{g}$-valued invariant 1-form on G

$$g^{-1}dg = \sum X_i \otimes \omega^i_g = \sum X_i \otimes \ell^*_{g^{-1}}(\omega^i)$$

where $\{X_i\}$ is a basis of $T_e(G)$ and $\{\omega^i\}$ be the dual basis of $T_e(G)^*$



Example 3.2. $G = GL(n, \mathbb{R})$, $\omega = A^{-1}dA$.

Let $\pi : P \to M$ be a principal G-bundle, with right action $(p, g) \mapsto r_g(p) := pg$. Then every $X \in \mathfrak{g} = T_e(G)$ induces a vector field $\tilde{X}$ on P as follows. By denoting a vector at a point of a manifold by a germ of smooth curves passing through that point, we denote $X = \{g_t\}$ where $t \mapsto g_t$ is a smooth curve in G with $g_0 = e$. Then define

$$p \mapsto \tilde{X}_p = \{pg_t\}$$

Let $V_p = \text{Span}\langle \tilde{X}_p \mid X \in \mathfrak{g}\rangle$, then $V = \cup_p V_p$ gives the vertical subbundle:

$$\begin{array}{ccc} V \subset TP & \longrightarrow & P \\ \downarrow & & \downarrow \pi \\ TM & \longrightarrow & M \end{array}$$

V is the vertical bundle since $\pi_* \tilde{X}_p = \pi_*\{pg_t\} = \{\pi(pg_t)\} = \{p\}$

Recall that the adjoint action $Ad(g) : G \to G$ induces an action $ad(g)$ on the tangent space $T_e(G) = \mathfrak{g}$ making the following commute

$$\begin{array}{ccc} \mathfrak{g} & \stackrel{ad(g^{-1})}{\longrightarrow} & \mathfrak{g} \\ \exp\downarrow & & \downarrow\exp \\ G & \stackrel{Ad(g^{-1})}{\longrightarrow} & G \end{array}$$

Lemma 3.3. *(a)* $r_{g*}(\tilde{X})_p = [\ \widetilde{ad(g^{-1})X}\ ]_{pg}$

(b) $[\tilde{X}, \tilde{Y}] = [\widetilde{X, Y}]$

Proof. To see (a), recall if $X = \{g_t\}$ then $\tilde{X}_p = \{pg_t\}$. Then

$$r_{g*}(\tilde{X}_p) = \{pg_t g\} = \{pg(g^{-1}g_t g)\} = [\ \widetilde{ad(g^{-1})X}\ ]_{pg}$$

For (b), first note that the map $\sigma_p : G \to P$, defined by $\sigma_p(g) = p.g$ gives

$$(\sigma_p)_* X = \tilde{X}$$

and the 1-parameter group of diffeomorphisms which induces the vector field $\tilde{X}$ is $r_{g_t}(p) = p.g_t$, where $X = \{g_t\}$ represents X. So in particular r_{g_t} is the 1-parameter group of diffeomorphisms for X. Clearly the following commutes:

$$\begin{array}{ccc} G & \xrightarrow{\sigma_{pg_t}} & P \\ ad(g_t)\downarrow & & \downarrow r_{g_t^{-1}} \\ G & \xrightarrow{\sigma_p} & P \end{array}$$

$$\begin{aligned} [\tilde{X}, \tilde{Y}]_p &= \lim_{t\to 0} \frac{1}{t} [\ \tilde{Y}_p - (r_{g_t^{-1}})_* \tilde{Y}_{pg_t}\ ] \\ &= \lim_{t\to 0} \frac{1}{t} [\ \tilde{Y}_p - (r_{g_t^{-1}})_* (\sigma_{pg_t})_* Y\ ] \\ &= \lim_{t\to 0} \frac{1}{t} [\ (\sigma_p)_* Y - (\sigma_p)_*\ ad(g_t)_* Y\ ] \\ &= (\sigma_p)_* \lim_{t\to 0} \frac{1}{t} [\ Y_e - (r_{g_t^{-1}})_* Y_{g_t}] \\ &= (\sigma_p)_* [X, Y] = [\widetilde{X, Y}]_p \end{aligned}$$

□

A principal G-bundle $P \xrightarrow{\pi} M$, gives an exact sequence of vector bundles over P

$$0 \longrightarrow ker\pi_* = V \longrightarrow TP \longrightarrow \pi^* TM \longrightarrow 0$$

G-acts each vector space by r_{g*}. This action maps V to itself.

Definition 3.4. *A connection A is a G-equivariant splitting of P*

$$0 \longrightarrow V \xrightarrow{\mathbf{i}} TP \underset{A}{\overset{q}{\rightleftarrows}} \pi^* TM \longrightarrow 0$$

such that $q \circ A = id$, where q is the projection.

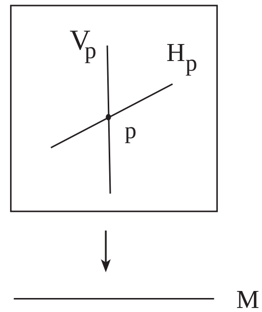


Figure 3.1: Vertical and horizontal fibers

Clearly $A(\pi^*TM) \subset TP$ is a horizontal bundle $H_A \to M$, i.e. $\pi_*|$ is an isomorphism $A(\pi^*TM) \to TM$. This gives two definitions of a connection:

(1) It is a subbundle $H = H_A \subset P$ such that

   (a) $T_p(P) = H_p \oplus V_p$

   (b) $H_{pg} = r_{g*}H_p$

(2) It is a $\mathfrak{g}$-valued 1-form A on P with

   (a) $A_p(\tilde{X}_p) = X$

   (b) $r_g^* A_{pg} = ad(g^{-1})A_p$

To see (1) $\Longrightarrow$ (2). Given H, we define A as follows: For any $Z_p \in T_p(P)$ with $Z_p = \tilde{X}_p + Y_p$ where $X \in \mathfrak{g}$, $Y_p \in H_p$, then we define $A_p(Z_p) = X$, so

$$(r_g^* A_{pg})(Y_p) = A_{pg}(r_{g*}Y_p) = 0$$

Since $Y_p \in H_p \Longrightarrow r_{g*}Y_p \in H_{pg}$, hence $ad(g^{-1})A_p(Y_p) = 0$, and also

$$\begin{aligned} (r_g^* A_{pg})(\tilde{X}_p) &= A_{pg}(r_{g*}\tilde{X}_p) = A_{pg}(ad\widetilde{(g^{-1})X_{pg}}) \\ &= ad(g^{-1})X \\ &= ad(g^{-1})A_p(\tilde{X}_p) \end{aligned}$$

$$\Rightarrow r_g^* A_{pg} = ad(g^{-1})A_p$$

To see (2) $\Longrightarrow$ (1), given A define H by

$$H_p = \ker A_p \Longrightarrow T_p(P) = H_p \oplus V_p,$$

$$\text{Then } Y_p \in H_p \Rightarrow A_{pg}(r_{g*}Y_p) = r_g^*(A_{pg})(Y_p) = ad(g^{-1})A_p(Y_p) = 0$$

Hence $r_{g*}Y_p \in H_{pg} \Rightarrow H_{pg} = r_{g*}H_p$.

Example 3.5. *If $P = M \times G$ then $A_g = g^{-1}dg = \sum X^i \otimes \omega_g^i$ is a connection, where $\{X^i\}$ is a basis of $\mathfrak{g}$ and $\{\omega_g^i\}$ are the left invariant 1-forms. Then*

$$\begin{aligned}
A_g &= \sum X^i \otimes \omega_g^i \Rightarrow \\
A_{gh} &= \sum X^i \otimes \ell_{h^{-1}}^*(\omega_g^i) \\
r_h^* A_{gh} &= \sum X^i \otimes r_h^* \ell_{h^{-1}}^*(\omega_g^i) \\
&= \sum X^i \otimes ad(h^{-1})(\omega_g^i) \\
&= ad(h^{-1}) \sum X^i \otimes \omega_g^i \\
&= ad(h^{-1})A_g
\end{aligned}$$

3.1 Local description of connections

Let A be a connection on a principal G-bundle $P \to M$ with local trivializations:

$$h_\alpha : P|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times G$$

Then $A_\alpha := (h_\alpha^{-1})^* A$ is a connection on $U_\alpha \times G$. On $U_\alpha \times G$ there is another connection $g_\alpha^{-1}dg_\alpha$ defined by Example 3.5. Let us consider the difference $a_\alpha = A_\alpha - g_\alpha^{-1}dg_\alpha$:

$$\begin{aligned}
a_\alpha(x, g_\alpha) &= A_\alpha(x, g_\alpha) - g_\alpha^{-1}dg_\alpha \Rightarrow \\
a_\alpha(x, g_\alpha h) &= ad(h^{-1})(A_\alpha(x, g_\alpha) - g_\alpha^{-1}dg_\alpha), \quad \Rightarrow \\
a_\alpha(x) := a_\alpha(x, 1) &= ad(g_\alpha)(A_\alpha(x, g_\alpha) - g_\alpha^{-1}dg_\alpha) \ \Rightarrow
\end{aligned}$$

$$A_\alpha(x, g_\alpha) = g_\alpha^{-1}dg_\alpha + g_\alpha^{-1}a_\alpha(x)g_\alpha \tag{3.1}$$

On $(U_\alpha \cap U_\beta) \times G$ we have $(x, g_\alpha) \sim (x, g_\beta) \Rightarrow A_\alpha(x, g_\alpha) = A_\beta(x, g_\beta) \Rightarrow$

$$g_\alpha^{-1} dg_\alpha + g_\alpha^{-1} a_\alpha(x) g_\alpha = g_\beta^{-1} dg_\beta + g_\beta^{-1} a_\beta(x) g_\beta \tag{3.2}$$

Also, since $g_\alpha = g_{\alpha\beta} g_\beta \;\Rightarrow\; dg_\alpha = dg_{\alpha\beta} g_\beta + g_{\alpha\beta} dg_\beta$

By multiplying both sides of this with $g_\alpha^{-1} = g_\beta^{-1} g_{\alpha\beta}^{-1}$ we get:

$g_\alpha^{-1} dg_\alpha = g_\alpha^{-1} dg_{\alpha\beta} g_\beta + g_\beta^{-1} dg_\beta$, then by plugging in (3.2) we obtain

$$a_\beta(x) = g_{\alpha\beta}^{-1} dg_{\alpha\beta} + g_{\alpha\beta}^{-1} a_\alpha(x) g_{\alpha\beta} \tag{3.3}$$

Hence a connection $A \in \mathcal{A}(P)$ can be described as a collection of $\mathfrak{g}$ valued 1-forms on local charts $A = \{a_\alpha : U_\alpha \to T^*U_\alpha \otimes \mathfrak{g}\}$ satisfying the condition (3.3). Given this we can define $A_\alpha(x, g_\alpha)$ by (3.1), then by piecing $h_\alpha^* A_\alpha$ by partions of unity we get A.

Therefore the difference of two connections $\theta_\alpha = A_\alpha - \bar{A}_\alpha = a_\alpha - \bar{a}_\alpha$ defines collection of maps $\theta_\alpha : U_\alpha \to T^*U_\alpha \otimes \mathfrak{g}$, satisfying $\theta_\beta = g_{\alpha\beta}^{-1} \theta_\alpha g_{\alpha\beta}$ on $U_\alpha \cap U_\beta$. Hence $\theta = \{\theta_a\}$ is a section of the bundle $T^*M \otimes ad(P)$, where $ad(P) \to M$ is the $\mathfrak{g}$- bundle associated to the principal bundle $P \to M$ by the conjugate representation $ad : G \to Aut(\mathfrak{g})$.

Definition 3.6. *Let $\mathcal{A}(P)$ to be the set of connections on a principle G-bundle $P \to M$. From the above discussion we see that, by fixing a connection A_0 we can identify:*

$$\mathcal{A}(P) = \{A_0\} + \Gamma(T^*M \otimes ad(P)) := \Omega^1(adP)$$

Hence the conections on principal circle bundles $P \to X$ are identified by $\Omega^1(X)$

3.2 Connections on vector bundles

Let $V \to E \to M$ be a (real or complex) vector bundle over M. A connection on E is a first order differential operator:

$$D : \Gamma(E) \longrightarrow \Gamma(T^*M \otimes E) \tag{3.4}$$

(a) $D(s_1 + s_2) = D(s_1) + D(s_2)$

(b) $D(fs) = fD(s) + df \otimes s$

Since $D(s)$ is a 1- form valued section of E, by evaluating the 1-form part on a vector field X, and denoting $D(s)(X) = \nabla_X(s)$ we can view a connection as a derivation:

$$\nabla_X : \Gamma(E) \to \Gamma(E) \tag{3.5}$$

Therefore by choosing a local frame $\{e_1, ..e_m\}$ for $T(M^m)$ and its dual basis of 1-forms $\{e^1, .., e^m\}$ we can express D as:

$$D = \sum e^i \otimes \nabla_{e_i}$$

Alternatively, by choosing a local frame frame $e_\alpha = \{e'_\alpha, \dots, e^n_\alpha\}$ of E on charts $\{U_\alpha\}$, consisting of n-linearly independent sections $e^i_\alpha \in \Gamma(E|_{U_\alpha})$. Then we can write $De^i_\alpha = \sum_{j=1}^n e^j_\alpha \alpha^\alpha_{ji}$. Let $a_\alpha = (a^\alpha_{ij})$, then $a_\alpha : U_\alpha \to T^*(U_\alpha) \otimes \mathfrak{g}$, $\mathfrak{g} = \mathfrak{g}\ell(n,k) = M_k(n)$, where $k = \mathbb{R}$ or $\mathbb{C}$. If $e_\beta = \{e'_\beta, \dots, e^n_\beta\}$ is another frame in $\Gamma|_{U_\beta}$ with $e_\alpha = g_{\alpha\beta} e_\beta$, then we can write:

$$De_\alpha = e_\alpha \alpha_\alpha \ \Rightarrow$$

$$\begin{aligned}
De_\alpha &= D(e_\beta g_{\beta\alpha}) = (De_\beta) g_{\beta\alpha} + e_\beta(dg_{\beta\alpha}) \\
e_\alpha a_\alpha &= e_\beta a_\beta g_{\beta\alpha} + e_\beta dg_{\beta\alpha} \\
e_\beta g_{\beta\alpha} a_\alpha &= e_\beta a_\beta g_{\beta\alpha} + e_\beta dg_{\beta\alpha} \\
g_{\beta\alpha} a_\alpha &= a_\beta g_{\beta\alpha} + dg_{\beta\alpha} \Rightarrow \\
a_\alpha &= g^{-1}_{\beta\alpha} a_\beta g_{\beta\alpha} + g^{-1}_{\beta\alpha} dg_{\beta\alpha}
\end{aligned}$$

Hence the differential operator D prescribes a connection $A \in \mathcal{A}(P)$ in the associated principle bundle $GL(n,k) \to P_E \to M$ to E, Conversely, by the above discussions, a connection on a principal G-bundle $P \to M$ gives such a differential operator D on the associated vector bundle $E \to M$. We denote

$$\Omega^p(E) = \Gamma(\Lambda^p T^* M \otimes E)$$

and by $D(\sigma \otimes \theta) = d\sigma \wedge \theta + (-1)^p \sigma \otimes D(\theta)$ we can extend D to differential operators:

$$\Omega^0(E) \xrightarrow{D} \Omega^1(E) \xrightarrow{D} \Omega^2(E) \longrightarrow \cdots \longrightarrow \Omega^p(E) \tag{3.6}$$

Furthermore, the set of connection $\mathcal{A}(E)$ on the bundle $E \to M$ is identified by

$$\mathcal{A}(E) = \{A_0\} + \Gamma(T^*M \otimes End(E)) := \Omega^1(End(E))$$

Because difference of two connection is an element of $\Omega^1(End(E))$ where $End(E)$ is the vector bundle $ad(P_E)$ associated principal bundle to $P_E \to M$ by adjoint representation.

