

# TWISTING 4-MANIFOLDS ALONG $\mathbb{R}\mathbb{P}^2$

SELMAN AKBULUT

ABSTRACT. We prove that the Dolgachev surface  $E(1)_{2,3}$  (which is an exotic copy of the elliptic surface  $E(1) = \mathbb{C}\mathbb{P}^2 \# 9\overline{\mathbb{C}\mathbb{P}^2}$ ) can be obtained from  $E(1)$  by twisting along a simple “plug”, in particular it can be obtained from  $E(1)$  by twisting along  $\mathbb{R}\mathbb{P}^2$ .

## 0. INTRODUCTION

Given a smooth 4-manifold  $M^4$ , what is the minimal genus  $g$  of an imbedded surface  $\Sigma_g \subset M^4$ , such that twisting  $M$  along  $\Sigma$  produces an exotic copy of  $M$ ? Here twisting means cutting out a tubular neighborhood of  $\Sigma$  and regluing back by a nontrivial diffeomorphism. Not much known about the case  $g > 1$ . The case  $g = 1$  is the well known “logarithmic transform” operation, which can change the smooth structure in some cases; in fact the first example of a closed exotic manifold found by Donaldson [D] was the Dolgachev surface  $E(1)_{2,3}$  which is obtained from  $E(1) = \mathbb{C}\mathbb{P}^2 \# 9\overline{\mathbb{C}\mathbb{P}^2}$  by two log transforms. The  $g = 0$  case is not well understood, twisting along  $S^2$  is usually called “Gluck construction” and we don’t know if this operation changes the smooth structure of an any orientable manifold, but there is an example of non-orientable manifold which the Gluck construction changes its smooth structure [A1]. The interesting case of  $\Sigma = \mathbb{R}\mathbb{P}^2$  was studied indirectly in [AY1] under the guise of *plugs*, which are more general objects. Recall that Figure 1 describes the tubular neighborhood  $W$  of  $\mathbb{R}\mathbb{P}^2$  in  $S^4$  as a disc bundle over  $\mathbb{R}\mathbb{P}^2$  (e.g. [A2]):

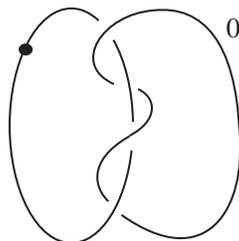


FIGURE 1.  $W$

If we attach a 2-handle to  $W$  as in Figure 2 we obtain an interesting manifold, which is the  $W_{1,2}$  “plug” of [AY1]. Recall [AY1], a *plug*  $(P, f)$  of  $M^4$  is a codimension zero Stein submanifold  $P \subset M$  with an involution  $f : \partial P \rightarrow \partial P$ , such that  $f$  does not extend to a homeomorphism inside; and the operation  $N \cup_{id} P \mapsto N \cup_f P$  of removing  $P$  from  $M$  and regluing it to its complement  $N$  by  $f$ , changes the smooth structure of  $M$  (this operation is called a “*plug twisting*”). For example the involution  $f : \partial W_{1,2} \rightarrow \partial W_{1,2}$  is induced from  $180^\circ$  rotation of the Figure 2, e.g. it maps the (red and blue) loops to each other  $\alpha \leftrightarrow \beta$ .

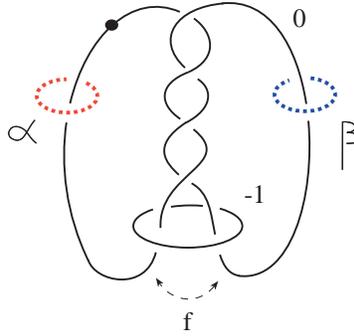


FIGURE 2.  $W_{1,2}$

Notice that the twisting along  $W_{1,2}$  is induced by twisting along  $\mathbb{RP}^2$  inside (i.e. cutting out  $W$  and regluing by the involution induced by the rotation). In [AY1] some examples of changing smooth structures via plug twisting were given, including twisting the  $W_{1,2}$  plug. Here we prove that by twisting along a  $W_{1,2}$  plug (in particular twisting along  $\mathbb{RP}^2$ ) we can completely decompose the Dolgachev surface  $E(1)_{2,3}$ . The following theorem should be considered as a structure theorem for the Dolgachev surface complementing Theorem 1 of [A3], where it was shown that a “*cork twisting*” also completely decomposes  $E(1)_{2,3}$ .