Now any connection $A = \{a_\alpha : U_\alpha \to T^*U_\alpha \otimes \mathfrak{g}\}$ on a principal G-bundle $P_G \to M$, and a homomorphism $\rho : G \to H$ induces the connection $A' = \{\rho_*(a_\alpha) : U \to T^*U_\alpha \otimes \mathfrak{h}\}$ on the principal H-bundle $P_H \to M$, where ρ_* is the derivative of ρ. Also A gives a connection on any associated vector bundle $E_\rho = P \times_\rho V \to M$, where $\rho : G \to \mathrm{Aut}(V)$ is a representation. We define the asociated differential operator $D_A : \Omega^0(E_\rho) \to \Omega^1(E_\rho)$ by using $\{\bar{a}_\alpha := \rho_*(a_\alpha) : U_\alpha \to \mathfrak{g}\ell(n) \otimes T^*(U_\alpha)\}$ where $\rho_* : \mathfrak{g} \to \mathrm{End}(V)$ is the derivative of the map ρ at the identity. So if e_α are local sections of E_ρ, we let $D_A(e_\alpha) = e_\alpha \cdot \bar{a}_\alpha$.

Example 3.7. *Let $\pi : P \to M$ be a principal G-bundle and ad $P \to X$ be the vector bundle associated to the adjoint representation $\rho : G \to Aut(\mathfrak{g})$. then $\rho_* : \mathfrak{g} \to End(\mathfrak{g})$ is $\rho_*(A)(B) = AB - BA = [A, B]$. Let $A \in \mathcal{A}(P) = \{a_\alpha : U_\alpha \to T^*U_\alpha \otimes \mathfrak{g}\}$. Then we get the differential operators*

$$\Omega^0(ad\ P) \xrightarrow{d_A^0} \Omega^1(ad\ P) \xrightarrow{d_A^1} \Omega^2(ad\ P) \xrightarrow{d_A^2} \cdots \to \Omega^p(ad\ P) \tag{3.7}$$

$$d_A^0(u) = du + [a_\alpha, u]$$
$$d_A^1(B) = dB + B \wedge a_\alpha + a_\alpha \wedge B$$
$$d_A^2(C) = dC + C \wedge a_\alpha - a_\alpha \wedge C$$

For example if $B = \theta \otimes \sigma \in \Omega'(ad\ P) = \Omega^0(T^*M \otimes ad\ P)$

$$\begin{aligned} d_A^1(\theta \otimes \sigma) &= d\theta \otimes \sigma - \theta \wedge d_A^0(\sigma) \\ &= d\theta \otimes \sigma - \theta \wedge (d(\sigma) + [a, \sigma]) \\ &= d(\theta \otimes \sigma) - \theta \wedge (a\sigma - \sigma a) \\ &= dB + a \wedge (\theta \otimes \sigma) + (\theta \otimes \sigma) \wedge a \\ &= dB + a \wedge B + B \wedge a \end{aligned}$$

If $C = \theta \otimes \sigma \in \Omega^2(ad\ P) = \Omega^0(\Lambda^2 T^*M \otimes ad\ P)$

$$\begin{aligned} d_A^2(\theta \otimes \sigma) &= d\theta \otimes \sigma + \theta \wedge d_A^0(\sigma) \\ &= d\theta \otimes \sigma + \theta \wedge (d(\sigma) + [a, \sigma]) \\ &= d(\theta \otimes \sigma) + \theta \wedge (a\sigma - \sigma a) \\ &= dC + a \wedge (\theta \otimes \sigma) - (\theta \otimes \sigma) \wedge a \\ &= dC + a \wedge C - C \wedge a \end{aligned}$$

Example 3.8. *Any principal G-bundle $\pi : P \to M$ with a G-invariant metric, i.e.*

$$\langle v, w\rangle_p = \langle r_{g_*}v, r_{g_*}w\rangle_{pg}$$

gives a connection $H_p \subset T_p(P) \cong V_p \oplus H_p$, where $\pi_* : H_p \overset{\cong}{\to} T_{\pi(p)}(M)$:

$$H_p = V_p^{\perp} \;=\; \{w \in T_p(P) \mid \langle v, w\rangle_p = 0 \;\; \forall \;\; v \in V_p\} \;\Rightarrow$$

$$\begin{aligned} r_{g_*}H_p &= \{r_{g*}(w) \mid \langle w, v\rangle_p = 0 \;\; \forall \;\; v \in V_p\} \\ &= \{\bar{w} \mid \langle r_{g_*^{-1}}\bar{w}, v\rangle_p = 0, \;\; \forall \;\; v \in V_p\} \\ &= \{\bar{w} \mid \langle \bar{w}, r_{g_*}v\rangle_{pg} = 0 \;\; \forall \;\; v \in V_p\} \\ &= \{\bar{w} \mid \langle \bar{w}, \bar{v}\rangle_{pg} = 0 \;\; \forall \;\; \bar{v} \in V_{pg}\} \\ &= H_{pg} \end{aligned}$$

Example 3.9. *Let $S^1 \to S^3 \to S^2$ be the Hopf fibration*

$$S^3 = \{(z, \omega) \in \mathbb{C}^2 \mid |z|^2 + |\omega|^2 = 1\}$$

$$r_\theta(z, \omega) \longmapsto (ze^{i\theta}, \omega e^{i\theta})$$

$$T_{(z,\omega)}(S^3) = \{(a, b) \in \mathbb{C}^2 \mid Re(\bar{a}z + \bar{b}\omega) = 0\}$$

The vertical bundle is $V_{(z,\omega)} = Span\left(\frac{\tilde{\partial}}{\partial\theta}\right)$, and $r_{\theta_*}(a, b)_{(z,\omega)} = (ae^{i\theta}, be^{i\theta})_{(ze^{i\theta},\omega e^{i\theta})}$. Then $\langle(a, b),(c, d)\rangle = Re(\bar{a}c + \bar{b}d)$ gives an S^1-invariant metric on S^3.

$$\begin{aligned} \left(\frac{\tilde{\partial}}{\partial\theta}\right)_{(z,\omega)} f &= \frac{d}{d\theta} f(ze^{i\theta}, \omega e^{i\theta})|_{\theta=0} \\ &= izf_z + i\omega f_\omega \\ &= (iz\frac{\partial}{\partial z} + i\omega\frac{\partial}{\partial\omega})f \quad \Rightarrow \end{aligned}$$

$$V_{(z,\omega)} = Span_{\mathbb{R}}(iz, i\omega) = Span\; Im(z\frac{\partial}{\partial z} + \omega\frac{\partial}{\partial\omega}) \;\Rightarrow$$

$$H_{(z,\omega)} = \{\; (a, b) \mid Re(\bar{a}z + \bar{b}\omega) = 0, \;\; Re\; i(\bar{a}z + \bar{b}\omega) = 0 \;\} \;\Rightarrow$$

$$\begin{aligned} H_{(z,\omega)} &= \{\ (a,b) \mid \bar{a}z + \bar{b}\omega = 0\} \\ &= Span_{\mathbb{R}}\{\ (\bar{\omega}, -\bar{z}), i(\bar{\omega}, -\bar{z})\ \} \\ &= Span\{\ Re(\bar{\omega}\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \omega}),\ Im(\bar{\omega}\frac{\partial}{\partial z} - \bar{z}\frac{\partial}{\partial \omega})\ \} \end{aligned}$$

Then $A = \bar{z}\ dz + \bar{\omega}\ d\omega$ *is a connection*

$$A\left(\frac{\tilde{\partial}}{\partial \theta}\right) = i = \frac{\partial}{\partial \theta}\ , \quad \textit{and } A \textit{ is equivariant.}$$

Recall the trivializations $h_{\pm} : \mathbb{C} \times S^1 \to S^3$

$$h_+(\lambda, e^{i\theta_+}) = \frac{1}{(1+|\lambda|^2)^{1/2}}(\lambda e^{i\theta_+}, e^{i\theta_+}) = (z, \omega) \tag{3.8}$$

$$h_-(\mu, e^{i\theta_-}) = \frac{1}{(1+|\mu|^2)^{1/2}}(e^{i\theta_-}, \mu e^{i\theta_-}) = (z, \omega) \tag{3.9}$$

$$\begin{aligned} \therefore \quad h_+^*(dz) &= dz = z_\lambda\ d\lambda + z_{\bar{\lambda}}\ d\bar{\lambda} + z_{\theta_+} d\theta_+ \\ h_+^*(d\omega) &= d\omega = \omega_\lambda\ d\lambda + \omega_{\bar{\lambda}}\ d\bar{\lambda} + \omega_{\theta_-} d\theta_- \end{aligned}$$

$$(3.8) \Rightarrow \begin{cases} z_\lambda = [\ (1+|\lambda|^2)^{-1/2} - \frac{\lambda}{2}(1+|\lambda|^2)^{-3/2}\ \bar{\lambda}\ ]\ e^{i\theta_+} = \lambda^{-1}z - z\ \dfrac{\bar{\lambda}}{2(1+|\lambda|^2)} \\ z_{\bar{\lambda}} = -\frac{\lambda}{2}(1+|\lambda|^2)^{-3/2}\lambda e^{i\theta_+} = -z\ \dfrac{\lambda}{2(1+|\lambda|^2)} \\ z_{\theta_+} = iz \end{cases}$$

$$\Rightarrow \quad \begin{cases} h_+^*(dz) = iz\ d\theta_+ + \lambda^{-1}z\ d\lambda - z\dfrac{\bar{\lambda}\ d\lambda + \lambda\ d\bar{\lambda}}{2(1+|\lambda|^2)} \\ h_+^*(d\omega) = i\omega\ d\theta_+ - \omega\dfrac{\bar{\lambda}\ d\lambda + \lambda\ d\bar{\lambda}}{2(1+|\lambda|^2)} \end{cases}$$

$$
\begin{aligned}
\Rightarrow \qquad h_+^*(A) &= h_+^*(\bar z\, dz + \bar\omega\, d\omega) \\
&= i|z|^2 d\theta_+ + \lambda^{-1}|z|^2 d\lambda - |z|^2 \frac{(\bar\lambda\, d\lambda + \lambda d\bar\lambda)}{2(1+|\lambda|^2)} \\
&\quad + i|\omega|^2 d\theta_+ \;-\; |\omega|^2 \frac{(\bar\lambda\, d\lambda + \lambda\, d\bar\lambda)}{2(1+|\lambda|^2)}. \\
&= i d\theta_+ + \lambda^{-1}|z|^2 d\lambda - \frac{\bar\lambda\, d\lambda + \lambda\, d\bar\lambda}{2(1+|\lambda|^2)}. \\
&= i\, d\theta_+ + \frac{\bar\lambda d\lambda - \lambda d\bar\lambda}{2(1+|\lambda|^2)} \quad \textit{(from (3.8))}
\end{aligned}
$$

$$\textit{Therefore}: \quad h_+^*(A) = i\left(d\theta_+ + Im\frac{\bar\lambda\, d\lambda}{1+|\lambda|^2}\right) := A_+$$

$$\textit{Similarly (3.9)} \Rightarrow h_-^*(A) = i\left(d\theta_- + Im\frac{\bar\mu\, d\mu}{1+|\mu|^2}\right) := A_-$$

$$\textit{Recall also that we have} \quad e^{i\theta_+} = \frac{\mu}{|\mu|}e^{i\theta_-} \quad \lambda = \frac{1}{\mu}$$

Example 3.10. *Consider the universal $O(n)$-bundle $O(n) \to EO(n) \to BO(n)$, which is given by the limit $N \to \infty$ of the frame bundle:*

$$
\begin{array}{l}
O(n+N) \\
\qquad \downarrow \\
O(n+N)/O(N) = V_n(\mathbf{R}^{n+N}) \\
\qquad \downarrow \\
O(n+N)/O(n)\times O(N) = G_n(\mathbf{R}^{n+N})
\end{array}
$$

$$
\begin{aligned}
\textit{where } O(n+N) &= \textit{the space of orthonormal frames } \{e_A\}_{a=1}^{n+N} \textit{ in } \mathbb{R}^{n+N} \\
EO(n) &= \textit{the space of orthonormal frames } \{e_i\}_{i=1}^{n} \\
BO(n) &= \textit{the space of planes spanned by } \{e_1,\ldots,e_n\}
\end{aligned}
$$

Let ω_{AB} *be the Cartan-Maruer forms in* $O(n+N)$, *where* $\omega_{AB} = (de_A, e_B)$ *and* (ω_{AB}) *is an* $o(n)$*-valued* 1*-form on* $O(n+N)$, *so* $\omega_{AB} = -\omega_{BA}$. *We can write*

$$de_A = \sum \omega_{AB} \otimes e_B$$

Let $\omega = (\omega_{ij})\ \ 1 \le i, j \le n$. *Consider orthogonal base change*