**Theorem 1.**  $E(1)_{2,3}$  is obtained by plug twisting  $E(1)$  along  $W_{1,2}$ , i.e. we can decompose  $E(1) = N \cup_{id} W_{1,2}$ , so that  $E(1)_{2,3} = N \cup_f W_{1,2}$ .

*Proof.* By cancelling the 1- and 2-handle pair of Figure 2 we obtain Figure 3, which is an alternative picture of  $W_{1,2}$ . By inspecting the diffeomorphism Figure 2  $\mapsto$  Figure 3 we see that the involution  $f$  twists the tubular neighborhood of  $\alpha$  once, while mapping to  $\beta$ .

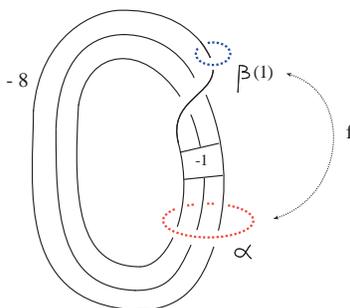


FIGURE 3.  $W_{1,2}$

By attaching a chain of eight 2-handles to  $-W_{1,2}$  (the mirror image of Figure 3) and a  $+1$  framed 2-handle to  $\alpha$ , we obtain Figure 4, which is a handlebody of  $E(1)$  given in [A3]. In Figure 4 performing  $W_{1,2}$  plug twist to  $E(1)$  has the effect of replacing the  $+1$ -framed 2-handle attached to  $\alpha$ , with a zero framed 2-handle attached to  $\beta$ . Here the complement of  $W_{1,2}$  in  $E(1)$  is the submanifold  $N$  consisting of the zero framed 2-handle (the cusp) and the chain of eight 2-handles, and the plug twisting is the operation:  $N \cup \alpha^{+1} \mapsto N \cup \beta^0$  (as seen from  $N$ ).

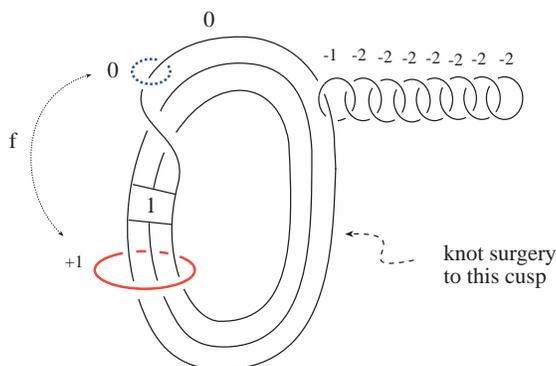


FIGURE 4.  $E(1)$

Therefore the plug twisting of  $E(1)$  along  $W_{1,2}$  gives Figure 5. After sliding over  $\beta$ , the chain of eight 2-handles become free from the rest of the figure, giving a splitting:  $Q \# 8\mathbb{C}P^2$ , where  $Q$  is the cusp with the trivially linking zero framed circle, hence  $Q = S^2 \times S^2$ . So the Figure 5 is just  $S^2 \times S^2 \# 8\bar{\mathbb{C}}P^2 = E(1)$ .