(1) $e_i^* = \sum_{k=1}^n a_{ki} e_k$

(2) $e_\alpha^* = \sum_{\beta=n+1}^N b_{\beta\alpha} e_\beta$

ω *is invariant under (2) so* ω *is an* $o(n)$*-valued* 1*-form on* $EO(n)$. *If we perform the transformation (1)*

$$\begin{aligned} \omega_{ij}^* &= (de_i^*, e_j^*) = (\sum_k a_{ki} de_k, \sum_\ell a_{\ell j} e_\ell) = \sum a_{ki}\omega_{k\ell}a_{\ell j} \\ &= (A^t \omega A)_{ij}\ ,\ \text{where}\ \ A = (a_{ij}) \end{aligned}$$

So $\omega^* = r_A^* \omega = A^t \omega A = A^{-1}\omega A = ad(A^{-1})\omega$. *Therefore* ω *is a connection on* $EO(n)$. *Now recall the construction of* $BGL(n)$ *as the quotient:*

$$E_N(n) = \{\ (m_1, \ldots, m_N) \in \prod_N M_{\mathbb{C}}(n) \mid \exists m_j \in GL(n, \mathbb{C})\}$$

$$\downarrow\ GL(n)$$

$$BGL_N(n)$$

$E_N(n)$ *has a* $GL(n)$*-invariant metric, i.e. on* $TE_N(n)|_{(m_i)} \cong \prod_N M_{\mathbb{C}}(n)$

$$\langle (u_i), (u_i') \rangle = Re\ tr(\sum_{i=1}^N u_i^* u_i')$$

The vertical space is: $V_{(m_i)} = Span\ \{(m_i h) \mid h \in M_{\mathbb{C}}(n)\} \subset TE_N|_{(m_i)}$. *The* $GL(n)$*-invariant metric gives a connection* A_N *with by the horizontal space:*

$$\begin{aligned} H_N|_{(m_i)} &= \{\ (u_i) \in \prod_N M_{\mathbb{C}}(n) \mid Re\ tr(\sum u_i^* m_i h) = 0 \\ &\qquad\qquad \forall\ h \in M_{\mathbb{C}}(n),\ \forall\ ih \in M_{\mathbb{C}}(n)\} \\ &= \{(u_i) \in \prod_N M_{\mathbb{C}}(n) \mid tr(\sum u_i^* m_i h) = 0\ ,\ \ \forall\ h \in M_{\mathbb{C}}(n)\} \end{aligned}$$

For $g \in GL(n)$ $(r_g)_*(u_i)|_{(m_i g)} = (u_i g)$, hence

$$\sum tr(g^* u_i^* m_i g h) = \sum tr(u_i^* m_i g h g^*) = 0$$

Therefore the horizontal spaces are mapped to horizontal spaces, and A_N is the $\mathfrak{gl}(n)$-valued 1-form

$$A_N = (\sum_k m_k^* m_k)^{-1} (\sum_j m_j^* dm_j)$$

If $h \in M_{\mathbb{C}}(n)$ then $\tilde{h} = \sum m_i h \frac{\partial}{\partial m_i}$

$$A_N(\tilde{h}) = (\sum_k m_k^* m_k)^{-1} (\sum_j m_j^* m_j h) = h$$

3.3 Holonomy

Let $P \xrightarrow{\pi} X$ be a principal G-bundle with a connection A. Then for any $p \in P$ and a smooth path $\gamma : I \longrightarrow X$ with $\pi(p) = g(0)$, there is a unique parallel lifting $\tilde{\gamma}_p : I \longrightarrow P$, i.e. satisfying $A(\tilde{\gamma}_p(t)') = 0$. It suffices to see this locally:

$$\begin{array}{ccccc} & & P|_{U_\alpha} & \cong & U_\alpha \times G \\ & \tilde{\gamma}_p \nearrow & \downarrow \pi & \swarrow & \\ I & \xrightarrow{\gamma} & U_\alpha & & \end{array}$$

$\tilde{\gamma}_p(t) = (\gamma(t), g_\alpha(t))$, $p = (\gamma(0), g)$, $A|_{U_\alpha} = A_\alpha = g_\alpha^{-1} dg_\alpha + g_\alpha^{-1} a_\alpha g_\alpha$. So we want $A_\alpha(\gamma'(t), g'_\alpha(t)) = 0$, therefore

$$\begin{array}{l} g_\alpha^{-1} dg_\alpha(g'_\alpha(t)) + g_\alpha(t)^{-1} a_\alpha(\gamma'(t)) g_\alpha(t) = 0 \\ \qquad \| \\ g_\alpha(t)^{-1} g'_\alpha(t) \end{array}$$

$$\Rightarrow \left\{ \begin{array}{l} g'_\alpha(t) = -a_\alpha(\gamma'(t)) g_\alpha(t) \\ g_\alpha(0) = g_\alpha \end{array} \right\}$$

This is an O.D.E. with initial condition, hence has a unique solution $g_\alpha(t)$. Also if $\rho : G \longrightarrow Aut(V)$ is any representation, then A determines a lifting to the associated vector bundle by the folowing:

$$\begin{array}{ccc} & & P \times_\rho V \\ \tilde{\gamma}_{[p,v]} & \nearrow & \downarrow \pi_1 \\ I & \xrightarrow{\gamma} & X \end{array}$$

defined by $\tilde{\gamma}_{[p,v]}(t) = [\tilde{\gamma}_p(t), v]$. This is well defined since

$$\begin{aligned} \tilde{\gamma}_{[pg^{-1},\rho(g)v]} &= [\tilde{\gamma}_{pg^{-1}}(t), \rho(g)v] \\ &= [\tilde{\gamma}_p(t)g^{-1}, \rho(g)v] \\ &= [\tilde{\gamma}_p(t), v] \\ &= \tilde{\gamma}_{[p,v]} \end{aligned}$$

So we can we parallel translate vectors in $P \times_\rho V \xrightarrow{\pi_1} X$

Chapter 4

Curvature

Let A be a connection on a principal G bundle $P \to M$. The curvature F_A of A is a $\mathfrak{g}$-valued 2-form on P defined by one of the conditions (1)-(3) below:

Lemma 4.1. *Let F_A be a $\mathfrak{g}$-valued 2-form on P then the following are equivalent.*

(1) $F_A = dA + A \wedge A = dA + \frac{1}{2}[A, A]$, *i.e.*
$F_A(X, Y) = dA(X, Y) + \frac{1}{2}[A(X), A(Y)]$.

(2) $F_A(X, \tilde{Y}) = 0$ *for any vertical vector* $\tilde{Y}$, *and*
$F_A(X, Y) = -\frac{1}{2} vert[X, Y]$ *for horizontal vectors* X, Y.

(3) Let $vert_A : TP \to V \cong \mathfrak{g}$ *be the vertical projection. For* $X, Y \in T_p(P)$, *if we extend* X, Y *to vector fields in the neighborhood of* p *then*

$$F_A(X, Y) = -\frac{1}{2}\, vert([X, Y] - [vert\, X, Y] - [X, vert\, Y] + [vert\, X, vert\, Y]).$$

Proof. Clearly (2) defines the 2-form F_A, and (3) has the same property as (2), hence (2) $\Longleftrightarrow$ (3). Now let us check *and* (1) $\Longleftrightarrow$ (2):

$$\begin{aligned} F_A(X, Y) &= dA(X, Y) + \frac{1}{2}[A(X), A(Y)] \\ &= \frac{1}{2}(X(AY) - Y(AX) + [A(X), A(Y)] - A[X, Y]) \end{aligned}$$

Case 1: If X, Y are horizontal

$$F_A(X, Y) = -\frac{1}{2}A[X, Y] = -\frac{1}{2}\,\mathrm{vert}[X, Y]$$

Case 2: If $X = \tilde{X},\ Y = \tilde{Y}$ are vertical with $X, Y \in \mathfrak{g}$, then

$$\begin{aligned} F_A(\tilde{X}, \tilde{Y}) &= \frac{1}{2}([X,Y] - A[\tilde{X}, \tilde{Y}]) \\ &= \frac{1}{2}([X,Y] - A\widetilde{[X,Y]}) \\ &= \frac{1}{2}([X,Y] - [X,Y]) \\ &= 0 \end{aligned}$$

Case 3: If X is horizontal, and $Y = \tilde{Y}$ vertical, then $F_A(X, \tilde{Y}) = -\frac{1}{2}A[X, \tilde{Y}] = 0$. This is because $[X, \tilde{Y}]$ is horizontal, which we can check by taking a 1-parameter subgroup $r_{\exp(tY)}$ generated by $\tilde{Y}$

$$[X, \tilde{Y}] = \lim_{t \to 0} \frac{1}{t}\, [\, r_{\exp(tY)_*} X - X] = \text{horizontal} \quad \square$$

Remark 4.2. *F_A vanishes on vertical vectors, so it is characterized on the horizontal vectors $Ker(A)$, by using the notation of Section 3.2 we calculate:*

$$\begin{aligned} F_A(X,Y) &= dA(X,Y) + A \wedge A\ (X,Y) \\ &= X(A(Y)) - Y(A(X)) - A[X,Y] + [A(X), A(Y)] \\ &= \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \end{aligned}$$

Proposition 4.3. *F_A has the following properties:*

(1) $F_A(X, fY) = f F_A[X,Y]$ *for* $f \in C^\infty(P)$

(2) $F_A(r_{g_*}X, r_{g_*}Y) = r_g^* F_A(X,Y) = ad(g^{-1}) F_A(X,Y)$

Proof. To prove (2), let $h_\alpha : P|_{U_\alpha} \xrightarrow{\cong} U_\alpha \times G$ be a local trivialization, and let $A_\alpha = (h_\alpha^{-1})^* A,\ \ F_\alpha = (h_\alpha^{-1})^* F_A$, then recall that

$$A_\alpha(x, g_\alpha) = g_\alpha^{-1} dg_\alpha + g_\alpha^{-1} a_\alpha g_\alpha\ , \ \ \text{where } a_\alpha \in C^\infty(T^*U_\alpha) \otimes \mathfrak{g}$$

with transition rules: $a_\beta(x) = g_{\alpha\beta}^{-1} a_\alpha(x) g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}$. So $F_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$

$$g_\alpha A_\alpha = dg_\alpha + a_\alpha g_\alpha \Rightarrow$$

$$dg_\alpha \wedge A_\alpha + g_\alpha \, dA_\alpha = da_\alpha \, g_\alpha - a_\alpha \wedge dg_\alpha$$

$$(g_\alpha A_\alpha - a_\alpha g_\alpha) \wedge A_\alpha + g_\alpha \, dA_\alpha = da_\alpha \, g_\alpha - a_\alpha \wedge (g_\alpha A_\alpha - a_\alpha g_\alpha)$$

Therefore $g_\alpha(dA_\alpha + A_\alpha \wedge A_\alpha) = (da_\alpha + a_\alpha \wedge a_\alpha)g_\alpha$, hence we can write

$$F_\alpha \;=\; g_\alpha^{-1}\Omega_\alpha g_\alpha \, g_\alpha = ad \, (g_\alpha^{-1}) \, \Omega_\alpha \tag{4.1}$$

$$\Omega_\alpha \;=\; da_\alpha + a_\alpha \wedge a_\alpha \tag{4.2}$$

where $\Omega_\alpha = da_\alpha + a_\alpha \wedge a_\alpha \in \Gamma(\Lambda^2 T^* U_\alpha \otimes \mathfrak{g})$. From the transition rules

$$g_{\alpha\beta} a_\beta - a_\alpha g_{\alpha\beta} = dg_{\alpha\beta}$$

$$dg_{\alpha\beta} \wedge a_\beta + g_{\alpha\beta} \, da_\beta - da_\alpha \, g_{\alpha\beta} + a_\alpha \wedge dg_{\alpha\beta} = 0$$

$$\begin{aligned}(g_{\alpha\beta} a_\beta - a_\alpha g_{\alpha\beta}) \wedge a_\beta + g_{\alpha\beta} da_\beta \\ -da_\alpha g_{\alpha\beta} + a_\alpha \wedge (g_{\alpha\beta} a_\beta - a_\alpha g_{\alpha\beta}) = 0 \Rightarrow\end{aligned}$$

$$g_{\alpha\beta}(da_\beta + a_\beta \wedge a_\beta) = (da_\alpha + a_\alpha \wedge a_\alpha) g_{\alpha\beta} \Rightarrow$$

$$\Omega_\beta = g_{\alpha\beta}^{-1} \Omega_\alpha g_{\alpha\beta}$$

Therefore F is given by $\mathfrak{g}$-valued 2-form $\Omega = \{\Omega_\alpha\} \in \Gamma(\Lambda^2 T^* M \otimes ad \; P) = \Omega^2(ad \; P)$. Differentiating both sides of 4.2, and then by plugging the value of da_α from 4.2 into the resulting expression gives the *Bianchi identity* (4.3), which is just $d_A(\Omega) = 0$ (3.7):

$$d\Omega_\alpha = \Omega_\alpha \wedge a_\alpha - a_\alpha \wedge \Omega_\alpha \tag{4.3}$$

Remark 4.4. *Let $E \to X$ be a vector bundle with a connection A*

$$\Omega^0(E) \xrightarrow{D} \Omega^1(E) \xrightarrow{D} \Omega^2(E)$$

be the associated sequence of 3.6, then $D \circ D(s) = F_A \otimes s$.

4.1 Chern-Weil theory

Let $E \to M$ be a $\mathbb{C}^n$- vector bundle with a connection $A = \{a_\alpha\}$ and curvture $F_A = \{A_a\}$. The the trace map gives the commuting diagram, from which we can define Chern classes:

$$\begin{array}{ccc} \Omega^p(End(E)) & \xrightarrow{d_A} & \Omega^{p+1}(End(E)) \\ \downarrow \text{tr} & & \downarrow \text{tr} \\ \Omega^p(M) & \xrightarrow{d} & \Omega^{p+1}(M) \end{array}$$

Following [C] to a curvature $\Omega \in \Omega^2(End(E)$ we associate $2k$-forms $c_k(\Omega) \in \Omega^{2k}(M)$

$$det(I + \frac{i}{2\pi}\Omega) = I + c_1(\Omega) + ... + c_n(\Omega) \tag{4.4}$$

More generally, to any invariant polynomial f we can associate differential forms

$$f(y^{-1}x_1y, ..., y^{-1}x_ny) = f(x_1, ..., x_n)$$

$$f(\Omega, ..., \Omega) \in \Omega^*(M)$$

Theorem 4.5. *([C]) The differential forms $c_k(\Omega)$ $(k = 1, 2, ..)$ are closed, hence they define classes $c_k(\Omega) \in H^{2k}(M, \mathbb{C})$, and their definition does not depend on Ω*

Proof. From the definition $c_k(\Omega_\alpha) = c_k(\Omega_\beta)$ on $U_\alpha \cap U_\beta$, hence we get globally defined $2k$ forms by $c_k(\Omega)$ defined by $c_k(\Omega)|_{U_\alpha} = c_k(\Omega_\alpha)$. Define another set of invariant forms

$$b_k(\Omega) = tr(\frac{i}{2\pi}\Omega^k)$$

When Ω is diagonal both sets of forms $\{c_k\}$ and $\{b_k\}$ are symmetrical functions on the diagonal elements, and it is easy to see that in general they satisfy the identity

$$\sum_{j=0}^{n} b_{k-j}(\Omega)c_j(\Omega) = 0$$

$\{c_k\}$ can be written as a linear combination of $\{b_k\}$ (and wice versa). So it suffices to show $db_k(\Omega) = 0$, and $b_k(\Omega)$ do not depend on Ω. By using Bianchi identity (4.3) we get

$$dtr(\Omega^k) = ktr(d\Omega \wedge \Omega^{k-1}) = ktr(\alpha \wedge \Omega^{k-1} - \Omega \wedge a \wedge \Omega^{k-2}) = 0$$

Let us show $b_k(\Omega)$ does not depend on Ω: Let $A = \{a\}$ and $A' = \{a'\}$ be two connections, and $\Omega = da + a \wedge a$ and $\Omega' = a' + a' \wedge a'$ be the corresponding connections. Let $A_t = \{a_t\}$, where $a_t = ta' + (1-t)a$, and $\Omega_t = da_t + a_t \wedge a_t$ be the corresponding curvature. Call $\gamma = a' - a$, then $a_t = t\gamma + a$. Let $\tau = tr(\gamma \wedge \Omega^{k-1})$, using (4.3) we calculate:

$$\Omega_t = a_t + a_t \wedge a_t = \Omega + t(d\gamma + a \wedge \gamma + \gamma \wedge a) + t^2 \gamma \wedge \gamma = \Omega + t d_A(\gamma) + t^2 \gamma \wedge \gamma$$

$$\frac{d}{dt} tr(\Omega_t^k) = k.tr(\beta \wedge \Omega_t^{k-1}) \ , \quad \text{where } \beta = d_A(\gamma) + 2t\gamma \wedge \gamma = d_{A_t}(\gamma)$$

$$d\tau = tr[d_{A_t}(\gamma) \wedge \Omega_t^k - \gamma \wedge d_{A_t}(\Omega_t^{k-1})] = tr(d_{A_t}(\gamma) \wedge \Omega_t^{k-1}) = tr(\beta \wedge \Omega_t^{k-1})$$

$$\frac{d}{dt} tr(\Omega_t^k) = k.d\tau \quad \Rightarrow \quad tr(\Omega_1^k) - tr(\Omega_0^k) = kd \int_0^t \tau dt \qquad \square$$

Definition 4.6. *$c_k(E) := c_k(\Omega)$ are called Chern classes of the bundle $E \to M$.*

4.2 Flat Bundles

Let $\{U_\alpha\}$ be an open cover of M and $P = \{U_\alpha, g_{\alpha\beta}\}$ be a principal G-bundle $\pi : P \to M$. We say P is *flat* if all $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ are constant functions.

Theorem 4.7. *The following are equivalent*

(1) $P \xrightarrow{\pi} M$ is flat.

(2) $P \xrightarrow{\pi} M$ has a connection with zero curvature.

(3) There is a homomorphism $\rho : \pi_1(M) \to G$ with $P \cong \tilde{M} \times_\rho G$

$$\text{i.e. } P \cong \tilde{M} \times G / \sim \ , \quad \text{where } (x, g) \sim (x\ell^{-1}, \rho(\ell) g)$$

Proof. (2) $\Rightarrow$ (3): Every smooth path $\lambda : [0,1] \to M$ induces an isomorphism $\lambda^\# : \pi^{-1}(0) \xrightarrow{\cong} \pi^{-1}(1)$ which is invariant of C^∞ homotopies relative to the end points of the path. The reason for this is:

$$\begin{aligned} 0 &= F_A = dA(X,Y) + \frac{1}{2}[A(X), A(Y)] \\ 0 &= X(AY) - Y(AX) - A[X,Y] + [A(X), A(Y)] \end{aligned}$$

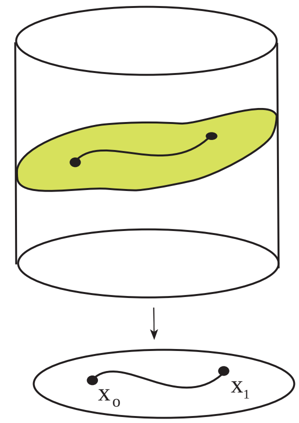


Figure 4.1:

Therefore If $X, Y \in \ker A \Rightarrow A[X,Y] = 0 \Rightarrow [X,Y] \in \ker A$, i.e. $\ker A$ deterines an involutive distribution on $P \Rightarrow \ker A$ is integrable, and hence it gives a foliation. Given any homotopy $\Lambda : [0,1] \times [0,1] \to M$ of the path λ, and its lifting $\tilde{\Lambda} : [0,1] \times [0,1] \to P$; then $\tilde{\Lambda}| : 1 \times [0,1] \to \pi^{-1}(x_1) \cap$ **a leaf**, being a continuous map from a connected space to a discrete space, maps $1 \times [0,1]$ to a single point. Therefore we can define:

$$\lambda^{\#}(p) = \tilde{\lambda}_p(1)$$

where $\tilde{\lambda}_p : [0,1] \to P$ is the lifing of the path λ with initial point $p \in P$. Therefore for a loop α based at x_0, we can define a map $\rho : \pi_1(M) \to G$ by

$$\tilde{\alpha}_p(1) = p.\rho[\alpha]$$

We can check that ρ is a homomorphism i.e. $\rho([\alpha][\beta]) = \rho[\alpha] \cdot \rho[\beta]$ by:

$$\begin{array}{rcl} \widetilde{(\alpha * \beta)}_p(1) & = & p \cdot \rho[\alpha * \beta] = \ p \cdot \rho([\alpha] \cdot [\beta]) \\ \| & & \\ \left(\tilde{\beta}_{\tilde{\alpha}_p(1)}\right)(1) & = & \tilde{\alpha}_p(1) \cdot \rho[\beta] = \ p \cdot \rho[\alpha]\rho[\beta] \end{array}$$

Recall that $\tilde{M} = \{\textit{Curves originating at } x_0\}/\textit{Homotopy}$, and also $\pi_1(M)$ acts on $\tilde{M}$, on the left, by the composition of paths $(\lambda, \alpha) \mapsto \lambda \cdot \alpha := \alpha * \lambda$. Now by fixing $q \in P$ we can define:

$$\eta : \tilde{M} \times G \longrightarrow P \quad \text{by}$$

$$\eta(\lambda, g) = \tilde{\lambda}_{q \cdot g}(1) \quad \Rightarrow$$

$$\begin{aligned}
\eta(\lambda \cdot \alpha, g) &= \eta(\alpha * \lambda, g) = (\alpha * \lambda)^{\#}_{q \cdot g}(1) \\
&= \left(\tilde{\lambda}_{\tilde{\alpha}_{q.g}(1)}\right)(1) = \left(\tilde{\lambda}_{\tilde{\alpha}_q(1).g}\right)(1) \\
&= \left(\tilde{\lambda}_{q.\rho(\alpha).g}\right)(1) = \eta(\lambda, \rho(\alpha)g)
\end{aligned}$$

Therefore $\eta(\lambda, \gamma) = \eta(\lambda.\alpha^{-1},\ \rho(\alpha)\gamma)$, and hence η induces an isomorphism.

$$\begin{array}{ccc}
\tilde{M} \times_\rho G & \xrightarrow{\eta} & P \\
& \searrow \quad \swarrow & \\
& M &
\end{array}$$

To see (3) $\Rightarrow$ (1), let $\eta : \tilde{M} \times_\rho G \xrightarrow{\cong} P$ be a bundle isomorphism corresponding to $\rho : \pi_1(M) \to G$. So if P is given by $\{U_\alpha, \rho(g_{\alpha\beta})\}$, where $\tilde{M} \xrightarrow{\pi} M$ is the principal $\pi_1(M)$-bundle $\{U_\alpha, g_{\alpha\beta} : U_\alpha \cap U_\beta \to \pi_1(M)\}$. Hence the bundle $P \to M$ reduces to the $\pi_1(M)$-bundle $\tilde{M} \longrightarrow M$ via

$$\rho(g_{\alpha\beta}) : U_\alpha \cap U_\beta \longrightarrow \pi_1(M) \underset{\rho}{\longrightarrow} G$$

Since $\pi_1(M)$ has discrete topology, these maps are constant.

To prove (1) $\Rightarrow$ (2), let $P = \{U_\alpha, g_{\alpha\beta}\}$ with $g_{\alpha\beta}$ constant. Let

$$P|U_\alpha \xrightarrow{h_\alpha} U_\alpha \times G$$

be the local trivializations. Define a connection A_α on $U_\alpha \times G$ by

$$A_\alpha(x, g_\alpha) = g_\alpha^{-1} dg_\alpha$$

That is, $a_\alpha = 0$ on this chart. Since each $g_{\alpha\beta}$ are constant, by the transformation rule this definition is valid on each chart. So for each α we have:

$$F_\alpha = g_\alpha^{-1} \Omega_\alpha g_\alpha = 0$$

□

Remark 4.8. *Let $P \to M$ be a principal G-bundle with a flat connection A. Then if M is 1-connected we can find a principal G-bundle isomorphism $\eta : P \xrightarrow{\cong} M \times G$, such that $A = \eta^*$ (product connection). This is because: A being flat, the distribution $H = \ker A$ comes from a foliation. So at each $p \in P$ there is a unique maximal integral manifold $M_p = M_{A,p} \subset P$ with:*

$$\begin{array}{ccc} M_p & \hookrightarrow & P \\ & \pi \searrow & \downarrow \pi \\ & & M \end{array} \qquad \begin{array}{c} T_p(M_p) = H_p \\ \pi| : H_p \xrightarrow{\cong} T_p(M) \implies \end{array}$$

$\pi|_{M_p}$ is a local diffeomorphism in fact $\pi| : M_p \to M$ is a covering space. Since H_p is g-equivariant $\Rightarrow M_{pg} = (M_p)g$. Since M is 1-connected and $\pi| : M_{A,p} \to M$ is a diffeomorphism. Therefore at each $x \in M$, $M_p \cap \pi^{-1}(x)$ is a unique point of P. Define $\eta^{-1} : M \times G \longrightarrow P$ by

$$(x, g) \longmapsto M_{pg} \cap \pi^{-1}(x)$$

This is a G-equivariant diffeomorphism.

There is also a uniqueness:

Proposition 4.9. *Let A, A' be flat connections on a given principal G-bundle $P \to M$. Then there is an equivariant map $\eta : P \to P$, such that $\eta^* A = A' \iff$ the corresponding representations $\rho_A \sim \rho_{A'}$ (i.e. they are conjugate).*

Proof. If $A' = \eta^*(A) \Rightarrow \eta(M_{A',p}) = M_{A,\eta(p)}$. For some fixed $p \in P$ write $\eta(p) = p.g$ (here g depends on p), then

$$\eta(\tilde{\alpha}_{A',p}(1)) = \tilde{\alpha}_{A,\eta(p)}(1) = \tilde{\alpha}_{A,pg}(1) = \tilde{\alpha}_{A,p}(1)g = p \cdot \rho_A[\alpha]g.$$

$$\text{Also:} \quad \eta(\tilde{\alpha}_{A',p}(1)) = \eta(p.\rho_{A'}[\alpha]) = \eta(p)\rho_{A'}[\alpha] = p \cdot g \cdot \rho_{A'}[\alpha]$$

$$\text{Therefore } \rho_A[\alpha]g = g\rho_{A'}[\alpha] \Rightarrow \rho_{A'}[\alpha] = g^{-1}\rho_A[\alpha]g.$$

Conversely, if $\rho_A \sim \rho_{A'}$, i.e. $\rho_{A'} = \ell \rho_A \ell^{-1}$ we can define

$$\varphi : \tilde{M} \times_{\rho_A} G \xrightarrow{\cong} \tilde{M} \times_{\rho_{A'}} G] \text{ by}$$

$$[x, g] \longrightarrow [x, \ell g],$$

This is well defined by:

$$\begin{array}{ccc} [x,g] & & [x,\ell g] \\ || & & || \\ [x\gamma^{-1},\rho_A(\gamma)g] & \longrightarrow & [x\gamma^{-1},\ell\rho_A(\gamma)g] = [x\gamma^{-1},\ell\rho_A(\gamma)\ell^{-1}\ell g] = [x\gamma^{-1},\rho_{A'}(\gamma)\ell g] \end{array}$$

Note that $\tilde{M}\times_{\rho_A} G$ has a natural flat connection A descending from the flat connection $\mathbb{A} = g^{-1}dg$ on $\tilde{M}\times G$ by the quotient map:

$$\begin{array}{ccc} \tilde{M}\times G & \longrightarrow & \tilde{M} \\ \downarrow & & \downarrow \\ \tilde{M}\times_{\rho_A} G & \longrightarrow & M \end{array}$$

$$\begin{aligned} \mathbb{A}_{(y\gamma^{-1},\rho_A(\gamma)g)} &= (\rho_A(\gamma)g)^{-1}d(\rho_A(\gamma)g) \\ &= g^{-1}(\rho_A(\gamma))^{-1}\rho_A(\gamma)dg \\ &= g^{-1}dg = \mathbb{A}_{(y,g)} \end{aligned}$$

So in particular $v \in H_A \subset T(\tilde{M}\times_{\rho_A} G) \iff v$ is the image under the projection $\tilde{M}\times G \longrightarrow \tilde{M}\times_{\rho_A} G$ of a vector in $H_{\mathbb{A}}$. Clearly $F_A \equiv 0$.