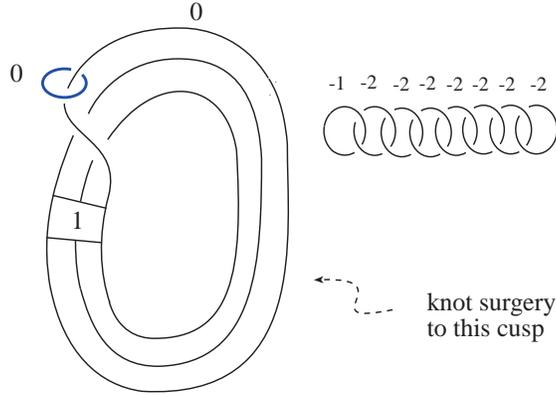


FIGURE 5

Next notice that if we first perform a “knot surgery” operation  $E(1) \mapsto E(1)_K$  by a knot  $K$ , along the cusp inside of Figure 4, and then do the plug twist along  $W_{1,2}$  (notice the cusp is disjoint from the plug since it lies in  $N$ ) we get the similar splitting except this time resulting:  $Q_K \# 8\bar{\mathbb{C}P}^2$ , where  $Q_K$  is the knot surgered  $Q$ . Notice the manifold  $Q = S^2 \times S^2$  is obtained by doubling the cusp, and  $Q_K$  is obtained by doing knot surgery to one of these cusps. In Theorem 4.1 of [A4] it was shown that when  $K$  is the trefoil knot then  $Q_K = S^2 \times S^2$ . Also recall that when  $K$  is the trefoil knot we have the identification with the Dolgachev surface  $E(1)_K = E(1)_{2,3}$  (e.g. [A3]).  $\square$

**Remark 1.** *If we could identify  $Q_K$  with  $S^2 \times S^2$  for infinitely many knots  $K$  with distinct Alexander polynomials, we would have infinitely many transforms  $E(1) \mapsto E(1)_K$  obtained by plug twistings along  $W_{1,2}$ . This would give infinitely many non-isotopic imbeddings  $W_{1,2} \subset E(1)$ , similar to the examples in [AY2]. In the absence of such identification we can only conclude that  $W_{1,2}$  is a plug of infinitely many distinct exotic copies  $E(1)_K$  of  $E(1)$ .*

**Remark 2.** *Recall that  $\partial W$  is the quaternionic 3-manifold, which is the quotient of  $S^3$  by the free action of the quaternionic group of order eight  $G = \langle i, j, k \mid i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j \rangle$  (e.g. [A2]). This manifold is a positively curved space-form and an  $L$  space (Floer homology groups vanish). Hence the change of smooth structure of  $E(1)$  by twisting  $W$  is due to the change of  $Spin^c$  structures, rather than permuting the Floer homology by the involution as in [A3], [AD].*

## REFERENCES

- [A1] S. Akbulut, *Constructing a fake 4-manifold by Gluck construction to a standard 4-manifold*, *Topology*, vol. 27, no. 2 (1988), 239-243.
- [A2] S. Akbulut, *Cappell-Shaneson's 4-dimensional s-cobordism*, *Geometry-Topology*, vol.6, (2002), 425-494
- [A3] S. Akbulut, *The Dolgachev surface*, arXiv:0805.1524v4 (2008).
- [A4] S. Akbulut, *A fake cusp and a fishtail*, *Turkish Jour. of Math* 1 (1999), 19-31. arXiv:math.GT/9904058.
- [A5] S. Akbulut, *Variations on Fintushel-Stern knot surgery*, *TJM* (2001), 81-92. arXiv:math.GT/0201156.
- [AD] S. Akbulut, and S. Durusoy. *An involution acting non-trivially on Heegaard-Floer homology*, *Fields Institute Communications*, vol 47, (2005),1-9
- [AY1] S. Akbulut and K.Yasui, *Corks, Plugs and exotic structures*, *Jour of GGT* 2 (2008), 40-82. arXiv:0806.3010
- [AY2] S. Akbulut and K.Yasui, *Knotting Corks*, *Journal of Topology*.
- [D] S.K. Donaldson, *Irrationality and the h-cobordism conjecture*, *Journal of Differential Geometry* 26 (1), (1987) 141168.
- [FS] R. Fintushel and R. Stern *Six lectures on 4-manifolds* , (2007) arXiv:math.GT/0610700v2.

DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, MI, 48824  
E-mail address: akbulut@math.msu.edu