Chapter 5

The gauge group

Definition 5.1. *A gauge transformation of a principal bundle* $P = \{U_\alpha, g_{\alpha\beta}\} \to X$ *is a fibre preserving map* $s : P \to P$*, which means the following diagram commutes*

$$\begin{array}{ccc} P & \xrightarrow{s} & P \\ \pi \searrow & & \swarrow \pi \\ & X & \end{array} \tag{5.1}$$

and also it is equivariant with respect to the action of G *which means* $s(pg) = s(p)g$*. The set of these transformations is called the gauge group and denoted by* $\mathcal{G}(P)$ *or* $Aut(P)$*.*

We can describe $s \in \mathcal{G}(P)$ by a local data $s = \{s_\alpha : U_\alpha \to G\}$. On each chart $U_\alpha \times G \to U_\alpha$ let s be given by by $(x, g_\alpha) \mapsto (x, s_\alpha g_\alpha)$. Hence when $g_\alpha \sim g_\beta$ we must have $s_\alpha g_\alpha \sim s_\beta g_\beta$. So when $g_\alpha = g_{\alpha\beta} g_\beta \implies s_\alpha g_\alpha = g_{\alpha\beta}(s_\beta g_\beta) \implies s_\alpha g_{\alpha\beta} g_\beta = g_{\alpha\beta} s_\beta g_\beta$ $\implies$ s satisfies $s_\alpha(x) = g_{\alpha\beta} s_\beta(x) g_{\alpha\beta}^{-1}(x) \implies s$ is a section of the bundle $Ad(P) \to X$

$$Ad(P) := P \times_{Ad} G = P \times G/_\sim \;\; (p, g) \sim (ph, hgh^{-1})$$

The sections $\Gamma(Ad(P))$ can also are identified with $\{f \in C^\infty(P, g) \mid f(pg) = g^{-1} f(p) g\}$. This identification can easily be checked by the association $s(p) = pf(p)$. To sum up

$$\mathcal{G}(P) = \{s : P \to P \mid s(pg) = s(p)g\} = \{f : P \to G \mid f(pg) = g^{-1} f(p) g\}$$

If $P \to X$ is the frame bundle of a vector bundle $E \to X$, then we can also identify

$$\mathcal{G}(P) = Aut(E) \subset \Gamma(E^* \otimes E) = End(E) \tag{5.2}$$

In this case $G = Aut(V)$, where V is the fiber of E, and we can think of $s = \{s_\alpha\}$ defined above, as a section of $Aut(E)$ by the association $v \mapsto s_\alpha(v)$.

$$\begin{array}{ccc} E & \xrightarrow{s} & E \\ \pi \searrow & & \swarrow \pi \\ & X & \end{array} \tag{5.3}$$

The tangent space of the gauge group at the identity gauge transformation I is

$$T_I\mathcal{G}(P) = \Omega^0(adP) = End(adP)$$

Recall that after fixing a base connection $A_0 \in \mathcal{A}(P)$ we identified

$$\mathcal{A}(P) \cong T_{A_0}\mathcal{A}(P) = \{A_0\} + \Omega^1(adP)$$

There is an action $\mathcal{G}(P)\times\mathcal{A}(P) \to \mathcal{A}(P)$ by pull back $(A, s) \to s^*A$. More specifically in our chart notation if $A = \{a_\alpha\}$ and $s = \{s_\alpha\}$ then we can express

$$s^*A = \{s_\alpha^{-1}ds_\alpha + s_\alpha^{-1}a_\alpha s_\alpha\} \tag{5.4}$$

The derivative of the obvious map $\mathcal{G}(P) \to \mathcal{A}(P)$, $s \mapsto s^*A_0$ at $s = I$

$$d_A : \Omega^0(adP) \to \Omega^1(adP) \text{ is given by } d_A(\dot{s}) = d\dot{s}_\alpha + [a_\alpha, \dot{s}_\alpha] \tag{5.5}$$

The derivative of the curvature map $\mathcal{A}(P) \to \Omega^2(P)$, $A \mapsto F_A$ at $A_0 = \{a_\alpha\}$

$$d_{A_0} : \Omega^1(adP) \to \Omega^2(adP) \text{ is given by } d_{A_0}(b) = db + [a_\alpha, b] = ds_\alpha + a_\alpha \wedge b + b \wedge a_\alpha$$

Example 5.2. *When $G = S^1$ then $\mathcal{G}(P) = Maps(X, S^1)$*

5.1 Gauge group of $SU(2)$ bundles over X^4

Let $P \to X^4$ be a principal $G = SU(2)$ bundle. Denote $\hat{G} = SU(2)/\{\pm I\} = SO(3)$. Clearly $-I \in \mathcal{G}(P)$ and $\{\pm I\} \subset \mathcal{G}(P)$ is the center. Define $\widehat{\mathcal{G}}(P) = \mathcal{G}(P)/\{\pm I\}$. We define *reducible connections*

$$\mathcal{A}_{red}(P) \cong \{A_0\} + \Omega^1(ad_{S^1}P)$$

where $ad_{S^1}P$ is the bundle associated to adjoint representation $\rho : S^1 \to End(\mathfrak{g})$, and $\mathfrak{g} = \mathfrak{su}(2)$. Considering the subgroup $S^1 \subset SU(2)$ the bundle $ad_{S^1}P$ is given by

$$g_{\alpha,\beta} = \begin{pmatrix} h_{\alpha,\beta} & 0 \\ 0 & \bar{h}_{\alpha,\beta} \end{pmatrix}$$

$$\mathfrak{g} = \{\begin{pmatrix} i\lambda & b \\ -\bar{b} & -i\lambda \end{pmatrix} \mid \lambda \in \mathbb{R}, b \in \mathbb{C}\} \cong \mathbb{R}^3$$

$$(b,\lambda) \cong \begin{pmatrix} i\lambda & b \\ -\bar{b} & -i\lambda \end{pmatrix} \mapsto g_{\alpha,\beta} \begin{pmatrix} i\lambda & b \\ -\bar{b} & -i\lambda \end{pmatrix} g_{\alpha,\beta}^{-1} = \begin{pmatrix} i\lambda & h_{\alpha,\beta}^2 b \\ -\bar{h}_{\alpha,\beta}^2 \bar{b} & -i\lambda \end{pmatrix} \cong (h_{\alpha,\beta}^2 b, \lambda)$$

Therefore we have the bundle identification $ad_{S^1}P \cong K^2 \oplus \mathbb{R}$, with K is the line bundle $\{h_{\alpha,\beta}\}$. Also note that the standard representation $S^1 \to End(\mathbb{C}^2)$ associates P_{S^1} a pair of complex line bundles K and $\bar{K}$. We define *irreducible connections* by the complement $\mathcal{A}^*(P) = \mathcal{A}(P) - \mathcal{A}_{red}(P)$.

Proposition 5.3. *$\widehat{\mathcal{G}}(P)$ acts on $\mathcal{A}^*(P)$ freely, and $\mathcal{A}^*(P)$ is contractible.*

Proof. If $s^*A = A$ for $s = \{s_\alpha : U_\alpha \to G\} \in \mathcal{G}(P)$ and $a_\alpha = s_\alpha^{-1} ds_\alpha + s_\alpha^{-1} a_\alpha s_\alpha$. Hence

$$d_A(s_\alpha) = ds_\alpha + [a_\alpha, s_\alpha] = ds_\alpha + \rho_*(a_\alpha)s_\alpha = 0 \tag{5.6}$$

Let λ_a, $\bar{\lambda}_\alpha$ be eigenvalues, and v_α be an eigenvector of s_α. By differentiating the both sides of $s_\alpha(v_\alpha) = \lambda_\alpha v_\alpha$ we get

$$\begin{aligned} s_\alpha(d_A v_\alpha) &= (d\lambda_\alpha)v_\alpha + \lambda_\alpha d_A(v_\alpha) \\ \langle s_\alpha(d_A v_\alpha), v_\alpha\rangle &= d\lambda_\alpha + \lambda_\alpha \langle d_A(v_\alpha), v_a\rangle \\ \langle s_\alpha(d_A v_\alpha), v_\alpha\rangle = \langle d_A(v_\alpha), s_\alpha^*(v_\alpha)\rangle &= \langle d_A(v_\alpha), \bar{\lambda}_\alpha(v_\alpha)\rangle = \lambda_\alpha \langle d_A(v_\alpha), v_\alpha\rangle \end{aligned} \tag{5.7}$$

Hence by (4.7) $d\lambda_\alpha = 0$ on U_α and λ_α is constant, also on $U_\alpha \cap U_\beta$, $\lambda_\alpha = \lambda_\beta$ because

$$\lambda_\alpha v_\alpha = s_\alpha(v_\alpha) = s_\alpha(g_{\alpha,\beta} v_\beta) = g_{\alpha\beta} s_\beta(v_\beta) = \lambda_\beta g_{\alpha\beta} v_\beta = \lambda_\beta v_\alpha$$

Call $\lambda_\alpha = e^{i\mu}$. Then $s : E \to E$ splits E into line bundles $E = K \oplus \bar{K}$, where

$$K = \{v \in E \mid s(v) = e^{i\mu} v\}$$

Since $s_\alpha(d_A v_\alpha) = \lambda_\alpha d_A(v_\alpha)$, d_A preserve K and $\bar{K}$, $A = A_0 + a$ is a reducible connection. From (4.6) we see the reducible connections are identified with $im(d_A)$ hence irreducible ones are identified by $ker(d_A^*)$; we will see this is infinite dimensional space. Hence $\mathcal{A}(P)$ contractible implies $\mathcal{A}^*(P)$ is contractible. ∎

$$\widehat{\mathcal{G}}(P) \to \mathcal{A}^*(P) \to \mathcal{A}^*(P)/\widehat{\mathcal{G}}(P) = \mathcal{B}^*(P)$$

Since $\mathcal{A}^*(P)$ is contractible, homotopically we can identify the base space as the classifying spaces of the corresponding gauge groups: $\mathcal{B}^*(P) = B_{\widehat{\mathcal{G}}}$

Proposition 5.4. *Let $P \to X^4$ be a principal G-bundle of a compact 4-manifold X. Then if $G = SO(3)$ there is a weak homotopy equivalence:*

$$\mathcal{B}^*(P) \backsimeq Map^P(X; B_{SO(3)})$$

where the right side is the component of the space of (unbased) maps inducing the bundle P. If $G = SU(2)$, then $\mathcal{B}^*(P)$ is the total space of a covering space with fiber $H^1(X; Z_2)$:

$$H^1(X; Z_2) \to \mathcal{B}^*(P) \to Map^P(X; B_{SO(3)}).$$

Proof. Let $G = SO(3)$; then consider the principal G-bundle:

$$\xi_P = \mathcal{A}^* \times_{\widehat{\mathcal{G}}} P \to \mathcal{B}^* \times X. \tag{5.8}$$

The total space is the equivalence classes of points $[A,p]$ in $\mathcal{A}^* \times P$ identified by the equivalence relation $(A,p) \sim (s^*A, p.s(p))$ where $s \in \widehat{\mathcal{G}}$. The $SO(3)$ action on the total space is given by the right multiplication: i.e., if $g \in SO(3)$ and $[A,p] \in \mathcal{A}^* \times_{\widehat{\mathcal{G}}} P$, then $[A,p]g = [A,pg]$. The action is well defined since $[A,p]g = [A,pg] = [s^*A, pg.s^*(pg)] = [s^*A, pg.g^{-1}s(p)g] = [s^*A, p.s(p)g] = [s^*A, p.s(p)]g$. It is free, since if $[A,p] = [A,pg]$, then $(A,p.g) = (s^*A, p.s(p))$ for some s, since A irreducible $s = I$, so $pg = p \implies g = I$.

Now consider the classifying maps to the universal $SO(3)$-bundle:

$$\begin{array}{ccc} \mathcal{A}^* \times_{\widehat{\mathcal{G}}} P & \xrightarrow{\tilde{\sigma}} & E_{SO(3)} \\ \downarrow & & \downarrow \\ \mathcal{B}^* \times X & \xrightarrow{\sigma} & B_{SO(3)} \end{array}$$

where $\tilde{\sigma}$ is an $SO(3)$-equivariant map and the diagram commutes. This in turn induces the following commutative diagram:

$$\begin{array}{ccc} \mathcal{A}^* & \xrightarrow{\tilde{\alpha}} & \mathrm{Map}_{SO(3)}(P, E_{SO(3)}) \\ \downarrow & & \downarrow \\ \mathcal{B}^* & \xrightarrow{\alpha} & \mathrm{Map}^P(X, B_{SO(3)}) \end{array} \tag{5.9}$$

where $\mathrm{Map}_{SO(3)}(P, E_{SO(3)})$ denotes the set of $SO(3)$-equivariant maps, and

$$\tilde{\alpha}(A)(p) = \tilde{\sigma}[A,p] \ \text{ and } \ \alpha([A])(x) = \sigma([A],x).$$

Since $\tilde{\alpha}(A)(pg) = \tilde{\sigma}[A,pg] = \tilde{\sigma}([A,p]g) = (\tilde{\sigma}[A,p])g = (\tilde{\sigma}(A)(p))g$, $\tilde{\alpha}$ is well defined. Furthermore, each vertical map of the diagram principal $\widehat{\mathcal{G}}$-bundle map, where the action of $\widehat{\mathcal{G}}$ on $\mathrm{Map}_{SO(3)}(P, E_{SO(3)})$ is defined by $(s, f) \mapsto s^* f$ with $(s^* f)(p) = f(p.s^{-1}(p))$; and $\tilde{\alpha}$ is $\widehat{\mathcal{G}}$ equivariant since

$$\begin{aligned}\tilde{\alpha}(s^* A)(p) &= \tilde{\sigma}[s^* A, p] = \tilde{\sigma}[(s^{-1})^* s^* A, p.s^{-1}(p)] \\ &= \tilde{\sigma}[A, p.s^{-1}(p)] = \tilde{\alpha}(A)(p.s^{-1}(p)) = s^*(\tilde{\alpha}(A))(p).\end{aligned}$$

Since $E_{SO(3)}$ is contractible, so is $\mathrm{Map}_{SO(3)}(P, E_{SO(3)})$ and hence both of these are universal principal $\widehat{\mathcal{G}}$-bundles. Hence, the base spaces are weakly homotopy equivalent.

In case $G = SU(2)$, we let $\rho : P \to \overline{P}$ denote the reduction map to the underlying $SO(3)$-bundle; then we have the basic exact sequence

$$Z_2 \to \mathcal{G}(P) \to \mathcal{G}(\overline{P}) \xrightarrow{\alpha} H^1(X; Z_2) \to 1$$

where α is defined as follows: for any element $h : \overline{P} \to \overline{P}$ of $\mathcal{G}(\overline{P})$ we let $\alpha(h)$ be the difference of the two spin structures ρ and $h \circ \rho$ on $\overline{P}$. Now the proof proceeds just as before. Since the center $\{\pm I\}$ of $\mathcal{G}(P)$ acts trivially on $\mathcal{A}^*$, we get an $SO(3)$-bundle as in (5.8). The only exception is that in the diagram (5.9) the left vertical map is a principal $\widehat{\mathcal{G}}(P)$-bundle and the right vertical map is a principal $\widehat{\mathcal{G}}(\overline{P})$-bundle. Hence to get a principal $\widehat{\mathcal{G}}(P)$-bundle on the right side we must divide $\mathrm{Map}_{SO(3)}(P, E_{SO(3)})$ only by the subgroup $\mathcal{G}(P)$ of $\mathcal{G}(\overline{P})$. This gives us $\mathcal{B}^*(P)$. Since $\mathrm{Map}^P(X, B_{SO(3)})$ is obtained from $\mathrm{Map}_{SO(3)}(P, E_{SO(3)})$ by dividing $\mathcal{G}(\overline{P})$, $\mathrm{Map}^P(X, B_{SO(3)})$ is obtained from $\mathcal{B}^*(P)$ by dividing $\mathcal{G}(\overline{P})/\mathcal{G}(P)$, which is $H^1(X; Z_2)$. □

We can now interpret the homotopy groups of $\mathcal{B}^*(P)$ in a geometric way:

$$\begin{aligned}\pi_k(\mathcal{B}^*) &= [S^k, s_0\ ;\ \mathrm{Map}^P(X,\ B_{SO(3)}), P] \\ &= \pi_0\ \mathrm{Map}_*(S^k,\ \mathrm{Map}^P(X,\ B_{SO(3)})) \\ &= \pi_0\ \mathrm{Map}(S^k \times X,\ B_{SO(3)})^P \\ &= [S^K \times X,\ B_{SO(3)}]^P\end{aligned}$$

where s_0 denotes the base point of S^k and the exponential P means the set of maps that restrict to P on $s_0 \times X$ and homotopic to P on each slice $s \times X$ (by viewing view P as a map). Thus we are reduced to studying isomorphism classes of bundles over $S^k \times X$, restricting to bundles isomorphic to P on each slice, and equal to P on a particular slice:

$$\pi_k(\mathcal{B}^*) = \left\{ \begin{array}{c} \xi \\ \downarrow \\ S^K \times X \end{array} \middle|\ \xi\big|_{s_0 \times X} = P,\ \ \xi\big|_{s \times X} \cong P \ \text{ for each } \ s \in S^k \right\}. \tag{5.10}$$

Chapter 6

Characteristic Classes

Let $V_k^n(X)$ be the isomorphism classes of k^n-vector bundles $E \to X$ over X, where $k = \mathbb{R}$ or $\mathbb{C}$, and denote $V_k(X) = \oplus V_k^n(X)$. Let $\widetilde{V_{\mathbb{R}}^n}(X)$ be the isomorphism class of oriented real vector bundles, and $\widetilde{V_{\mathbb{R}}}(X) = \oplus \widetilde{V_{\mathbb{R}}^n}(X)$, and let $H^*(X;\mathbb{R}) = \oplus H^i(X;\mathbb{R})$.

Definition 6.1. *The total Chern class is a map:*

$$c : V_{\mathbf{C}}(X) \to H^*(X;\mathbb{Z})$$

where $c(E) = 1 + c_1(E) + c_2(E) + ..$ *with* $c_i(E) \in H^{2i}(X, \mathbb{Z})$, $i = 0, 1, ..$ *satisfying:*

(a) Naturality: If $f : X \to Y$ *is continuous function then* $c(f^*E) = f^*(cE)$

(b) Whitney sum: $c(E \oplus F) = c(E)c(F)$

(c) Finiteness: If $E \in Vec_k^n(X)$ *then* $c_i(E) = 0$ *for* $i > n$

(d) Nontriviality: If $L \to \mathbb{CP}^1$ *is the tautological line bundle, then* $c_1(L) = -\alpha$*, where* $\alpha \in H^2(\mathbb{CP}^1, \mathbb{Z})$ *is the genertator*

These properties uniquely determine Chern classes by the following proposition:

Proposition 6.2 (Splitting principle)**.** *For every $E \in V_k^n(X)$, then is some space Y and a map $f : Y \to X$ satisfying:*

- $f^*(E) = L_1 \oplus ... \oplus L_n$*, where each* L_i *is a* k*-line bundle.*
- $f^* : H^*(X; R) \to H^*(Y; R)$ *is an injection, where* $R = \mathbb{Z}_2$ *or* $\mathbb{Z}$ *depending upon* $k = \mathbb{R}$ *or* $\mathbb{C}$ *repectively.*

By this proposition any two maps $P, Q : V(X) \to H^*(X)$, satisfying (a), (b), (c) above, has to agree if they agree on line bundles. This is because by choosing f as above, we get $f^*(PE) = P(f^*E) = Q(f^*E) = f^*(QE)$, where f^* is an injection.

Example 6.3. *The classes defined by Theorem 4.5 satisfy these axioms, so they are the integral classes defined by these axioms. For example to see (d) we cover* S^2 *with two copies of* $\mathbb{C}$ *(complements of north and south pole)* $\{U_-, U_+\}$*. Let* $L \to S^2$ *be the line bundle given by the transition function* $z \mapsto z^n$*. Let* $A = \{a_-, a_+\}$ *be a connection on* L*,* $a_+ = a_- + z^{-n}d(z^n) = a_- + nz^{-1}dz$*,* $F_A = \{da_-, da_+\}$*, then we compute* $\langle c_1(L), [S^2] \rangle$ *by*

$$\frac{i}{2\pi}\int_{S^2} F_A = \frac{i}{2\pi}(\int_{D_+} da_+ + \int_{-D_-} da_-) = \frac{i}{2\pi}\int_{S^1}(a_+ - a_-) = \frac{i}{2\pi}\int_{S^1} n\frac{dz}{z} = -n$$

6.0.1 Pontryagin classes

Let $E \in \widetilde{V_{\mathbb{R}}}(X)$ be a real vector bundle and $E \otimes \mathbb{C} \to X$ be its complexification.

Definition 6.4. *The total Pontryagin class and Pontyagin classes are defined by* $p(E) = 1 + p_1(E) + p_2(E) + ..$ *where*

$$p_i(E) = (-1)^i c_{2i}(E \otimes \mathbb{C}) \in H^{4i}(X; \mathbb{Z})$$

Example 6.5. *If* $E = L_1 \oplus ... \oplus L_n$*, with each* L_i *is a complex line bundle , then*

$$c(E) = \prod_{i=1}^{n}(1 + x_i)$$

where $x_i = c_1(L_i) \in H^2(X)$*. Hence Chern classes given by the elementary symetric polynomials* $c_r(E) = s_i(x_1, .., x_n) = \sum_{j_1 < j_2 < .. < j_r} x_{j_1} x_{j_2} .. x_{j_r}$*. If each* L_i *is an oriented 2-plane bundle, then* $L_i \otimes \mathbb{C} = L_i \oplus \bar{L}_i$*, and hence*

$$p(E) = \prod_{i=1}^{n}(1 + x_i^2)$$

Hence $p_i(E) = s_i(x_1^2, .., x_n^2)$ are elementary symetric functions in $x_1^2, .., x_n^2$

6.0.2 Steifel Whitney classes

$w(E) = 1 + w_1(E) + w_2(E)...$ where

$$w : V_{\mathbb{R}}(X) \to H^*(X, \mathbb{Z}_2)$$

with $w_i(E) \in H^i(X, \mathbf{Z}_2)$. These are the real version of Chern classes, they are defined by the naturality axioms (a), (b), (c) and the real version of the fourth axiom (d'): For the tautological line bundle $L \to \mathbb{RP}^2$, $w_1(L)$ is the generator of $H^1(\mathbb{RP}^2, \mathbb{Z}_2)$.

6.0.3 Natural classes

Definition 6.6. *Let $\mathbb{Q}[[t]]$ denote the ring of power series. The additive (or multiplicative) natural class associated to an element $q \in \mathbb{Q}[[t]]$ is a map:*

$$Q : V_{\mathbb{C}}(X) \to H^*(X, \mathbb{Q}) \ \ \textit{satisfying}:$$

(a) If $f : X \to Y$ is a continuous function then $Q(f^*E) = f^*(QE)$

(b) $Q(E \oplus F) = Q(E) + Q(F)$ (or $Q(E \oplus F) = Q(E)Q(F)$)

(c) $Q(\gamma) = q(c_1(\gamma))$, if $\gamma \in Vec^1_{\mathbb{C}}(X)$

Q is uniquely defined and it can be expressed in terms of Chern classes. This is because by the splitting principle we can assume $E = L_1 \oplus L_2 \oplus ..L_n$, where each L_i is a complex line bundle. Hence $Q(E)$ is the sum (or the product) of $q(x_i)$'s, since this expression is symmetric in x_i's, it can be written in terms of elementery symmetric functions $s_i(x_1, .., x_n) = c_i(E)$.

The additive natural class corresponding to the power series $q(t) = e^t$ is called the *Chern character* denoted by ch, and the multiplicative natural class corresponding to $q(t) = t/(1 - e^{-t})$ is called the *Todd class* and denoted by td.

$$ch(E) = \sum e^{x_i} = n + c_1 + \frac{1}{2}(c_1^2 - 2c_2) + .. \tag{6.1}$$

$$td(E) = \prod \frac{x_i}{1 - e^{-x_i}} = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_2 + c_1^2) + .. \tag{6.2}$$

Chern character $ch : Vec_{\mathbb{C}}(X) \to H^*(X,\mathbb{Q})$ in fact is a ring homomorphism:

$$ch(E \oplus F) = ch(E) + ch(F) \tag{6.3}$$

$$ch(E \otimes F) = ch(E)ch(F) \tag{6.4}$$

Let $\widetilde{Vec}_{\mathbb{R}}(X)$ denote the real oriented vector bundles over X. Similarly to an element $q \in \mathbb{Q}[[t]]$ we can define a *real natural classes* as a map

$$Q : \widetilde{Vec}_{\mathbb{R}}(X) \to H^*(X,\mathbb{Q})$$

satisfying properties (a), (b) above and the real version of the third axiom

(c') $Q(\gamma) = 1$ if $\gamma \in \widetilde{Vec}^1_{\mathbb{R}}(X)$, and $Q(\gamma) = q(p_1(\gamma))$ if $\gamma \in \widetilde{Vec}^2_{\mathbb{R}}(X)$

In this case Q is also uniquely defined and it can be expressed as a combination of Ponrtyagin classes. This is because: $Q(E \oplus \epsilon) = Q(E)$ where ϵ is the trivial line bundle, so we can assume E is even dimensional, and by the splitting principle we can assume $E = L_i \oplus L_2 \oplus ..L_n$ is direct sum of 2-dimensional oriented bundles. Then we can compute the multiplicative real natural class $Q(E)$ by:

$$Q(E) = \prod Q(L_i) = \prod q(p_1(L_i)) = \prod q(x_i^2) = Q(p_1(E), ..., p_n(E))$$

Since next to last expression is symmetric in x_i^2's, it is be written in terms of elementery symmetric functions $s_i(x_1^2, .., x_n^2) = p_i(E)$. In literature $Q(p_1, .., p_n)$ is usually called q-genus, where q is the defining power series. For example

$$l(t) = \frac{\sqrt{t}}{\tanh\sqrt{t}}$$

gives the L-genus of Hirzebruch. From the expression below we can compute

$$L(E) = \prod \frac{x_i}{\tanh\, x_i} = 1 + L_1(E) + L_2(E) + ..$$

$$\begin{aligned} L_1(E) &= \frac{1}{3}p_1 \\ L_2(E) &= \frac{1}{45}(7p_2 - p_1^2) \\ L_3(E) &= \frac{1}{945}(62p_3 - 13p_2p_1 + 2p_1^3) \text{ ... etc.} \end{aligned}$$

Another useful real natural class is **A**-genus which is associated to

$$a(t) = \frac{\frac{\sqrt{t}}{2}}{\tanh \frac{\sqrt{t}}{2}}$$

$$A(E) = \prod \frac{\frac{x_i}{2}}{\tanh \frac{x_i}{2}} = 1 + A_1(E) + A_2(E) + ..$$

$$\begin{aligned} A_1(E) &= -\frac{1}{24}p_1 \\ A_2(E) &= \frac{1}{5760}(-4p_2 + 7p_1^2) \\ A_3(E) &= \frac{1}{967680}(16p_3 - 44p_2p_1 + 31p_1^3) \text{ ... etc.} \end{aligned}$$

Any multiplicative real natural class defines a ring homomorphism

$$Q : \Omega^* \to \mathbb{Q}$$

where $\Omega^* = \oplus \Omega^i$ is the orinted cobordisim ring: by $Q[M] =< Q(TM), [M] >$, where $TM \to M$ is the tangent bundle, and $[M]$ denotes the fundamental homology class. This is because if $i : \partial W \to W$ is the inclusion, then

$$< Q(\partial W), [\partial W] >=< i^* Q(W), [\partial W] >=< Q(W), i_*[\partial W] >= 0.$$

Theorem 6.7 (Hirzebruch)**.** *The signature $\sigma(M)$ of a closed oriented manifold M can be expressed as:*

$$I(M) = L(M)$$

Proof. Since the signature σ, and $L : \Omega \otimes \mathbb{Q} \to \mathbb{Q}$ are two ring homomorphisims on the polynomial ring $\Omega \otimes \mathbb{Q} = \mathbb{Q}[[\mathbb{CP}^2], .., [\mathbb{CP}^{2n}], ..]$, which is known to be generated by the even dimensional complex projective spaces, it suffices to show:

$$L[\mathbb{CP}^{2n}] = \sigma[\mathbb{CP}^{2n}] = 1$$

$T(\mathbb{CP}^{2n}) \oplus \varepsilon^1 = \bar{L} \oplus .. \oplus \bar{L}$ where $L \to \mathbb{CP}^{2n}$ is the canonical line bundle, hence

$$L[\mathbb{CP}^{2n}] =< (\frac{z}{\tanh z})^{2n+1}, \mathbb{CP}^{2n} >$$

where $z \in H^2(\mathbb{CP}^{2n})$ is the generator. Hence $L[\mathbb{CP}^{2n}]$ is the coefficient of z^{2n} in the power series, which we can compute by the residue

$$\frac{1}{2\pi i}\int (\frac{z}{\tanh\, z})^{2n+1}\frac{1}{z^{2n+1}}dz = \frac{1}{2\pi i}\int \frac{dz}{(\tanh\, z)^{2n+1}} = \frac{1}{2\pi i}\int \frac{du}{u^{2n+1}(1-u^2)} = 1$$

where $u = \tanh\, z$ and $dz = du/(1-u^2)$

□

Let $P \to X$ be a principal G-bundle, and $R(G)$ be the complex representation space of the Lie group G. The association $\rho \to E_\rho$ gives a map:

$$R(G) \to \mathrm{Vect}_{\mathbb{C}}(X)$$

Let $T \subset G$ be the maximal torus; then $R(T) = \mathbb{Z}[t_1, t_1^{-1}, ..., t_k, t_k^{-1}]$, where $k =$ rank (T), and t_i is the representation of T given by

$$(v_1, ...v_k).v \to (v_1, .., t_i v_i, ..., v_k)$$

Let $W = W(G) = N(T)/T$ be the Weyl group, where

$$N(T) = \{g \in G \mid gTg^{-1} \subset T\ \}$$

W acts on T, hence on $R(T)$. Since inner automorphisms act as identity on $R(G)$ the restriction map $R(G) \to R(T)$ factors to the subspace fixed by W, by a monomorphisim

$$R(G) \hookrightarrow R(T)^W$$

So we can describe any associated vector bundle $\rho \to E_\rho$, by a polynomial $f(t_1, t_1^{-1}, ..., t_k, t_k^{-1})$ in $R(T)$ invariant under the Weyl group.

The inclusion $T \subset G$ induces a map $BT \to BG$ which in turn gives: $H^*(BG;\mathbb{Q}) \to H^*(BT;\mathbb{Q})$, and W acts on $H^*(BT;\mathbb{Q})$. Borel's theorem says that this inclusion factors through the subspace fixed by W, by an isomorphisim

$$H^*(BG;\mathbb{Q}) \xrightarrow{\cong} H^*(BT;\mathbb{Q})^W$$

$B(T) = E(T)/T = E(S^1)/S^1 \times .. \times E(S^1)/S^1 = \prod_{i=1}^k \mathbb{CP}^\infty$, in particular:

$$H^*(BT;\mathbb{Q}) = \mathbb{Q}[x_1, x_1^{-1}, ..., x_k, x_k^{-1}]$$

For example with these identifications the Chern character of the vector bundle E_ρ corresponding to the invariant polynomial $f(t_1, t_1^{-1}, ..., t_k, t_k^{-1})$ can be written as:

$$ch(E_\rho) = f(e^{x_1}, e^{-x_1}, ..., e^{x_k}, e^{-x_k}) \in H^*(X;\mathbb{Q})$$

where $e^{x_i} = ch(L_i)$, and $L_i = E_{t_i}$ is the line bundle associated to the representation t_i, i.e. $e_i = c_1(L_i)$

6.1 Clifford Algebra

Consider $\mathbb{R}^n$ with the standard inner product $<,>$, define Clifford Algebra:

$$C_n = \left(\sum_{k=0}^{\infty} \otimes^k \mathbb{R}^n \right) / \mathcal{I}$$

where $\mathcal{I}$ = ideal generated by $x \otimes x + \|x\|^2$. If we choose an orthogonal basis $\{e_1, ..., e_n\}$ of $\mathbb{R}^n$, then $e_i^2 = -1$ and $e_i e_j = -e_j e_i \quad i \neq j$, and we can identify

$$C_n = \text{span}_{\mathbb{R}} \{ \, e_{i_1} .... e_{i_r} \mid i_1 < i_1 < ... < i_r \, \}$$

In particular as $SO(n)$ vector spaces $C_n \cong \Lambda^*(R^n)$, so then dim $C_n = 2^n$.
C_n is an $\mathbb{R}$ algebra with the universal property:
For every linear map $f : \mathbb{R}^n \longrightarrow A$ to an $\mathbb{R}$-algebra A, with the property $f(x)^2 = -\|x\|^2 1$, factors uniquely through an $\mathbb{R}$- algebra homomorphism $\tilde{f}$

$$\begin{array}{ccc} C_n & \xrightarrow{\tilde{f}} & A \\ \uparrow & \nearrow & \\ \mathbb{R}^n & & \end{array}$$

Define $\mathbb{C}_n = C_n \otimes \mathbb{C}$. For example $\mathbb{C}_1 \cong \mathbb{C}$ and $\mathbb{C}_2 \cong \mathbb{H}$, by maps:

$$\begin{array}{rcl} 1 & \longmapsto & 1 \\ e_1 & \longmapsto & i \end{array} \qquad \begin{array}{rcl} 1 & \longmapsto & 1 \\ e_1 & \longmapsto & i \\ e_2 & \longmapsto & j \\ e_1 e_2 & \longmapsto & k \end{array}$$

Proposition 6.8. *There exists a complex vector space Δ of dimension 2^l and a $\mathbb{C}$-algebra isomorphism:*

$$\mathbb{C}_{2l} \cong End(\Delta) \cong \Delta^* \otimes \Delta$$

Proof. We can see this several different ways: Consider the involutions

$$Q_i : \mathbb{C}_{2l} \to \mathbb{C}_{2l}$$

given by the right multiplication by $i.e_{2i-1}e_{2i}$, they have the properties $Q_i^2 = id$ and $Q_iQ_j = Q_jQ_i$. Then ±1 eigenspaces of Q_i split $\mathbb{C}_{2l}$ into 2^l subpaces, i.e.

$$\mathbb{C}_{2l} = \oplus_\epsilon \Delta_\epsilon$$

where $\epsilon = (\epsilon_1, ..., \epsilon_l)$ with $\epsilon_i = \pm1$, and $\Delta_\epsilon = \{a \in \mathbb{C}_{2l} \mid Q_j a = \epsilon_j a\ \}$.

By right multiplication of elements of $\mathbb{C}_{2l}$ all Δ_ϵ's are isomorphic to each other; let Δ denote the isomorphisim class of these modules. So Δ is a complex vector space of dimension 2^l which we can write

$$\mathbb{C}_{2l} \cong 2^l\Delta \cong \Delta^* \otimes \Delta \cong \mathrm{End}(\Delta)$$

Considering $\mathbb{C}_{2l}$ as a $\mathbb{C}_{2l}$-module by left multiplication, we see that $\mathbb{C}_{2l}$ preserves each Δ_ϵ. Hence Δ is a $\mathbb{C}_{2l}$ module giving the representation

$$\rho : \mathbb{C}_{2l} \to GL(\Delta)$$

There is also a more concrete proof; it proceeds by first identifying

$$\begin{array}{rcl} \mathbb{C}_2 & \overset{\cong}{\underset{\Psi}{\longmapsto}} & \mathfrak{g}\ell(2,\mathbb{C}) \\ 1 & \longmapsto & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ e_1 & \longmapsto & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ e_2 & \longmapsto & \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \text{hence}\quad e_1e_2 & \longmapsto & \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix} \end{array}$$

$$\Psi(e_1)^2 = -1$$
$$\Psi(e_2)^2 = -1$$
$$\Psi(e_1)\Psi(e_2) = -\Psi(e_2)\Psi(e_1)$$

We claim that: $\mathbb{C}_{2l} \cong \mathbb{C}_{2l-2} \otimes \mathbb{C}_2$. This claim finishes the proof by induction:

$$\mathbb{C}_{2l} \cong \bigotimes^{l} \mathbb{C}_2 \cong \bigotimes^{l} g\ell(2,\mathbb{C}) \cong g\ell(2^l,\mathbb{C})$$

We can see the claim by writing the isomorphisim $\theta : \ \mathbb{C}_{2l} \to \mathbb{C}_{2l-2} \otimes \mathbb{C}_2$

$$\begin{aligned} 1 &\longmapsto 1 \otimes 1 \\ e_1 &\longmapsto 1 \otimes e_1 \\ e_2 &\longmapsto 1 \otimes e_2 \\ e_j &\longmapsto e_{j-2} \otimes i(e_1 e_2) \ \text{ for } \ j = 3, 4, \ldots, 2m \end{aligned}$$

It is easy to check that $\theta(e_j)^2 = -1$ and $\theta(e_j)\theta(e_k) = -\theta(e_k)\theta(e_j)$ □

There is also a further splitting of the complex vector space

$$\Delta = \Delta_- \oplus \Delta_+$$

as the ± 1 eigenspaces of of the involution $\tau : \Delta \to \Delta$ given by left multiplication with the volume form $\tau = i^l e_1 e_2 ... e_{2l}$.

Definition 6.9. *$Spin(n) \subset C_n \subset \mathbb{C}_n$ is the subgroup of units in C_n generated by even number of vectors in $\mathbb{R}^n$, i.e.,*

$$Spin(n) \ = \ \{x_1 \cdots x_{2r} \mid x_j \in \mathbb{R}^n, \ ||x_j|| = 1, \ r = 0, 1, 2, \ldots\}$$

For $v = x_1 x_2 \cdots x_{2r} \in Spin(n)$, define $v^t = x_{2r} \cdots x_2 x_1$, notice $vv^t = 1$. Let

$$ad : Spin(n) \to GL(C_n)$$

be adjoint representation $ad(v)(a) = vav^t$. This representation preserves the subspace $\mathbb{R}^n \subset C_n$ as orthogonal transformations. This gives a homomorphisim

$$\pi : Spin(n) \to SO(n)$$

which is a two fold covering map. We can check these statements by choosing a unit vector $x \in \mathbb{R}^n$ and completing it to orthogonal basis $\{x, f_2, \ldots, f_n\}$. If $v = ax + b_2 f_2 + \cdots + b_n f_n$, then we can compute the adjoint map

$$\begin{aligned} ad(x)(v) &= x(ax + b_2 f_2 + \cdots + b_n f_n)x^t \\ &= -ax + b_2 f_2 + \cdots + b_n f_n \end{aligned}$$

since $x^2 = -1$, $xf_i = -f_i x$, we have $ad(x)v =$ reflection through x. Hence for $v \in Spin(n)$, $\pi(v)$ is a product of even number of reflections, i.e. $\pi(v) \in SO(n)$.

Example 6.10. *From the following identity we can identify* $Spin(2)$ *with* S^1*:*

$$(e_1 Cos\ \alpha + e_2 Sin\ \alpha)(e_1 Cos\ \beta + e_2 Sin\ \beta) = Cos\ (\beta - \alpha) + e_1 e_2 Sin\ (\beta - \alpha)$$

$$Spin(2) = \{Cos\ \theta + e_1 e_2 Sin\ \theta\}$$

and the map $\pi : Spin(2) = S^1 \to S^1 = SO(2)$ *becomes multiplication by two* $\theta \mapsto 2\theta$

$$\pi(\theta) = \begin{pmatrix} Cos\ 2\theta & -Sin\ 2\theta \\ Sin\ 2\theta & Cos\ 2\theta \end{pmatrix}$$

Left multiplication $Spin(2l) \times \Delta \to \Delta$ preserves complex subspaces $\Delta_{\pm}$, this gives 2^{l-1} dimensional complex repesentations

$$\rho_{\pm} : Spin(2l) \to GL(\Delta_{\pm})$$

Also notice that $\rho \otimes \rho : \mathbb{C}_{2l} \to GL(\Delta \otimes \Delta)$ and $ad : \mathbb{C}_{2l} \to GL(\mathbb{C}_{2l})$ being irreducible representations of the same dimension they have to be isomorphic. This gives yet another proof of the proposition that as $Spin(n)$ modules

$$\Lambda^*(\mathbb{C}^{2l}) \cong \Delta \otimes \Delta$$

Similarly by the same dimension counting we can write

$$\mathbb{C}_{2l} = End(\Lambda^*(\mathbb{C}^l))$$

Now let us specialize to the four dimensional case, i.e., $l = 2$. Also for brevity let us, when convenient, denote direct sum and tensor products of modules as sum and products. The isomorphism $\mathbb{C}_4 \cong \mathbb{C}_2 \times \mathbb{C}_2$ gives the splitting

$$Spin(4) = Spin(3) \times Spin(3)$$

Each $Spin(3) = SU(2) = S^3$ sits in $\mathbb{C}_2 \cong \mathbb{H} \cong \mathbb{C}^2$ as unit quaternions, and the adjoint action of $Spin(4)$ on $\mathbb{R}^4$ via $\pi : Spin(4) \to SO(4)$ is given by:

$$(x, y) \in S^3 \times S^3, \text{ and } v \in \mathbb{H} \qquad (x, y)v \longmapsto xvy^t$$

By complexifying $\mathbb{H} \otimes \mathbb{C} = \mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$ we see $\mathbb{C}^4$ as a tensor product of two $SU(2)$ representations; these in fact come by restricting the usal $Spin(4)$ representations $\rho_{\pm}$ to the two of the factors

$$SU(2) \to Spin(4) \to GL(\Delta_{\pm})$$

We have $\Lambda^*(\mathbb{C}^4) = 2(\mathbb{C}+\mathbb{C}^4)+\Lambda^2(\mathbb{C}^4) = \mathbb{C}^4+\mathbb{C}^4+(\Lambda^2_-(\mathbb{C}^4)+\mathbb{C})+(\Lambda^2_+(\mathbb{C}^4)+\mathbb{C})$, which is isomorphic to $(\Delta_-+\Delta_+)(\Delta_-+\Delta_+)$ as $Spin(4)$ modules, in fact

$$\begin{array}{rcl} \mathbb{C}^4 & \cong & \Delta_+\Delta_- \\ \Lambda^2_-(\mathbb{C}^4)+\mathbb{C} & \cong & \Delta_-\Delta_- \\ \Lambda^2_+(\mathbb{C}^4)+\mathbb{C} & \cong & \Delta_+\Delta_+ \end{array}$$

These facts can be verified by identifying the representations in corresponding maximal tori. Write $S^1 = \mathbb{R}/\mathbb{Z} = \{e^{2\pi i\theta} \mid 0 \le \theta \le 1\}$, consider maps

$$S^1\times S^1 \xrightarrow{\omega} Spin(4) \xrightarrow{\pi} S^1\times S^1 \subset SO(4)$$

$$\omega(\theta_1,\theta_2) = (Cos\ 2\pi\theta_1 + e_1e_2 Sin\ 2\pi\theta_1)(Cos\ 2\pi\theta_2 + e_3e_4 Sin\ 2\pi\theta_2)$$

$$\pi\circ\omega(\theta_1,\theta_2) = \begin{pmatrix} Cos\ 4\pi\theta_1 & -Sin\ 4\pi\theta_1 & & \\ Sin\ 4\pi\theta_1 & Cos\ 4\pi\theta_1 & & \\ & & Cos\ 4\pi\theta_2 & -Sin\ 4\pi\theta_2 \\ & & Sin\ 4\pi\theta_2 & Cos\ 4\pi\theta_2 \end{pmatrix} = (2\theta_1, 2\theta_2)$$

Let $t = e^{2\pi i\theta} \in S^1$ $(0 \le \theta < 2\pi)$, we can identify $\mathrm{Ker}(\pi\circ\omega) = \{(\pm1,\pm1)\}$, and hence we obtain the maximal tori by factoring ω, they are images of f and g

$$\begin{array}{lll} (S^1\times S^1)\times\mathbb{C} & im(f) \cong (S^1\times S^1)/\mathbb{Z}_2 \hookrightarrow Spin(4) & \\ \uparrow f & & \\ S^1\times S^1 & & \downarrow\pi \\ \downarrow g & & \\ S^1\times S^1 & im(g)\cong S^1/\mathbb{Z}_2\times S^1/\mathbb{Z}_2 \hookrightarrow SO(4) & \end{array}$$

where $f(t_1,t_2) = (t_1^2, t_2^2, t_1t_2)$, and $g(t_1,t_2) = (t_1^2, t_2^2)$. Therefore we get the representation spaces of maximal tori, and the corresponding maps as follows

$$\begin{array}{ccc} R(Spin(4)) & \longleftarrow & R(SO(4)) \\ \bigcap & & \bigcap \\ R(T) & & R(T) \\ \| & & \| \\ \mathbb{Z}[t_1^{1/2}, t_1^{-1/2}, t_2^{1/2}, t_2^{-1/2}] & \xleftarrow{\theta} & \mathbb{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}] \quad , \quad \theta(t_i) = t_i \end{array}$$

$$\begin{array}{lcl} R(Spin(4)) & = & \mathbb{Z}[1,\Delta_-,\Delta_+] \qquad (\text{ here } 1 = \mathbb{C}) \\ R(SO(4)) & = & \mathbb{Z}[1,\mathbb{C}^4,\Lambda^2_-,\Lambda^2_+]\ , \ \text{with relation } (1+\Lambda^2_-)(1+\Lambda^2_+) = (\mathbb{C}^4)^2 \end{array}$$

$$\Delta = (t_1^{1/2} + t_1^{-1/2})(t_2^{1/2} + t_2^{-1/2}) \qquad \mathbb{C}^4 = t_1 + t_1^{-1} + t_2 + t_2^{-1}$$
$$\Delta_+ = t_1^{1/2} t_2^{1/2} + t_1^{-1/2} t_2^{-1/2} \qquad \Lambda^2_+ = 1 + t_1 t_2 + t_1^{-1} t_2^{-1}$$
$$\Delta_- = t_1^{1/2} t_2^{-1/2} + t_1^{-1/2} t_2^{1/2} \qquad \Lambda^2_- = 1 + t_1 t_2^{-1} + t_1^{-1} t_2$$

Now we can check all above the above identities. For example we can verify

$$\Lambda^*(\mathbb{C}^4) = (1 + t_1)(1 + t_1^{-1})(1 + t_2)(1 + t_2^{-1}) = \Delta^2$$

As $SO(3)$ modules we also have $ad(\Delta_\pm) \cong \Lambda^2_\pm(\mathbb{R}^4)$, for example $ad(\Delta_-) = 1 + t_1 t_2^{-1}$ and hence $ad(\Delta_-) \otimes \mathbb{C} = 1 + t_1 t_2^{-1} + t_1^{-1} t_2$. A useful fact: the obvious isomorphisim

$$so(4) \cong \Lambda^2 T^*(X) = \Lambda^2_- + \Lambda^2_+$$

identifies $ad_{\mathbb{C}}$, the complexified adjoint representation $ad : SO(4) \to GL(so(4))$.

Remark 6.11. *If M^4 is spin then $\mathbb{C}^4$ corresponds to $TM_{\mathbb{C}}$, then by 6.4*

$$ch(TM_{\mathbb{C}}) = ch(\Delta_+ \Delta_-) = ch(\Delta_+) ch(\Delta_-) = (2 - c_2(\Delta_-))(2 - c_2(\Delta_+)) \implies$$

$$3\sigma(M) = p_1(TM)[M] = -c_2(TM_{\mathbb{C}})[M] = -2c_2(\Delta_+)[M] - 2c_2(\Delta_-)[M] \tag{6.5}$$

where $[M]$ is the fundamental class. Also by Atiyah-Singer index theorem we get

$$\chi(M) = -c_2(\Delta_+)[M] + c_2(\Delta_-)[M] \tag{6.6}$$

$$c_2(\Delta_+)[M] = (-3\sigma(M) - 2\chi(M))/4$$
$$c_2(\Delta_-)[M] = (-3\sigma(M) + 2\chi(M))/4$$

and $c_2(\Lambda_\pm(M)) = 4c_2(\Delta_\pm)$. Also if $L \to X$ is a complex line bundle, by then using 6.4 we get $ch(\Delta_+ L) = ch(\Delta_+) ch(L)$ and from these we get the corresponding Chern classes.

$$c_1(\Delta_\pm L) = 2c_1(L) = c_1(L^2)$$

$$c_2(\Delta_+ L)[M] = (c_1^2(L^2)[M] - 3\sigma(M) - 2\chi(M))/4$$
$$c_2(\Delta_- L)[M] = (c_1^2(L^2)[M] - 3\sigma(M) + 2\chi(M))/4$$

6.2 Fundamental group of Gauge group

Theorem 6.12. *([AMR]) Let $P \to X$ be a principal $SO(3)$-bundle over a compact smooth 4-manifold X, with $H_1(X) = 0$; then*

$$\pi_1(\mathcal{B}^*(P)) = \begin{cases} Z_2 & \text{if } \left\{ \begin{array}{c} p_1(P) = \sigma(X) \ mod\ 8 \\ w_2(P) = w_2(TX) \ mod\ 2 \end{array} \right\} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By the description 5.10, the fundamental group is identified by the isomorphism class of $SO(3)$-bundles over $S^1 \times X$ which restrict to P on $s_0 \times X$ and isomorphic to P on all other slices. To count these bundles we cut $S^1 \times X$ open along $s_0 \times X$ and count the bundles on $I \times X$ which restrict to P on the boundary.

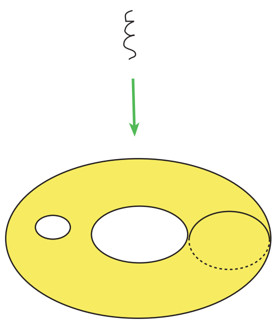


Figure 6.1:

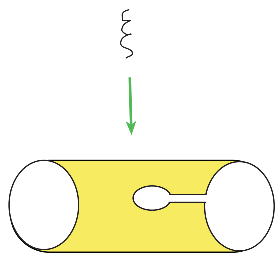


Figure 6.2:

But since $H^4(X \times S^1) \approx H_1(X) \oplus Z$ has no 2-torsion, by [DW] the restriction of these bundles over the 4-skeleton $X \times S^1 - B^5$ are determined by their first Pontryagin and the second Steifel-Whitney classes. But since $H_1(X) = 0$, the only possibility for these classes are $p_1(P) \times 1$ and $w_2(P) \times 1$. Therefore the restrictions of these bundles to $X \times S^1 - B^5$ are unique. Furthermore, there are two different ways of extending any bundle over $X \times S^1 - B^5$ to the top skeleton since $\pi_4 SO(3) = Z_2$. Hence if $\eta \to S^5$ is the nontrivial bundle given by the generator of $\pi_4 SO(3) = Z_2$, then

$$\pi_1(\mathcal{B}^*) = \begin{cases} 0 & \text{if } P\#\eta \cong P, \\ Z_2 & \text{otherwise} \end{cases}$$

where $P \to S^1 \times X$ also denotes the bundle pulled back from $P \to X$ by the projection map and $P\#\eta$ is obtained from P by clutching P with η in a chart. The following is an observation of J. Morgan [M]:

Lemma 6.13. *If there is an $SO(3)$-bundle $\xi \to S^1 \times S^1 \times X - int(B^6)$, such that the restriction to the boundary $\xi\big|_{S^5} = \eta$ and $\xi\big|_{0\times S^1\times X} = P$, then $\pi_1(\mathcal{B}^*) = 0$ otherwise $\pi_1(\mathcal{B}^*) = Z_2$.*

Proof. Given such a bundle ξ we cut $S^1 \times S^1 \times X - \mathrm{int}(B^6)$ open to get a bundle of $I \times S^1 \times X - int(B^6)$ which restricts to P on the two outside boundary components and to η over S^5. Then by connecting summing S^5 to one of the outside boundary components as in Figure 6.2 we get a bundle over $I \times X$ restricting P and $P\#\eta$ over the two ends; hence, these two bundles are isomorphic. The converse of this also holds. The result follows from the above description of $\pi_1(\mathcal{B}^*)$. □

Now the proof of Theorem 6.12 follows from the following lemma:

Lemma 6.14. *Every bundle $\xi \to S^1 \times S^1 \times X - int(B^6)$ with $\xi\big|_{0\times S^1\times X} = P$ extends over $S^1 \times S^1 \times X$ if and only if $p_1(P) = c_1^2(P)$ mod 8, and $c_1(P) = c_1(TX)$ mod 2, where $c_1(P)$ is an integral lift of $w_2(P)$.*

Proof. Call $(Z, Z_0) = (S^1 \times S^1 \times X,\ S^1 \times X^1 \times X - \mathrm{int}(B^6))$. Since w_2 has an integral lift, ξ comes from a $U(2)$-bundle. So we can assume that ξ is a $U(2)$-bundle. In particular, $c_2(P) = (c_1^2(P) - p_1(P))/4$. The obstruction to extending the bundle $\xi \to Z_0$ over the whole six-skeleton Z is given by k_6 where k_6 is the first k-invariant of $BU(2)$. By the definition, $k_6 \in H^6(K(Z,2) \times K(Z,4), Z_2)$ such that its pull-back to $BU(2)$ vanishes. $H^6(K(Z,2) \times K(Z,4), Z_2)$ is generated by $c_1^3, Sq^2c_2,\ c_1c_2$. It is easy to check that

$$k_6 = Sq^2c_2(\xi) + c_1(\xi)c_2(\xi)$$

is the only choice. We can write

$$\begin{aligned} c_2(\xi) &= 1 \times 1 \times c_2(P) + [S^1] \times [S^1] \times \alpha, \\ c_1(\xi) &= 1 \times 1 \times c_1(P) + \varepsilon([S^1] \times [S^1] \times 1) \end{aligned}$$

where $\alpha \in H^2(X)$ and $\varepsilon = 0$ or 1, and obtain

$$k_6 = [S^1] \times [S^1] \times (\alpha^2 + \alpha c_1(P) + \varepsilon c_2(P)).$$

Hence k_6 is zero for all such bundles only if for all $\alpha \in H^2(X)$

$$\alpha^2 + \alpha c_1(P) = 0 \text{ mod } 2 \ \text{ and } \ c_2(P) = 0 \text{ mod } 2.$$

By Wu's formula the first condition is equivalent to $c_1(P) = c_1(TX)$ mod 2, and the second condition is obviously equivalent to $p_1(P) = c_1^2(P)$ mod 8.

It remains to show that vanishing k_6 is sufficient for extending ξ to Z. Since if $k_6 = 0$, the obstruction theory says that ξ can be extended only after possibly changing it over the 5-cells, let us show that this is not necessary. That is, if ξ does not extend, then it cannot be extended even after readjusting it over the 5-cells. Otherwise we would have two bundles over the complement of the four-skeleton:

$$Z_0 - T^2 \times X^{(2)} \cup \mathrm{int}(B^2) \times Y = T_0^2 \times Y - \mathrm{int}(B^6)$$

agreeing over the outside boundary, and one extending the other not extending over B^6. Here $X^{(2)}$ is the two-skeleton of $X, Y = X - X^{(2)}$, T^2 is the 2-torus, B^2 is a ball in T^2, and $T_0 = T - \mathrm{int}(B^2)$. By putting these bundles together on the double of this manifold (two copies of the manifold glued along the common boundary) we obtain a bundle on

$$W = \mathrm{Double}(T_0^2 \times Y) - \mathrm{int}(B^6)$$

which is nontrivial on the boundary. Since $H_1(X) = 0$, Y is a homology 4-ball, and hence W is a homology $S^1 \times S^5 \# S^1 \times S^5$ minus a point. Hence we get a map $f : W \to B_{SO(3)}$ which restricts to the nontrivial element of $\pi_5(B_{SO(3)}) = Z_2$ on the boundary S^5. By surgering $\pi_1(W)$ we can turn f into a map from a homology ball $f : B^6 \to B_{SO(3)}$, contradicting with the fact that f is essential on the boundary S^5 (Figures 6.3 and 6.4). $\square$

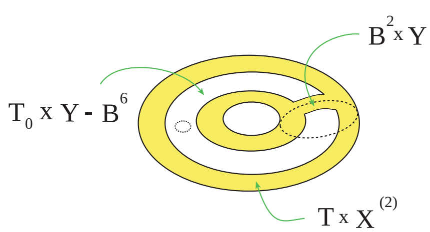


Figure 6.3:

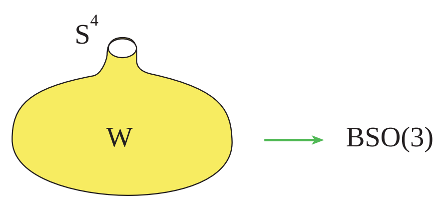


Figure 6.4:

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